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Robert A. McDonald, *California Polytechnic State University - San Luis Obispo*

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Fundamental Sizing Implications of Constant or Increasing Weight Aircraft

Robert A. McDonald*

California Polytechnic State University, San Luis Obispo, CA, 93407

As the energy storage capabilities of batteries and fuel cells advance, these technologies are increasingly being considered for aircraft primary propulsion. In addition to other fundamental differences from conventional systems, these concepts may have constant or even increasing mass throughout the mission. In this paper, the implications of constant or increasing mass on aircraft sizing are considered with the aim of generating insight for the design of these systems.

Nomenclature

$(\cdot)_i$	Quantity $(\cdot)$ at moment i
$\Delta(\cdot)$	Change in $(\cdot)$
$\Delta(\cdot)_{i,j}$	Change in $(\cdot)$ from i to j
$\dot{X}$	Time derivative of X
η	Overall efficiency
η_c	Cycle efficiency
η_e	Electrical efficiency
η_p	Propulsive or propeller efficiency
$\mathcal{M}/\mathcal{MP}$	Combined mission capacity fraction
$BSFC$	Brake specific fuel consumption
CP	Climb parameter
EP	Endurance parameter
MP	Mission parameter
RP	Range parameter
$TSFC$	Thrust specific fuel consumption
μ	Proportion of reactants created as fuel is consumed $\mu \equiv -\dot{W}_r/\dot{W}_f$
$d(\cdot)$	Differential of $(\cdot)$
D	Drag
E	Energy
e	Weight specific energy
F_N	Net thrust
k	Aircraft weight change coefficient $k \equiv 1 - \mu$
L	Lift
M	Mission extent
P_s	Specific excess power
P_T	Thrust power
R	Range
t	time
u	Ratio of drag to thrust $u \equiv D/F_N$
V	Velocity
W	Gross weight

*Associate Professor, Aerospace Engineering, One Grand Avenue, AIAA Senior Member.

W_e	Empty weight
W_f	Fuel weight
W_p	Payload weight
W_r	Retained products weight
z_e	Energy height

I. Introduction

THROUGH the course of their mission, conventional aircraft consume fuel and consequently lose weight. This weight change forms the basis for the derivation of the familiar Breguet range equation and leads to the natural logarithm in the resulting form. Advanced concepts using sealed batteries and regenerative fuel cells for primary propulsion do not lose weight throughout the mission, while concepts using metal-air batteries or water retaining hydrogen-air fuel cells gain weight throughout the mission. This difference leads to a different form of the mission analysis equations and has direct implications for the sizing of these aircraft.

In this paper, the mission analysis equations for a constant or increasing weight aircraft are developed. The mission capability fraction is then introduced as a unified measure of the extent that an aircraft can conduct its mission. The mission capability fraction can represent different segments of flight (say climb, cruise, and loiter) and can combine them into a composite mission capability fraction. The relationship between mission capability fraction and fuel fraction for a vehicle of different energy storage technologies is developed.

II. Motivation

Advances in energy storage, motor, and controller technology have sparked growing interest in electric aircraft of different forms. Battery electric systems have become the dominant technology for radio control aircraft and small tactical UAV's. There is interest in battery, fuel cell, and hybrid systems for aircraft of all scales from small UAV's to general aviation, high altitude long endurance (HALE) UAV's, and even transport aircraft.

The unique characteristics of electric systems can provide an attractive design alternative for many vehicle classes. Small tactical UAV's benefit from the low noise, high reliability, and simplified logistics and integration of battery-electric systems. Solar regenerative or beamed energy systems using either batteries or fuel cells for energy storage enable the indefinite duration goals of HALE missions. Increased environmental regulations in the form of emissions and noise may drive consideration of electric systems. For vehicles where the cost and availability of fuel are of prime concern, electric systems have the potential to greatly reduce operating cost and enable independence from fuel.

The increased design freedom of hybrid and electric systems have exciting potential to enable new aircraft concepts, configurations, and missions. Hybrid systems have the potential to maximize the strengths and minimize the weaknesses of disparate technologies. Electric systems can fundamentally change the approach to propulsion airframe integration; thereby changing the approach to configuration design.

