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2024

## Conventions, Definitions, Identities, and Other Useful Formulae

Robert A. McNees IV

Loyola University Chicago, [rmcnees@luc.edu](mailto:rmcnees@luc.edu)

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# Conventions, Definitions, Identities, and Formulas

*Last modified: August 31, 2023.*

A collection of results that are useful enough for me to keep them all in one place. [Please let me know](#) if you find any typos or mistakes!

Several people have contacted me to say they used some of these results in published papers. That's great! If you use this reference for something that ends up in a publication, please consider including a citation with my name, the page title, and the url. [Here's how to do it in REVTeX](#) (thanks, [Tim Wiser](#)), and here's the [Bib-TeX code](#) I used in a recent paper.

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## 1. Curvature Tensors

Consider a  $d + 1$  dimensional spacetime  $(\mathcal{M}, g)$ . The covariant derivative  $\nabla$  is metric-compatible with  $g$ .

- Christoffel Symbols

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\rho} (\partial_{\mu} g_{\rho\nu} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu}) \quad (1)$$

- Riemann Tensor

$$R^{\lambda}_{\mu\sigma\nu} = \partial_{\sigma} \Gamma_{\mu\nu}^{\lambda} - \partial_{\nu} \Gamma_{\mu\sigma}^{\lambda} + \Gamma_{\mu\nu}^{\kappa} \Gamma_{\kappa\sigma}^{\lambda} - \Gamma_{\mu\sigma}^{\kappa} \Gamma_{\kappa\nu}^{\lambda} \quad (2)$$

- Ricci Tensor

$$R_{\mu\nu} = \delta^{\sigma}_{\lambda} R^{\lambda}_{\mu\sigma\nu} \quad (3)$$

- Schouten Tensor

$$S_{\mu\nu} = \frac{1}{d-1} \left( R_{\mu\nu} - \frac{1}{2d} g_{\mu\nu} R \right) \quad (4)$$

$$\nabla^\nu S_{\mu\nu} = \nabla_\mu S^\nu{}_\nu \quad (5)$$

- Weyl Tensor

$$C^\lambda{}_{\mu\sigma\nu} = R^\lambda{}_{\mu\sigma\nu} + g^\lambda{}_\nu S_{\mu\sigma} - g^\lambda{}_\sigma S_{\mu\nu} + g_{\mu\sigma} S^\lambda{}_\nu - g_{\mu\nu} S^\lambda{}_\sigma \quad (6)$$

- Commutators of Covariant Derivatives

$$[\nabla_\mu, \nabla_\nu] A_\lambda = R_{\lambda\sigma\mu\nu} A^\sigma \quad (7)$$

$$[\nabla_\mu, \nabla_\nu] A^\lambda = R^\lambda{}_{\sigma\mu\nu} A^\sigma \quad (8)$$

- Bianchi Identities

$$\nabla_\kappa R_{\lambda\mu\sigma\nu} - \nabla_\lambda R_{\kappa\mu\sigma\nu} + \nabla_\mu R_{\kappa\lambda\sigma\nu} = 0 \quad (9)$$

$$\nabla^\nu R_{\lambda\mu\sigma\nu} = \nabla_\mu R_{\lambda\sigma} - \nabla_\lambda R_{\mu\sigma} \quad (10)$$

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2} \nabla_\mu R \quad (11)$$

- Bianchi Identities for the Weyl Tensor

$$\nabla^\nu C_{\lambda\mu\sigma\nu} = (d-2) (\nabla_\mu S_{\lambda\sigma} - \nabla_\lambda S_{\mu\sigma}) \quad (12)$$

$$\begin{aligned} \nabla^\lambda \nabla^\sigma C_{\lambda\mu\sigma\nu} = & \frac{d-2}{d-1} \left[ \nabla^2 R_{\mu\nu} - \frac{1}{2d} g_{\mu\nu} \nabla^2 R - \frac{d-1}{2d} \nabla_\mu \nabla_\nu R \right. \\ & - \left( \frac{d+1}{d-1} \right) R_\mu{}^\lambda R_{\nu\lambda} + C_{\lambda\mu\sigma\nu} R^{\lambda\sigma} + \frac{(d+1)}{d(d-1)} R R_{\mu\nu} \\ & \left. + \frac{1}{d-1} g_{\mu\nu} \left( R^{\lambda\sigma} R_{\lambda\sigma} - \frac{1}{d} R^2 \right) \right] \end{aligned} \quad (13)$$

## 2. Differential Forms

- p-Form Components

$$\mathbf{A}_{(p)} = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \quad (14)$$

- Exterior Derivative

$$\left( d\mathbf{A}_{(p)} \right)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} \quad (15)$$

$$B_{[\mu_1 \dots \mu_n]} := \frac{1}{n!} \left( B_{\mu_1 \dots \mu_n} + \text{permutations} \right) \quad (16)$$

- Hodge-Star

$$\left( \star \mathbf{A}_{(p)} \right)_{\mu_1 \dots \mu_{d+1-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{d+1-p}}^{\nu_1 \dots \nu_p} A_{\nu_1 \dots \nu_p} \quad (17)$$

$$\star \star = (-1)^{p(d+1-p)+1} \quad (18)$$

- Wedge Product

$$\left( \mathbf{A}_{(p)} \wedge \mathbf{B}_{(q)} \right)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p! q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]} \quad (19)$$