Electric systems are very different from conventional fuel burning aircraft systems. Although battery technology is advancing quickly, it still represents the critical technology for these systems; the energy storage density of current battery systems (250 Wh/kg or 0.9 MJ/kg) is approximately 1/50th that of jet fuel (18,500 Btu/lb or 43 MJ/kg). On the other hand, in electric systems, high efficiency motors and controllers replace the relatively low cycle efficiency of combustion engines. Perhaps the most fundamental difference between these systems is that electric systems may have constant or even increasing weight as energy is consumed whereas combustion systems lose weight as energy is consumed.

Each of these differences fundamentally influences the overall aircraft sizing problem. The vehicle sizing problem is not just about determining the size of the vehicle. The sizing problem determines the growth factor – an indicator of risk in the system. The sizing problem determines whether the vehicle closes for a given mission – determining the feasibility of the system. Of course, the sizing problem also determines the empty and gross weight of the system – each of which are commonly used as surrogates for cost, thereby determining the viability of the system.

Although the other benefits of advanced technology concepts may allow them to tolerate some size and cost growth relative to their conventional counterparts, to a rough order, in order to be viable, a vehicle of certain capability (range/speed/payload) should be of similar size. In previous work,[1] the author has explored the possibility of designing unconventional aircraft to a reduced capability set based on the patterns of aircraft use rather than capability; thereby mitigating some of the penalties inherent to these unconventional systems.

III. Generalized Aircraft Sizing

The derivation of the formulae in this paper generally follow the derivation of the conventional mission sizing formulae presented by Mattingly.[2] This derivation is extended to vehicles with constant or increasing weight following the presentation by Nam,[3] but with some simplifications to the nomenclature. This analysis is extended to the endurance, range, and climb parameters – which are generalized to the mission parameter. These parameters are observed to be the ultimate mission performance possible for a constant weight aircraft made entirely of fuel. This leads to the interpretation of the mission capacity fraction as a fundamental driver of mission performance.

III.A. Weight Breakdown

During the mission sizing phase of the conceptual design process of conventionally fueled vehicles, the aircraft weight, W, is often broken into the empty weight W_e, payload weight W_p, and fuel weight W_f. Unconventional aircraft can be conceived which have constant weight or even gain weight as energy is consumed. Following Nam,[3] these aircraft can be considered by also including W_r, the weight of retained products of reaction from the energy consumption process.

$$W = W_e + W_p + W_f + W_r \tag{1}$$

In this paper, any consumable finite energy source is considered 'fuel' and its weight is book-kept as W_f. A fully charged sealed (constant weight) battery would be considered as fuel at the start of the mission. As energy is consumed, the fuel weight is reduced, and the retained products weight W_r is correspondingly increased; the sum $W_f + W_r$ is constant and equal to the battery weight.

The derivative of Equation 1 with respect to time can be taken to arrive at Equation 2. In arriving at Equation 2, the empty weight and payload weight are considered constant. A mid-mission payload drop is beyond the scope of this paper, but can be considered in the same way as for a conventional vehicle sizing analysis.[2]

$$\dot{W} = \dot{W}_f + \dot{W}_r \tag{2}$$

The products of reaction are assumed to accumulate in proportion to the consumption of fuel as shown in Equation 3 where μ is the constant of proportionality and the negative sign reflects the fact that the consumption of fuel results in an accumulation of reactants.

$$\dot{W}_r = -\mu\, \dot{W}_f \tag{3}$$

Defining $k \equiv 1 - \mu$ and substitution of Equation 3 into Equation 2 results in Equation 4, a simplified expression for the rate of change in aircraft weight. A conventional vehicle which retains none of the products of reaction will have $k = 1$. A vehicle powered by a sealed battery with constant weight will have $k = 0$. A vehicle powered by a metal-air battery or other system that gains weight as energy is consumed will have $k < 0$.

$$\dot{W} = k\, \dot{W}_f \tag{4}$$

Equation 4 can be integrated to Equation 5 for any mission segment, where ΔW is the vehicle weight change for the segment and ΔW_f is the fuel weight consumed for that segment.

$$\Delta W = k\, \Delta W_f \tag{5}$$

Table 1 presents the values for μ and k for some candidate energy storage systems. All conventional systems which do not retain any products have $k = 1$. All sealed battery or regenerative fuel cell systems with constant mass have $k = 0$. The chemistry for two metal-air batteries and a hydrogen-air fuel cell which retains the produced water are presented. The lighter the molecular weight of the fuel, the more negative the resulting k value; consequently, the water retaining hydrogen-air case represents a lower bound on k. The water retaining fuel cell concept has been studied as an emissionless aircraft concept.[4]

Table 1. Values of μ and k for energy storage systems.

Process	Chemical Formula	μ	k
Conventional		0.0	1.0
Sealed Battery		1.0	0.0
Zn-Air	$2Zn + O_2 \rightarrow 2ZnO$	1.245	-0.245
Li-Air	$4Li + O_2 \rightarrow 2Li_2O$	2.153	-1.153
H_2-Air (Retaining H_2O)	$2H_2 + O_2 \rightarrow 2H_2O$	8.936	-7.936

III.B. Specific Energy

The weight specific energy, e, of a fuel source is defined as the ratio of the stored energy, E, to the weight W_f of a given source.

$$E = e\, W_f \tag{6}$$

Taking the time derivative of Equation 6 gives the rate of energy produced by consumption of a given weight of fuel.

$$\dot{E} = e\dot{W}_f \tag{7}$$

III.C. Aircraft Mission Analysis

The production of thrust power, P_T, is the predominate energy consumer for an aircraft. Thrust power is equal to the product of net thrust, F_N, and the aircraft velocity, V, i.e. $P_T = F_N\, V$. Other sources of power consumption are ignored in this analysis, but could be included in the same way as for conventional vehicles. Definition of η as the overall conversion efficiency from stored energy to thrust power results in Equation 8 for the energy balance of an aircraft.

$$\dot{E} = -\frac{P_T}{\eta} = -\frac{F_N\, V}{\eta} \tag{8}$$

Equation 8 can be combined with Equation 7 and Equation 4 to form Equation 9 for the rate of change of the weight of an aircraft in flight.

$$\dot{W} = -k\frac{F_N\, V}{e\,\eta} \tag{9}$$

Both sides of Equation 9 can be multiplied by dt/W, resulting in Equation 10, which is the fundamental equation for mission analysis. Integration of this equation under appropriate conditions can yield a weight fraction equation for any mission segment.

$$\frac{dW}{W} = -k\frac{F_N\, V}{e\,\eta\, W}dt \tag{10}$$

As an example, Equation 10 applied to the conditions of equilibrium cruising flight, i.e. $F_N = D$ and $W = L$, results in Equation 11.

$$\frac{dW}{W} = -k\frac{D\, V}{e\,\eta\, L}dt \tag{11}$$

Observation of this equation leads to the definition of the endurance parameter, EP, shown in Equation 12.

$$EP \equiv \frac{e\,\eta}{V}\frac{L}{D} \tag{12}$$

Defining the endurance parameter allows Equation 11 to be written as Equation 13.

$$\frac{dW}{W} = -k\frac{1}{EP}dt \tag{13}$$

Equation 10 is extended to the range parameter and climb parameter in Appendix A; this equation and those presented in the appendix are generalized to the mission parameter, MP, in Equation 14. At this point, generalization to the mission parameter can be viewed as direct substitution of symbols without physical justification. The following procedure could be conducted for the endurance, range, or climb parameters with identical results.

$$\frac{dW}{W} = -k\frac{1}{MP}dM \tag{14}$$

Equation 14 can then be integrated as in Equation 15 from the start 0 to the end 1 of a notional mission segment. The variable of integration dM indicates that integration is over M, the extent of the mission. Note that M and MP must always share the same units.

$$\int_0^1 \frac{dW}{W} = \int_0^1 -k\frac{1}{MP}dM \tag{15}$$

Assuming constant k and MP allows them to be pulled from the integral; k is dependent on the energy storage technology and is constant throughout flight. Though there is no reason to expect constant MP, any mission may be broken into shorter segments such that MP may be considered constant for a segment. Taking $M = \int_0^1 dM$ and the exponential of both sides yields Equation 16.

$$\frac{W_1}{W_0} = \exp\left\{-k\frac{M}{MP}\right\} \tag{16}$$

Equation 16 gives the mission segment weight fraction W_1/W_0 for a mission segment flown at the conditions corresponding to the mission parameter MP, for the extent M. Subtracting both sides of Equation 16 from 1.0 leads to Equation 17 after some algebraic manipulation.