### 3. Lie Derivatives

Let  $T$  be a rank  $(n, m)$  tensor and  $\xi$  be a vector. The Lie derivative of  $T$  along  $\xi$  is also a rank  $(n, m)$  tensor, with components

$$\begin{aligned} \mathcal{L}_\xi T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} &= \xi^\lambda \partial_\lambda T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \\ &\quad - T^{\lambda \mu_2 \dots \mu_n}_{\nu_1 \dots \nu_m} \partial_\lambda \xi^{\mu_1} - \dots - T^{\mu_1 \dots \mu_{n-1} \lambda}_{\nu_1 \dots \nu_m} \partial_\lambda \xi^{\mu_n} \\ &\quad + T^{\mu_1 \dots \mu_n}_{\lambda \nu_2 \dots \nu_m} \partial_{\nu_1} \xi^\lambda + \dots + T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_{m-1} \lambda} \partial_{\nu_m} \xi^\lambda \end{aligned} \quad (20)$$

Any derivative operator can be used here.

### 4. Euler Densities

Let  $\mathcal{M}$  be a manifold with dimension  $d + 1 = 2n$  an even number. Our normalization gives  $\chi(S^{2n}) = 2$ .

- Curvature Two-Form

$$\mathbf{R}^a_b = \frac{1}{2} R^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \quad (21)$$

- Euler Density

$$\mathbf{e}_{2n} = \frac{1}{(4\pi)^n \Gamma(n+1)} \epsilon_{a_1 \dots a_{2n}} \mathbf{R}^{a_1 a_2} \wedge \dots \wedge \mathbf{R}^{a_{2n-1} a_{2n}} \quad (22)$$

$$\mathcal{E}_{2n} = \frac{1}{(8\pi)^n \Gamma(n+1)} \epsilon_{\mu_1 \dots \mu_{2n}} \epsilon_{\nu_1 \dots \nu_{2n}} R^{\mu_1 \mu_2 \nu_1 \nu_2} \dots R^{\mu_{2n-1} \mu_{2n} \nu_{2n-1} \nu_{2n}}$$

- Euler Number

$$\begin{aligned}\chi(\mathcal{M}) &= \int_{\mathcal{M}} d^{2n}x \sqrt{g} \mathcal{E}_{2n} \\ &= \int_{\mathcal{M}} \mathbf{e}_{2n}\end{aligned}\quad (23)$$

- Examples

$$\begin{aligned}\mathcal{E}_2 &= \frac{1}{8\pi} \epsilon_{\mu\nu} \epsilon_{\lambda\rho} R^{\mu\nu\lambda\rho} \\ &= \frac{1}{4\pi} R\end{aligned}\quad (24)$$

$$\begin{aligned}\mathcal{E}_4 &= \frac{1}{128\pi^2} \epsilon_{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} R^{\mu\nu\alpha\beta} R^{\lambda\rho\gamma\delta} \\ &= \frac{1}{32\pi^2} (R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} - 4 R^{\mu\nu} R_{\mu\nu} + R^2) \\ &= \frac{1}{32\pi^2} C^{\mu\nu\lambda\rho} C_{\mu\nu\lambda\rho} - \frac{1}{8\pi^2} \left( \frac{d-2}{d-1} \right) \left( R^{\mu\nu} R_{\mu\nu} - \frac{d+1}{4d} R^2 \right)\end{aligned}\quad (25)$$

## 5. Hypersurface Formed by a Spacelike or Timelike Vector

Let  $\Sigma \subset \mathcal{M}$  be a  $d$  dimensional hypersurface whose embedding in  $\mathcal{M}$  is normal to a unit vector  $n^\mu$ . The vector  $n^\mu$  is assumed to be spacelike or timelike, but not null:  $n^\mu n_\mu = e$  with  $e = \pm 1$ . If the vector is spacelike then it is taken to be “outward pointing,” while (contravariant) timelike unit vectors are taken to be “forward pointing.”

Indices are lowered and raised using  $g_{\mu\nu}$  and  $g^{\mu\nu}$ , though in some cases this is identical to lowering or raising indices with the metric  $h_{\mu\nu}$  induced on the hypersurface (first fundamental form). Symmetrization of indices is implied when appropriate.

- First Fundamental Form / Induced Metric on  $\Sigma$

$$h_{\mu\nu} = g_{\mu\nu} - e n_\mu n_\nu \quad (26)$$

- Projection of a tensor along  $\Sigma$

$$\perp T^{\mu \cdots \nu \cdots} = h^\mu{}_\lambda \cdots h^\sigma{}_\nu \cdots T^{\lambda \cdots \sigma \cdots} \quad (27)$$

- Covariant Derivative on  $\Sigma$  compatible with  $h_{\mu\nu}$

$$D_\mu T^{\alpha \cdots \beta \cdots} = \perp \nabla_\mu T^{\alpha \cdots \beta \cdots} \quad \forall \quad T = \perp T \quad (28)$$

- Intrinsic Curvature of  $(\Sigma, h)$

$$[D_\mu, D_\nu] A^\lambda = \mathcal{R}^\lambda{}_{\sigma\mu\nu} A^\sigma \quad \forall \quad A^\lambda = \perp A^\lambda \quad (29)$$