$$\frac{W_0 - W_1}{W_0} = 1 - \exp\left\{-k\frac{M}{MP}\right\} \tag{17}$$

Recognizing $W_0 - W_1$ as the vehicle weight change for the mission segment allows Equation 5 to be substituted. This gives Equation 18 for the fuel fraction consumed during the mission segment.

$$\frac{\Delta W_{f_{0,1}}}{W_0} = \frac{1 - \exp\left\{-k\frac{M}{MP}\right\}}{k} \tag{18}$$

III.D. Constant Weight Vehicle ($k = 0$)

The constant weight ($k = 0$) case can be investigated by substituting Equation 7 (but not Equation 4) into Equation 8 and then directly integrating. Alternatively, the limiting result can be obtained by application of L'Hôpital's Rule to Equation 18.

$$\lim_{k\to 0}\left(\frac{\Delta W_{f_{0,1}}}{W_0}\right) = \frac{-\exp\left\{-k\frac{M}{MP}\right\}\left(-\frac{M}{MP}\right)}{1} \tag{19}$$

Simplification of Equation 19 for $k = 0$ results in Equation 20, the fuel fraction consumed during a mission segment for a constant weight aircraft.

$$\frac{\Delta W_{f_{0,1}}}{W_0} = \frac{M}{MP}; \text{For } k = 0 \tag{20}$$

Solving Equation 20 for M results in Equation 21 which gives the mission extent, M, that can be flown by a constant weight aircraft expending a given fuel fraction, $\Delta W_{f_{0,1}}/W_0$ at the mission parameter, MP, corresponding to some specific flight conditions.

$$M = \frac{\Delta W_{f_{0,1}}}{W_0}MP; \text{For } k = 0 \tag{21}$$

III.E. Combined Mission Capacity Fraction

Examination of Equation 21 reveals that a constant weight ($k = 0$) aircraft consuming its entire weight in fuel $\Delta W_{f_{0,1}}/W_0 = 1$ would have mission extent, M, equal to the mission parameter, MP. The mission parameter is therefore the ultimate mission extent possible for a constant weight vehicle; it will be called the constant weight ultimate mission capacity and M/MP can be interpreted as the constant weight ultimate mission capacity fraction. These names are shortened to the mission capacity and the mission capacity fraction respectively.

Appendix B demonstrates that arbitrary number and types of sequential mission segments may be combined by summing the appropriate capacity fractions for each mission segment. For example, a mission comprising a climb, a cruise, and a loiter segment may be combined into a common mission capacity fraction as shown in Equation 22.

$$\frac{\mathcal{M}}{\mathcal{MP}} = \frac{\Delta z_e}{CP} + \frac{R}{RP} + \frac{t}{EP} \tag{22}$$

Interpretation of $\mathcal{M}/\mathcal{MP}$ as the full-fuel mission capacity fraction for a given aircraft allows Equations 18 and 20 to be applied to the aircraft fuel fraction as in Equations 23 and 24 below.

$$\frac{W_f}{W_0} = \frac{1 - \exp\left(-k\frac{\mathcal{M}}{\mathcal{MP}}\right)}{k} \tag{23}$$

$$\frac{W_f}{W_0} = \frac{\mathcal{M}}{\mathcal{MP}}; \text{For } k = 0 \tag{24}$$

Equation 23 can be solved for $\mathcal{M}/\mathcal{MP}$ and specialized to the conventional fuel burning case ($k = 1$) giving Equation 25.

$$\frac{\mathcal{M}}{\mathcal{MP}} = -\ln\left(1 - \frac{W_f}{W_0}\right); \text{For } k = 1 \tag{25}$$

$\mathcal{M}/\mathcal{MP}$ can be viewed as a measure of the fuel carrying ability of an aircraft. Decreasing W_e/W_0 or W_p/W_0 will increase W_f/W_0 and thereby increase $\mathcal{M}/\mathcal{MP}$ for the same gross weight. Alternately, for given gross, empty, and payload weights, the mission capability fraction $\mathcal{M}/\mathcal{MP}$ will take a fixed value which balances the mission extent, $\mathcal{M}$, with the mission efficiency, $\mathcal{MP}$. Increases in $\mathcal{MP}$ can be matched with increases in $\mathcal{M}$ while decreases in $\mathcal{MP}$ must me accompanied by decreases in $\mathcal{M}$.