- “Acceleration” Vector

$$a^\mu = n^\nu \nabla_\nu n^\mu \quad (30)$$

- Second Fundamental Form / Extrinsic Curvature of  $\Sigma$

$$K_{\mu\nu} = \frac{1}{2} \perp (\nabla_\mu n_\nu + \nabla_\nu n_\mu) = \frac{1}{2} h_\mu^\lambda h_\nu^\sigma (\nabla_\lambda n_\sigma + \nabla_\sigma n_\lambda) \quad (31)$$

$$= \frac{1}{2} \mathcal{L}_n h_{\mu\nu} \quad (32)$$

$$= \frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu - e n_\mu a_\nu - e n_\nu a_\mu) \quad (33)$$

- Trace of Extrinsic Curvature

$$K = \nabla_\mu n^\mu \quad (34)$$

- Given a local description of the surface by the condition  $f(x) = 0$ , for some sufficiently differentiable function  $f$ , the unit vector can be defined as a normalized gradient

$$n_\mu = \sqrt{\frac{e}{g^{\alpha\beta} \partial_\alpha f \partial_\beta f}} \partial_\mu f \quad (35)$$

- Useful results for surface-forming  $n_\mu$ :

$$\perp (\nabla_\mu n_\nu - \nabla_\nu n_\mu) = 0 \quad (36)$$

$$\perp (\nabla_\mu a_\nu - \nabla_\nu a_\mu) = \mathcal{D}_\mu a_\nu - \mathcal{D}_\nu a_\mu = 0 \quad (37)$$

$$\nabla_\mu n_\nu = K_{\mu\nu} + e n_\mu a_\nu \quad (38)$$

- Gauss-Codazzi

$$\perp R_{\lambda\mu\sigma\nu} = \mathcal{R}_{\lambda\mu\sigma\nu} + e (K_{\mu\sigma} K_{\nu\lambda} - K_{\lambda\sigma} K_{\mu\nu}) \quad (39)$$

$$\perp (R_{\lambda\mu\sigma\nu} n^\lambda) = \mathcal{D}_\nu K_{\mu\sigma} - \mathcal{D}_\sigma K_{\mu\nu} \quad (40)$$

$$\perp (R_{\lambda\mu\sigma\nu} n^\lambda n^\sigma) = -\mathcal{L}_n K_{\mu\nu} + K_\mu^\lambda K_{\lambda\nu} + \mathcal{D}_\mu a_\nu - e a_\mu a_\nu \quad (41)$$

- Projections of the Ricci tensor

$$\begin{aligned} \perp (R_{\mu\nu}) &= \mathcal{R}_{\mu\nu} + e (\mathcal{D}_\mu a_\nu - e a_\mu a_\nu) - e \mathcal{L}_n K_{\mu\nu} - e K K_{\mu\nu} \\ &\quad + 2e K_\mu^\lambda K_{\nu\lambda} \end{aligned} \quad (42)$$

$$\perp (R_{\mu\nu} n^\mu) = \mathcal{D}^\mu K_{\mu\nu} - \mathcal{D}_\nu K \quad (43)$$

$$R_{\mu\nu} n^\mu n^\nu = -\mathcal{L}_n K - K^{\mu\nu} K_{\mu\nu} + \mathcal{D}_\mu a^\mu - e a_\mu a^\mu \quad (44)$$

- Decomposition of the Ricci scalar

$$R = \mathcal{R} - e K^2 - e K^{\mu\nu} K_{\mu\nu} - 2e \mathcal{L}_n K + 2e (\mathcal{D}_\mu a^\mu - e a_\mu a^\mu) \quad (45)$$

- Lie Derivatives along  $n^\mu$  for any  $\mathcal{F} = \perp \mathcal{F}$

$$\perp (\mathcal{L}_n \mathcal{F}^\mu \cdots \mathcal{F}_\nu \cdots) = \mathcal{L}_n \mathcal{F}^\mu \cdots \mathcal{F}_\nu \cdots \quad (46)$$

$$h_\mu{}^\nu \mathcal{L}_n \mathcal{F}_{\nu \dots} = \mathcal{L}_n \mathcal{F}_{\mu \dots} \quad (47)$$

$$h_\mu{}^\nu \mathcal{L}_n \mathcal{F}^{\mu \dots} = \mathcal{L}_n \mathcal{F}^{\nu \dots} \quad (48)$$

$$\mathcal{L}_n K_{\mu\nu} = n^\lambda \nabla_\lambda K_{\mu\nu} + K_{\lambda\nu} \nabla_\mu n^\lambda + K_{\mu\lambda} \nabla_\nu n^\lambda \quad (49)$$

$$h^{\mu\nu} \mathcal{L}_n K_{\mu\nu} = \mathcal{L}_n K + 2 K^{\mu\nu} K_{\mu\nu} \quad (50)$$

## 6. Sign Conventions for the Action

These conventions follow Weinberg (after accounting for his definition of the Riemann tensor, which has a minus sign relative to our definition). They are appropriate when using signature  $(-, +, \dots, +)$ . The  $d + 1$ -dimensional Newton's constant is  $2\kappa^2 = 16\pi G_{d+1}$ . The boundary  $(\partial\mathcal{M}, h)$  is formed by a spacelike unit vector  $n^\mu$ , as in the previous section, and the sign on the boundary term follows from our definition of the extrinsic curvature.

- Gravitational Action

$$I_G = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (R - 2\Lambda) + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{h} K \quad (51)$$

$$= \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (\mathcal{R} + K^2 - K^{\mu\nu} K_{\mu\nu} - 2\Lambda) \quad (52)$$

- Gauge Field Coupled to Particle

$$I_{Maxwell} + I_{Matter} = -\frac{1}{4} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} F^{\mu\nu} F_{\mu\nu} \quad (53)$$

$$- \sum_n m_n \int dp \sqrt{-g_{\mu\nu}(x_n(p))} \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp} \quad (54)$$

$$+ \sum_n e_n \int dp \frac{dx_n^\mu(p)}{dp} A_\mu(x_n(p)) \quad (55)$$

- Gravity Minimally Coupled to a Gauge Field

$$I_G + I_{Maxwell} = \int_{\mathcal{M}} d^{d+1}x \sqrt{g} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{h} K \quad (56)$$

## 7. Hamiltonian Formulation for Evolution Along a Spatial Direction

The canonical variables are the metric  $h_{\mu\nu}$  on  $\Sigma$  and its conjugate momentum  $\pi^{\mu\nu}$ , which is defined with re-

spect to evolution along the spacelike direction  $n^\mu$  normal to  $\Sigma$ . Even though the evolution is along a space-like direction we will use the same terminology as for timelike evolution: “Hamiltonian”, “momentum constraint”, etc.

- Bulk Lagrangian Density

$$\mathcal{L}_{\mathcal{M}} = \frac{1}{2\kappa^2} (K^2 - K^{\mu\nu} K_{\mu\nu} + \mathcal{R} - 2\Lambda) \quad (57)$$

- Momentum Conjugate to  $h_{\mu\nu}$

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial (\mathcal{L}_n h_{\mu\nu})} = \frac{1}{2\kappa^2} (h^{\mu\nu} K - K^{\mu\nu}) \quad (58)$$

$$\pi = \pi^\mu{}_\mu = \frac{d-1}{2\kappa^2} K \quad (59)$$

- Momentum Constraint

$$\mathcal{H}_\mu = \frac{1}{\kappa^2} \perp (n^\nu G_{\mu\nu}) = 2\mathcal{D}^\nu \pi_{\mu\nu} = 0 \quad (60)$$

- Hamiltonian Constraint

$$\mathcal{H} = -\frac{1}{\kappa^2} n^\mu n^\nu G_{\mu\nu} = 2\kappa^2 \left( \pi^{\mu\nu} \pi_{\mu\nu} - \frac{1}{d-1} \pi^2 \right) + \frac{1}{2\kappa^2} (\mathcal{R} - 2\Lambda) = 0 \quad (61)$$

## 8. Conformal Transformations

The dimension of spacetime is  $d+1$ . Indices are raised and lowered using the metric  $g_{\mu\nu}$  and its inverse  $g^{\mu\nu}$ .

- Metric

$$\hat{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu} \quad (62)$$

- Christoffel

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \Theta^\lambda_{\mu\nu} \quad (63)$$

$$\Theta^\lambda_{\mu\nu} = \delta^\lambda_\mu \nabla_\nu \sigma + \delta^\lambda_\nu \nabla_\mu \sigma - g_{\mu\nu} \nabla^\lambda \sigma \quad (64)$$

- Riemann Tensor

$$\hat{R}^\lambda_{\mu\rho\nu} = R^\lambda_{\mu\rho\nu} + \delta^\lambda_\nu \nabla_\mu \nabla_\rho \sigma - \delta^\lambda_\rho \nabla_\mu \nabla_\nu \sigma + g_{\mu\rho} \nabla_\nu \nabla^\lambda \sigma - g_{\mu\nu} \nabla_\rho \nabla^\lambda \sigma \quad (65)$$

$$+ \delta^\lambda_\rho \nabla_\mu \sigma \nabla_\nu \sigma - \delta^\lambda_\nu \nabla_\mu \sigma \nabla_\rho \sigma + g_{\mu\nu} \nabla_\rho \sigma \nabla^\lambda \sigma - g_{\mu\rho} \nabla_\nu \sigma \nabla^\lambda \sigma \quad (66)$$

$$+ (g_{\mu\rho} \delta^\lambda_\nu - g_{\mu\nu} \delta^\lambda_\rho) \nabla^\alpha \sigma \nabla_\alpha \sigma \quad (67)$$

- Ricci Tensor

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} \nabla^2 \sigma - (d-1) \nabla_\mu \nabla_\nu \sigma + (d-1) \nabla_\mu \sigma \nabla_\nu \sigma \quad (68)$$

$$- (d-1) g_{\mu\nu} \nabla^\lambda \sigma \nabla_\lambda \sigma \quad (69)$$

- Ricci Scalar

$$\hat{R} = e^{-2\sigma} (R - 2d \nabla^2 \sigma - d(d-1) \nabla^\mu \sigma \nabla_\mu \sigma) \quad (70)$$

- Schouten Tensor

$$\hat{S}_{\mu\nu} = S_{\mu\nu} - \nabla_\mu \nabla_\nu \sigma + \nabla_\mu \sigma \nabla_\nu \sigma - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \sigma \nabla_\lambda \sigma \quad (71)$$

- Weyl Tensor

$$\hat{C}^\lambda{}_{\mu\rho\nu} = C^\lambda{}_{\mu\rho\nu} \quad (72)$$

- Normal Vector

$$\hat{n}^\mu = e^{-\sigma} n^\mu \quad \hat{n}_\mu = e^\sigma n_\mu \quad (73)$$