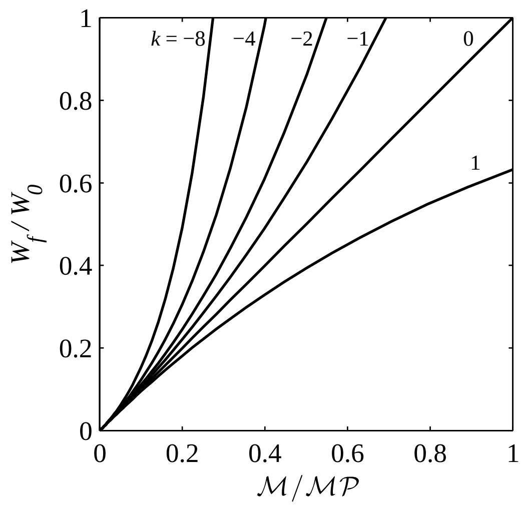
IV. Results

Equation 23 was evaluated for a variety of energy storage technologies (k values) and plotted as Figure 1. The dramatic difference between conventional ($k = 1$, curved down), constant weight ($k = 0$, straight line), and increasing weight ($k < 0$, curved up) systems is clearly evident. The penalty for these systems can be observed as either higher W_f/W_0 required for a given $\mathcal{M}/\mathcal{MP}$, or as lower $\mathcal{M}/\mathcal{MP}$ capability for a given W_f/W_0.

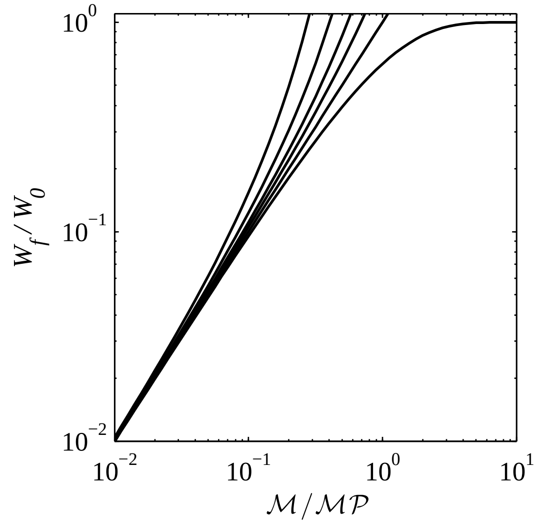
Of course, Figure 1 does not tell the whole story; the exceptional specific energy of hydrogen (approximately three times that of jet fuel) can do much to improve $\mathcal{MP}$ such that its extreme $k = -7.93$ may still be tenable. However, for systems with relatively poor specific energy, Figure 1 can be viewed as a relatively direct penalty.

When comparing only conventional and constant weight systems ($k = 1$ and 0), the price paid for retaining weight is not severe for vehicles with low W_f/W_0 or short mission extent (low $\mathcal{M}/\mathcal{MP}$). Systems with mission capacity fraction greater than about 0.4 pay significant penalty for retaining weight; while systems with mission capacity fraction less than about 0.15 pay almost no such penalty.

Equation 26 is a simple empty weight fraction regression for passenger aircraft proposed by Mattingly.[2] It was used to solve a simple sizing problem in terms of payload weight (W_p) and mission capability fraction $\mathcal{M}/\mathcal{MP}$. Figure 2 depicts payload weight plotted against $\mathcal{M}/\mathcal{MP}$ for a range of gross weights; conventional aircraft are depicted with the black solid curves and constant weight aircraft are depicted with the grey dashed curves.



(a) Linear scale.



(b) Log scale.