- Extrinsic Curvature

$$\hat{K}_{\mu\nu} = e^\sigma (K_{\mu\nu} + h_{\mu\nu} n^\lambda \nabla_\lambda \sigma) \quad (74)$$

$$\hat{K} = e^{-\sigma} (K + d n^\lambda \nabla_\lambda \sigma) \quad (75)$$

## 9. Small Variations of the Metric

Consider a small perturbation to the metric of the form  $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ . All indices are raised and lowered using the unperturbed metric  $g_{\mu\nu}$  and its inverse. All quantities are expressed in terms of the perturbation to the metric, and never in terms of the perturbation to the inverse metric. As in the previous sections,  $\nabla_\mu$  is the covariant derivative compatible with the metric  $g_{\mu\nu}$ , and  $\mathcal{D}_\mu$  is the covariant derivative along a hypersurface  $\Sigma \subset \mathcal{M}$  (normal to the spacelike or timelike unit vector  $n^\mu$ ) compatible with the induced metric  $h_{\mu\nu}$ .

- Inverse Metric

$$g^{\mu\nu} \rightarrow g^{\mu\nu} - g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} + \dots \quad (76)$$

- Square Root of Determinant of Metric

$$\sqrt{g} \rightarrow \sqrt{g} \left( 1 + \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} + \dots \right) \quad (77)$$

- Variational Operator

$$\delta(g_{\mu\nu}) = \delta g_{\mu\nu} \quad \delta^2(g_{\mu\nu}) = \delta(\delta g_{\mu\nu}) = 0 \quad (78)$$

$$\delta(g^{\mu\nu}) = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \quad (79)$$

$$\delta^2(g^{\mu\nu}) = \delta(-g^{\mu\lambda} g^{\nu\rho} \delta g_{\lambda\rho}) = 2 g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} \quad (80)$$

$$\mathcal{F}(g + \delta g) = \mathcal{F}(g) + \delta \mathcal{F}(g) + \frac{1}{2} \delta^2 \mathcal{F}(g) + \dots + \frac{1}{n!} \delta^n \mathcal{F}(g) + \dots \quad (81)$$

- Christoffel

$$\delta \Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\rho} (\nabla_{\mu} \delta g_{\rho\nu} + \nabla_{\nu} \delta g_{\mu\rho} - \nabla_{\rho} \delta g_{\mu\nu}) \quad (82)$$

$$\delta^2 \Gamma_{\mu\nu}^{\lambda} = -g^{\lambda\alpha} g^{\rho\beta} \delta g_{\alpha\beta} (\nabla_{\mu} \delta g_{\rho\nu} + \nabla_{\nu} \delta g_{\mu\rho} - \nabla_{\rho} \delta g_{\mu\nu}) \quad (83)$$

$$\delta^n \Gamma_{\mu\nu}^{\lambda} = \frac{n}{2} \delta^{n-1} (g^{\lambda\rho}) (\nabla_{\mu} \delta g_{\rho\nu} + \nabla_{\nu} \delta g_{\mu\rho} - \nabla_{\rho} \delta g_{\mu\nu}) \quad (84)$$

- Riemann Tensor

$$\delta R_{\mu\sigma\nu}^{\lambda} = \nabla_{\sigma} \delta \Gamma_{\mu\nu}^{\lambda} - \nabla_{\nu} \delta \Gamma_{\mu\sigma}^{\lambda} \quad (85)$$

- Ricci Tensor

$$\delta R_{\mu\nu} = \nabla_{\lambda} \delta \Gamma_{\mu\nu}^{\lambda} - \nabla_{\nu} \delta \Gamma_{\mu\lambda}^{\lambda} \quad (86)$$

$$= \frac{1}{2} (\nabla^{\lambda} \nabla_{\mu} \delta g_{\lambda\nu} + \nabla^{\lambda} \nabla_{\nu} \delta g_{\mu\lambda} - g^{\lambda\rho} \nabla_{\mu} \nabla_{\nu} \delta g_{\lambda\rho} - \nabla^2 \delta g_{\mu\nu}) \quad (87)$$

- Ricci Scalar

$$\delta R = -R^{\mu\nu} \delta g_{\mu\nu} + \nabla^{\mu} (\nabla^{\nu} \delta g_{\mu\nu} - g^{\lambda\rho} \nabla_{\mu} \delta g_{\lambda\rho}) \quad (88)$$

- Surface Forming Unit Vector

$$\delta n_{\mu} = \frac{e}{2} n_{\mu} n^{\nu} n^{\lambda} \delta g_{\nu\lambda} = \frac{1}{2} \delta g_{\mu\nu} n^{\nu} + \frac{1}{2} c_{\mu} \quad (89)$$

$$c_{\mu} = e n_{\mu} n^{\nu} n^{\lambda} \delta g_{\nu\lambda} - \delta g_{\mu\nu} n^{\nu} = -h_{\mu}^{\lambda} \delta g_{\lambda\nu} n^{\nu} \quad (90)$$

- Acceleration vector

$$\delta a^{\mu} = -\frac{1}{2} D^{\mu} (n^{\nu} n^{\lambda} \delta g_{\nu\lambda}) + c^{\mu} \quad (91)$$

- Extrinsic Curvatures

$$\begin{aligned} \delta K_{\mu\nu} &= \frac{e}{2} n^{\alpha} n^{\beta} \delta g_{\alpha\beta} K_{\mu\nu} + e \delta g_{\lambda\rho} n^{\rho} (n_{\mu} K_{\nu}^{\lambda} + n_{\nu} K_{\mu}^{\lambda}) \\ &\quad - \frac{1}{2} h_{\mu}^{\lambda} h_{\nu}^{\rho} n^{\alpha} (\nabla_{\lambda} \delta g_{\alpha\rho} + \nabla_{\rho} \delta g_{\lambda\alpha} - \nabla_{\alpha} \delta g_{\lambda\rho}) \end{aligned} \quad (92)$$

$$\delta K = -\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} - \frac{1}{2}n^\mu (\nabla^\nu \delta g_{\mu\nu} - g^{\nu\lambda}\nabla_\mu \delta g_{\nu\lambda}) + \frac{1}{2}D_\mu c^\mu \quad (93)$$

- Gibbons-Hawking-York Lagrangian (  $e = \pm 1$  )

$$\delta(e\sqrt{h}K) = e\sqrt{h} \left[ \frac{1}{2}(h^{\mu\nu}K - K^{\mu\nu})\delta g_{\mu\nu} + \frac{1}{2}D_\mu c^\mu - \frac{1}{2}n^\mu (\nabla^\nu \delta g_{\mu\nu} - g^{\nu\lambda}\nabla_\mu \delta g_{\nu\lambda}) \right] \quad (94)$$

## 10. The ADM Decomposition

The conventions and notation in this section (and the next) differ from the preceding sections. In this section we consider a  $d$ -dimensional spacetime with metric  $h$  and metric-compatible covariant derivative  ${}^d\nabla$ . Note that, among other differences, the definition of the extrinsic curvature here differs from the one used in the [section on hypersurfaces](#) by a sign:  $\theta_{\mu\nu} = -K_{\mu\nu}$ .

We start by identifying a scalar field  $t$  whose isosurfaces  $\Sigma_t$  are normal to the timelike unit vector given by

$$u_a = -\alpha \partial_a t, \quad (95)$$

where the lapse function  $\alpha$  is

$$\alpha := \frac{1}{\sqrt{-h^{ab} \partial_a t \partial_b t}}. \quad (96)$$

An observer whose worldline is tangent to  $u_a$  experiences an acceleration given by the vector

$$a_b = u^c \cdot {}^d\nabla_c u_b, \quad (97)$$

which is orthogonal to  $u_a$ . The (spatial) metric on the  $d - 1$  dimensional surface  $\Sigma_t$  is given by

$$\sigma_{ab} = h_{ab} + u_a u_b. \quad (98)$$

The intrinsic Ricci tensor built from this metric is denoted by  $\mathcal{R}_{ab}$ , and its Ricci scalar is  $\mathcal{R}$ . The covariant derivative on  $\Sigma_t$  is defined in terms of the  $d$  dimensional covariant derivative as

$$D_a V_b := \sigma_a^c \sigma_b^e ({}^d\nabla_c V_e) \quad \text{for any} \quad V_b = \sigma_b^c V_c. \quad (99)$$

The extrinsic curvature of  $\Sigma_t$  embedded in the ambient  $d$  dimensional spacetime is

$$\theta_{ab} := -\sigma_a^c \sigma_b^d ({}^d\nabla_c u_d) = -{}^d\nabla_a u_b - u_a a_b = -\frac{1}{2} \mathcal{L}_u \sigma_{ab}. \quad (100)$$

This definition has an overall minus sign compared to the extrinsic curvature in [section 4](#), where we considered surfaces normal to a spacelike vector.

Now we consider a ‘time flow’ vector field  $t^a$ , which satisfies the condition

$$t^a \partial_a t = 1 . \quad (101)$$

The vector  $t^a$  can be decomposed into parts normal and along  $\Sigma_t$  as

$$t^a = \alpha u^a + \beta^a , \quad (102)$$

where  $\alpha$  is the lapse function (96) and  $\beta^a := \sigma^a_b t^b$  is the shift vector. An important result in the derivations that follow relates the Lie derivative of a scalar or spatial tensor (one that is orthogonal to  $u^a$  in all of its indices) along the time flow vector field, to Lie derivatives along  $u^a$  and  $\beta^a$ . Let  $S$  be a scalar. Then

$$\mathcal{L}_t S = \mathcal{L}_{\alpha u} S + \mathcal{L}_\beta S = \alpha \mathcal{L}_u S + \mathcal{L}_\beta S . \quad (103)$$

Rearranging this expression then gives

$$\mathcal{L}_u S = \frac{1}{\alpha} (\mathcal{L}_t S - \mathcal{L}_\beta S) . \quad (104)$$

Similarly, for a spatial tensor with all lower indices we have

$$\mathcal{L}_t W_{a\dots} = \alpha \mathcal{L}_u W_{a\dots} + \mathcal{L}_\beta W_{a\dots} . \quad (105)$$

This is not the case when the tensor has any of its indices raised. In a moment, these identities will allow us to express certain Lie derivatives along  $u^a$  in terms of regular time derivatives and Lie derivatives along the shift vector  $\beta^a$ .