Figure 1. Relationship between fuel and mission capacity fractions for different energy conversion technologies.

$$\frac{W_e}{W_0} = 1.02\,(W_0)^{-.06} \tag{26}$$

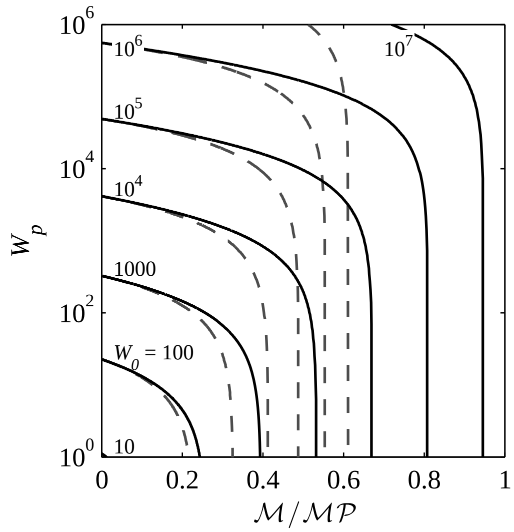


Figure 2. Passenger aircraft sizing for conventional ($k = 1$, solid) and constant weight ($k = 0$, dashed) aircraft.

For example, a conventional aircraft carrying 1,000 *lb* of payload with an $\mathcal{M}/\mathcal{MP}$ of 0.4 will weigh about 10,000*lb*. For either category of aircraft, there is a $\mathcal{M}/\mathcal{MP}$ beyond which it is not practical to design a vehicle. In conventional sizing parlance, the growth factor of the concept is too large and the vehicle does not close. In Figure 2 this is evident when the constant gross weight contours become vertical lines – note that these lines are vertical on a log scale. Vehicles designed near the vertical portion of the curve are very aggressive in terms of growth factor and are going to face severe challenges meeting both a payload and range requirement.

Figure 2 clearly depicts that vehicles which retain their fuel reach the area of severe growth factor at significantly lower $\mathcal{M}/\mathcal{MP}$ than do conventional vehicles. However, once again, we see that the penalty for retaining fuel weight for vehicles with $\mathcal{M}/\mathcal{MP}$ below 0.2 is slight and the curves are indistinguishable for vehicles with $\mathcal{M}/\mathcal{MP}$ less than about 0.1.

From these results, we see that the most attractive systems to design with a fuel retaining technology are those which are accomplished today with relatively low values of $\mathcal{M}/\mathcal{MP}$ which places them near the flatter portion of the curves shown in Figure 2.

Figure 3 is similar to Figure 2 except that data for actual fuel burning aircraft are depicted. Each aircraft is represented by a curve and a data point. The data point represents the aircraft at maximum fuel and maximum gross weight conditions. All data for the aircraft are at maximum gross weight. The curves which extend from the data points represent the potential trade of fuel for payload for each aircraft. The black lines which extend up and to the left of the data points are for aircraft with published values for the maximum payload. For these vehicles, the curve corresponds to the maximum gross weight edge of a payload-range diagram. For vehicles with a grey line extending up and to the left of the data points, no maximum payload weight was given, so the lines were extended enough to indicate where those systems fell on their notional curve. Finally, vehicles with a grey line extending to both sides of the data point are unmanned systems which are normally considered to have a fixed payload. The curve thereby represents a notional payload-fuel trade for those systems.

Figure 3 has two horizontal axes. The bottom axis corresponds to $\mathcal{M}/\mathcal{MP}$ and utilizes the solid guide lines. The top axis corresponds to W_f/W_0 and utilizes the dashed guide lines. Because all of the aircraft depicted are conventional fuel consuming systems ($k = 1$), there is a direct correlation between these parameters.

Prior analysis indicated that the best candidate aircraft to develop constant weight replacements were those at low $\mathcal{M}/\mathcal{MP}$ and those which are near the flat region of the curves on the payload weight vs. $\mathcal{M}/\mathcal{MP}$ curve. Observation of real aircraft data indicates that further considerations are needed – aircraft such as the Boeing 737 (especially when considering the BBJ variants) can be operated in diverse ways which correspond to dramatically different points on the payload-range diagram. For these vehicles, it is clear that aircraft which are most frequently operated near the max payload limit of the payload-range diagram are better suited to constant weight concepts than those aircraft which are most frequently operated near the maximum fuel limit.

V. Acknowledgements

This work was initiated while the author served as a member of the Aeronautics Systems Analysis Branch at NASA Langley Research Center, through an Intergovernmental Personnel Act agreement.