Next, we construct the coordinate system that we will use for the decomposition of the equations of motion. The adapted coordinates  $(t, x^i)$  are defined by

$$\partial_t x^a := t^a . \quad (106)$$

The  $x^i$  are coordinates along the  $d - 1$  dimensional hypersurface  $\Sigma_t$ . If we define

$$P_i^a := \frac{\partial x^a}{\partial x^i} , \quad (107)$$

then it follows from the definition of the coordinates that  $P_i^a \partial_a t = 0$  and we can use  $P_i^a$  to project tensors onto  $\Sigma_t$ . For example, in the adapted coordinates the spatial metric, extrinsic curvature, and acceleration and shift vectors are

$$\sigma_{ij} = P_i^a P_j^b \sigma_{ab} \quad (108)$$

$$\theta_{ij} = P_i^a P_j^b \theta_{ab} \quad (109)$$

$$a_j = P_j^b a_b \quad (110)$$

$$\beta_i = P_i^a \beta_a = P_i^a t_a . \quad (111)$$

The line element in the adapted coordinates takes a familiar form:

$$h_{ab}dx^a dx^b = h_{ab} \left( \frac{\partial x^a}{\partial t} dt + \frac{\partial x^a}{\partial x^i} dx^i \right) \left( \frac{\partial x^b}{\partial t} dt + \frac{\partial x^b}{\partial x^j} dx^j \right) \quad (112)$$

$$= h_{ab} (t^a dt + P_i^a dx^i) (t^b dt + P_j^b dx^j) \quad (113)$$

$$= t^a t_a dt^2 + 2t_a dt P_i^a dx^i + h_{ab} P_i^a P_j^b dx^i dx^j \quad (114)$$

$$= (-\alpha^2 + \beta^i \beta_i) dt^2 + 2\beta_i dt dx^i + \sigma_{ij} dx^i dx^j \quad (115)$$

$$\Rightarrow h_{ab}dx^a dx^b = -\alpha^2 dt^2 + \sigma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt) . \quad (116)$$

Thus, in the adapted coordinate system we can express the components of the spacetime metric  $h_{ab}$  and its inverse  $h^{ab}$  as

$$h_{ab} = \left( \begin{array}{c|c} -\alpha^2 + \beta^i \beta_i & \sigma_{ij} \beta^j \\ \hline \sigma_{ij} \beta^j & \sigma_{ij} \end{array} \right) \quad (117)$$

$$h^{ab} = \left( \begin{array}{c|c} -\frac{1}{\alpha^2} & \frac{1}{\alpha^2} \beta^i \\ \hline \frac{1}{\alpha^2} \beta^i & \sigma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \end{array} \right) \quad (118)$$

$$\det(h_{ab}) = -\alpha^2 \det(\sigma_{ij}) \quad (119)$$

Obtaining the components of the inverse is a short algebraic calculation. Note that the spatial indices  $i, j, \dots$  in the adapted coordinates are lowered and raised using the spatial metric  $\sigma_{ij}$  and its inverse  $\sigma^{ij}$ .

In adapted coordinates there are several results concerning the projections of Lie derivatives of scalars and tensors which will be important in what follows. The first, which is trivial, is that the Lie derivative of a scalar  $S$  along the time-flow vector  $t^a$  is just the regular time-derivative

$$\mathcal{L}_t S = t^a \partial_a S = \frac{\partial x^a}{\partial t} \frac{\partial S}{\partial x^a} = \partial_t S . \quad (120)$$

Next, we consider the projector  $P_i^a$  applied to the Lie derivative along  $t^a$  of a general vector  $W_a$ , which gives

$$P_i^a \mathcal{L}_t W_a = \partial_t W_a \quad \forall \quad W_a . \quad (121)$$

The important point is that this applies not just to spatial vectors but to *any* vector  $W_a$ , as a consequence of the result

$$P_i^a \mathcal{L}_t u_a = 0 . \quad (122)$$

Finally, we can show that the Lie derivative along  $t^a$  of any contravariant spatial vector satisfies

$$P^i_a \mathcal{L}_t V^a = \partial_t V^i \quad \forall \quad V^i = P^i_a V^a . \quad (123)$$

This follows from a lengthier calculation than what is required for the first two results.

Given these results, we can express various geometric quantities and their projections normal to and along  $\Sigma_t$  in terms of quantities intrinsic to  $\Sigma_t$  and simple time derivatives. First, the extrinsic curvature is

$$\theta_{ij} = -\frac{1}{2} P_i^a P_j^b \mathcal{L}_u \sigma_{ab} \quad (124)$$

$$= -\frac{1}{2} P_i^a P_j^b \left( \frac{1}{\alpha} (\mathcal{L}_t \sigma_{ab} - \mathcal{L}_\beta \sigma_{ab}) \right) \quad (125)$$

$$\Rightarrow \theta_{ij} = -\frac{1}{2\alpha} \left( \partial_t \sigma_{ij} - (D_i \beta_j + D_j \beta_i) \right). \quad (126)$$

Since  $\theta_{ab}$  is a spatial tensor, projections of its Lie derivative along  $u^a$  can be expressed in a similar manner

$$P_i^a P_j^b \mathcal{L}_u \theta_{ab} = \frac{1}{\alpha} (\partial_t \theta_{ij} - \mathcal{L}_\beta \theta_{ij}). \quad (127)$$