A. Generalized Mission Segments

The variable of integration in Equation 10, repeated below for convenience, can be substituted to form mission equations in terms of different mission extent (distance, climb altitude, etc.).

$$\frac{dW}{W} = -k\frac{F_N\,V}{e\,\eta\,W}dt$$

The aircraft velocity can be written as the time derivative of range, $V = dR/dt$, which can be solved for dt and substituted into Equation 11. Recall that Equation 11 has had $F_N = D$ and $W = L$ substituted into Equation 10 for the equilibrium flight conditions.

$$\frac{dW}{W} = -k\frac{D}{e\,\eta\,L}dR \tag{27}$$

Observation of Equation 27 leads to the definition of the range parameter, RP, shown in Equation 28.

$$RP \equiv e\,\eta\frac{L}{D} \tag{28}$$

For climbing and/or accelerating flight, thrust is not equal to drag. As shown in Equation 29, the specific excess power, P_s, is the time derivative of the energy height, z_e.

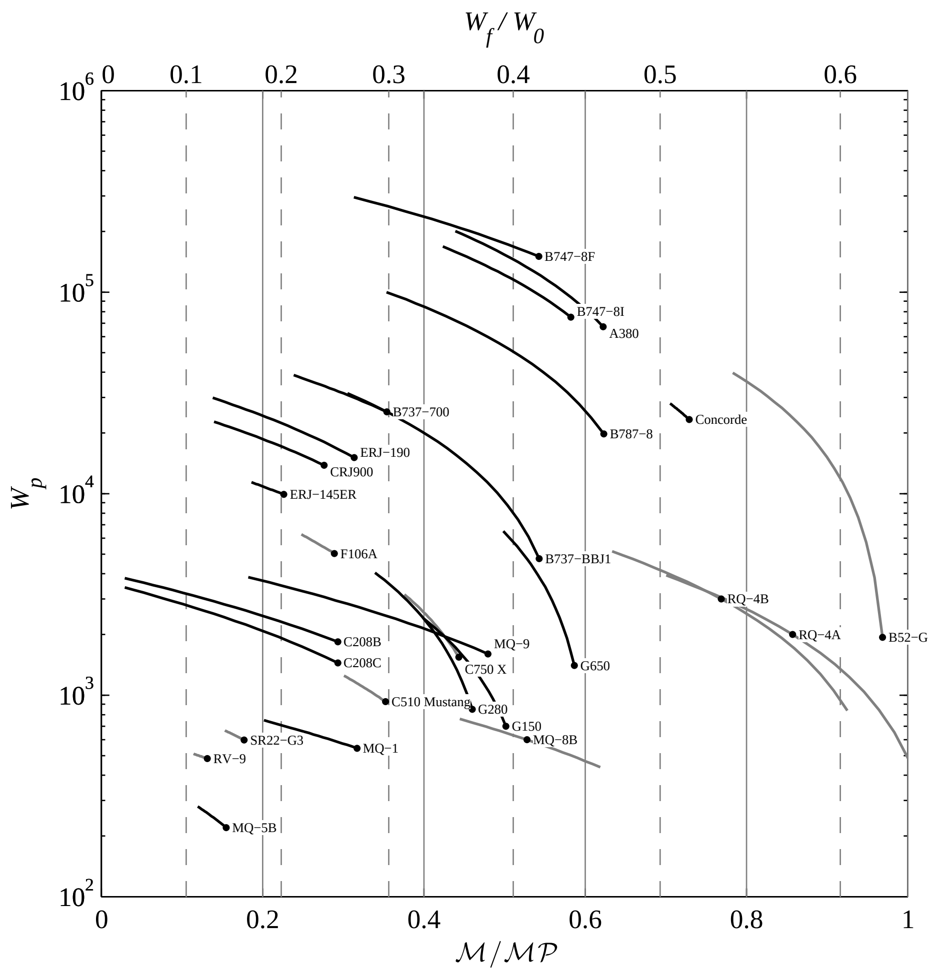


Figure 3. Payload weight, fuel fraction, and mission capacity fraction for various aircraft.

$$P_s = \frac{dz_e}{dt} = \frac{(F_N - D)\,V}{W} \tag{29}$$

Multiplying the numerator of Equation 29 by F_N/F_N, introducing u as the ratio of drag to thrust, $u \equiv D/F_N$, simplifying, and solving for dt results in Equation 30.

$$dt = \frac{W}{F_N\,V\,(1-u)} dz_e \tag{30}$$

Equation 30 can be substituted into Equation 10 and simplified to result in Equation 31.

$$\frac{dW}{W} = -k \frac{1}{e\,\eta\,(1-u)} dz_e \tag{31}$$

Observation of Equation 31 leads to the definition of the climb parameter, CP, shown in Equation 32.

$$CP \equiv e\,\eta\,(1-u) \tag{32}$$

Table 2 summarizes formulae for the endurance, range, and climb parameter for various propulsion systems which may be considered. As appropriate, the mission parameters are placed in terms of the canonical fuel consumption metric used for each system.

Table 2. Mission parameter formulae for different propulsion systems.

	Fundamental	Electric-Propeller	Piston-Propeller	Jet
Fuel Consumption Metric			$BSFC = \frac{1}{\eta_c LHV}$	$TSFC = \frac{V}{\eta_c \eta_p LHV}$
Endurance Parameter	$\frac{e\,\eta}{V}\frac{L}{D}$	$\frac{e\,\eta_p \eta_e}{V}\frac{L}{D}$	$\frac{\eta_p}{V\,BSFC}\frac{L}{D}$	$\frac{1}{TSFC}\frac{L}{D}$
Range Parameter	$e\,\eta\frac{L}{D}$	$e\,\eta_p \eta_e \frac{L}{D}$	$\frac{\eta_p}{BSFC}\frac{L}{D}$	$\frac{V}{TSFC}\frac{L}{D}$
Climb Parameter	$e\,\eta\,(1-u)$	$e\,\eta_p \eta_e\,(1-u)$	$\frac{\eta_p}{BSFC}(1-u)$	$\frac{V}{TSFC}(1-u)$

B. Combining Multiple Mission Segments

Arbitrary number and types of sequential mission segments may be combined to form a composite mission. In this appendix, an example two-segment mission is considered to illustrate how a composite mission influences the mission equations derived in the body of the paper. In this example, mission segment 1 runs from state 0 to state 1, while mission segment 2 runs from state 1 to state 2. Each mission segment has its own mission extent M_1 & M_2 and is executed with its own mission parameter MP_1 & MP_2. As shown in Equation 33, the product of the mission segment weight fractions forms the combined mission weight fraction.

$$\frac{W_2}{W_0} = \frac{W_1}{W_0}\frac{W_2}{W_1} \tag{33}$$

Equation 16 is substituted for each weight fraction in Equation 33 resulting in Equation 34.

$$\frac{W_2}{W_0} = \exp\left(-k\frac{M_1}{MP_1}\right)\exp\left(-k\frac{M_2}{MP_2}\right) \tag{34}$$

The product of exponentials in Equation 34 can be combined to form Equation 35.

$$\frac{W_2}{W_0} = \exp\left(-k\left(\frac{M_1}{MP_1} + \frac{M_2}{MP_2}\right)\right) \tag{35}$$

The steps which resulted in Equation 18 from Equation 16 can be applied to Equation 35 to arrive at Equation 36.

$$\frac{\Delta W_{f_{0,2}}}{W_0} = \frac{1 - \exp\left\{-k\left(\frac{M_1}{MP_1} + \frac{M_2}{MP_2}\right)\right\}}{k} \tag{36}$$

Similarly, the limiting procedure that resulted in Equation 20 from Equation 18 can be applied to Equation 36 to arrive at Equation 37

$$\frac{\Delta W_{f_{0,2}}}{W_0} = \frac{M_1}{MP_1} + \frac{M_2}{MP_2}; \text{For } k = 0 \tag{37}$$

Equations 35, 36, and 37 demonstrate that arbitrary number and types of sequential mission segments may be combined by summing the mission capacity fractions for each mission segment. In this paper, the calligraphic $\mathcal{M}/\mathcal{MP}$ will be used as in Equation 38 to indicate a combined mission capacity fraction.

$$\frac{\mathcal{M}}{\mathcal{MP}} = \frac{M_1}{MP_1} + \frac{M_2}{MP_2} \tag{38}$$

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