Now we present the Gauss-Codazzi and related equations in adapted coordinates:

$$P_i^a P_j^b ({}^d R_{ab}) = \mathcal{R}_{ij} + \theta \theta_{ij} - 2\theta_i^k \theta_{jk} - \frac{1}{\alpha} (\partial_t \theta_{ij} - \mathcal{L}_\beta \theta_{ij}) - \frac{1}{\alpha} D_i D_j \alpha \quad (128)$$

$$P_i^a ({}^d R_{ab} u^b) = D_i \theta - D^j \theta_{ij} \quad (129)$$

$${}^d R_{ab} u^a u^b = \frac{1}{\alpha} (\partial_t \theta - \beta^i \partial_i \theta) - \theta^{ij} \theta_{ij} + \frac{1}{\alpha} D_i D^i \alpha \quad (130)$$

$${}^d R = \mathcal{R} + \theta^2 + \theta^{ij} \theta_{ij} - \frac{2}{\alpha} (\partial_t \theta - \beta^i \partial_i \theta) - \frac{2}{\alpha} D_i D^i \alpha. \quad (131)$$

These allow us to write out the various projections of the Einstein equations  $G_{ab} = \kappa^2 T_{ab}$ .

- Hamiltonian Constraint

$$\mathcal{R} + \theta^2 - \theta^{ij} \theta_{ij} = 2\kappa^2 \rho \quad (132)$$

$$\rho := T_{ab} u^a u^b \quad (133)$$

- Momentum Constraint

$$D^j \theta_{ij} - D_i \theta = \kappa^2 j_i \quad (134)$$

$$j_i := -P_i^a (T_{ab} u^b) \quad (135)$$

- ADM Evolution Equations

$$\begin{aligned} \mathcal{R}_{ij} - \frac{1}{2} \sigma_{ij} \mathcal{R} - \frac{1}{\alpha} \left( \partial_t \theta_{ij} - \frac{1}{2} \sigma_{ij} \partial_t \theta \right) + \frac{1}{\alpha} \left( \mathcal{L}_\beta \theta_{ij} - \frac{1}{2} \sigma_{ij} \mathcal{L}_\beta \theta \right) \\ + \theta \theta_{ij} - 2\theta_i^k \theta_{jk} - \frac{1}{2} \sigma_{ij} (\theta^2 + \theta^{kl} \theta_{kl}) - \frac{1}{\alpha} (D_i D_j \alpha - \sigma_{ij} D^k D_k \alpha) \\ = \kappa^2 P_i^a P_j^b T_{ab} \end{aligned} \quad (136)$$

Or, if we use the trace of this equation to rewrite  $\mathcal{R}$  and define the spatial stress tensor as  $S_{ij} := P_i^a P_j^b T_{ab}$ , this can be written as

$$\begin{aligned} \partial_t \theta_{ij} = & \mathcal{L}_\beta \theta_{ij} + \alpha (\mathcal{R} + \theta \theta_{ij} - 2 \theta_i^k \theta_{jk}) - D_i D_j \alpha \\ & - \kappa^2 \alpha \left( S_{ij} - \frac{1}{d-2} \sigma_{ij} S^k_k \right) - \kappa^2 \frac{1}{d-2} \sigma_{ij} \alpha \rho \end{aligned} \quad (137)$$

## 11. Converting to ADM Variables

The line element is often presented in the form

$$h_{ab} dx^a dx^b = h_{tt} dt^2 + 2h_{ti} dt dx^i + h_{ij} dx^i dx^j. \quad (138)$$

We would like to relate the components of the metric to the ADM variables: the lapse function  $\alpha$ , the shift vector  $\beta_i$ , and the spatial metric  $\sigma_{ij}$ . This is a straightforward exercise in linear algebra. Comparing with (116), we first note that

$$\sigma_{ij} = h_{ij}. \quad (139)$$

The inverse spatial metric,  $\sigma^{ij}$ , is literally the inverse of  $h_{ij}$ , which is not in general the same thing as  $h^{ij}$  (see, for instance, equation (118))

$$\sigma^{ij} = (\sigma_{ij})^{-1} = (h_{ij})^{-1} \neq h^{ij}. \quad (140)$$

For the shift vector we have

$$h_{ti} = \sigma_{ij} \beta^j \quad \rightarrow \quad \sigma^{ik} h_{tk} = \sigma^{ik} \sigma_{kl} \beta^l = \beta^i \quad (141)$$

$$\Rightarrow \beta^i = \sigma^{ij} h_{tj}. \quad (142)$$

Finally, for the lapse we obtain

$$\alpha^2 = \sigma^{ij} h_{ti} h_{tj} - h_{tt}. \quad (143)$$

## Recent Changes

August 31, 2023

- Updated [hypersurface](#) results to include both spacelike and timelike unit normals.
- Added useful simplifications associated with surface-forming (normalized gradient) unit normals.
- Reorganized the order of results in the section on [hypersurfaces](#).
- Changed definition of the vector  $c^\mu$  in the section on [small variations of the metric](#).
- Updated variations of  $n^\mu$ ,  $K_{\mu\nu}$ , and  $K$  to accommodate both spacelike and timelike hypersurfaces.
- Added explicit variation of the Gibbons-Hawking-York Lagrangian, including the (usually neglected) corner term.
- Added a note about the different conventions used for the extrinsic curvature in the [ADM](#) and

[hypersurface](#) sections.

May 5, 2023

- Added a short section with the explicit form of the Lie derivative of an arbitrary tensor.

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