May, 2013

On Random Time and on the Relation Between Wave and Telegraph Equations

Ramakrishna Janaswamy, *University of Massachusetts - Amherst*

On Random Time and on the Relation Between Wave and Telegraph Equations

Ramakrishna Janaswamy, *Fellow, IEEE*

Abstract—**Kac's conjecture relating the solution of wave and telegraph equations in higher dimensions through a Poisson-process-driven random time is established through the concepts of stochastic calculus. New expression is derived for the probability density function of the random time. We demonstrate how the relationship between the solution of a lossy wave- and that of a lossless wave equation can be exploited to derive some statistical identities. Relevance of the results presented to the study of pulse propagation in a dispersive medium characterized by a Lorentz or Drude model is discussed and new evolution equations for 2D Maxwell's equations are presented for the Drude medium. It is shown that the computational time required for updating the electric field using the stochastic technique is expected to go up as $O(\sqrt{t})$.**

Index Terms—**Poisson processes, Stochastic differential equations, telegrapher's equation, random time, dispersive media, FDTD.**

I. Introduction

There has been an interest lately in developing probabilistic, stochastic, or Monte Carlo methods for field computation [1]. The basic idea is to formulate a stochastic process, producing an appropriate random variable, whose expected value is the solution of a certain deterministic problem. Among the computational advantages of the stochastic method that various researchers have advanced are (i) the method is local in that the field can be calculated at a desired point only without having to obtain the global solution by solving a linear system of equations, (ii) the method needs no meshing, (iii) the degree of accuracy can be controlled by varying the sample size (or number of realizations), (iv) the method in inherently parallel because the intermediate results generated by each random realization are independent of each other. The resurgence of interest in the method is likely triggered by the last feature. Garcia and Sadiku [2] applied the random walk method with a variable step-size to solve Poisson's equation. Le Coz *et al.* [3] determined the frequency-dependent capacitance and inductance of high-speed computer interconnects using the random walk method and quasi-static approximations. Subsequent researchers have developed random walk method for the Helmholtz and wave equations [4], [5], and [6]. Budaev and Bogy [7] applied the stochastic differential equation technique to solve the complete transport equation pertaining to the amplitude part of the solution of the scalar Helmholtz equation in an inhomogeneous medium. The same technique was used in [8] and [9] to address the problem of wedge diffraction with impedance boundary conditions. The basic differential equations that form the subject matter of the present paper are the 3D version of the telegrapher's equation and the wave equation and certain stochastic quantities that relate the two.

The connection between partial differential equations (PDE) and random dispersion of particles in space-time has a long history; see [10], [11], [12], [13] for some of the initial work. Majority of the works that followed these papers concentrated on random walks tied to the Brownian motion. The research has been documented in a number of books and research monographs such as [14], [15], [16], [17]. It has previously been shown by Kac [18] (demonstrated for 1D with zero initial velocity conditions and conjectured for higher dimensions) that the solution of a lossy wave equation (which is a modified telegrapher's equation) can be obtained from that of a corresponding lossless wave equation by simply changing the time variable t in the latter by a randomized time variable $\zeta(t)$ that depends on a Poisson process and then performing an averaging operation with respect to the underlying Poisson process. The statistics of the Poisson process are governed by the loss present in the medium. This resulted in a very interesting stochastic representation for the solution of telegrapher equation, which could be conveniently implemented on a parallel type of computing machine. The link between wave and telegrapher's equations via random time is very interesting and implies that macroscopic loss mechanisms can be thought of as arising from randomized time and an averaging process. Following Kac's work, a number of researchers advanced the idea further and attempted to build random walk models for the telegrapher's equation in higher dimensions. Some of this latter work is nicely summarized in [19] as well as in the recent paper [20].

Unlike some of the recent work, the present paper is not concerned with building random walk models for the telegrapher's equations. Rather, the contributions of the present paper are (i) presenting a mathematical proof of Kac's conjecture in higher space dimensions (section II-C) using concepts from stochastic calculus [21]; no such formal proof currently exists in the literature to the best knowledge of the present author, (ii) given the important role played by the random time in such stochastic representations, presenting a new expression for the first order statistics of the random time (II-D), and (iii) demonstrating how the stochastic representation can be utilized to arrive at some useful schemes for computing multi-dimensional fields in dispersive media (subsection II-E). In particular, the Lorentz and Drude media are considered in subsection II-E. Hence, apart from being pedagogical in nature, the paper contains new results that are of interest to researchers working in the

The author is with the Department of Electrical & Computer Engineering, 100 Natural Resources Road, University of Massachusetts, Amherst, MA 01003, USA. Email: janaswamy@ecs.umass.edu

IEEE Transactions on Antennas & Propagation, vol. 61(5), pp. 2735-2744, May 2013.

area of wave propagation. If the idea of random time can be extended to more complicated dispersive media than the ones considered in the paper, the concept could provide an alternate means to computing the time-instantaneous field in those media where analytical expressions are not available. For the benefit of readers unfamiliar with the techniques of stochastic processes and the associated calculus, we first present some basic properties of Poisson processes and preliminaries of stochastic calculus in section II-A. The relevance of the stochastic model presented here to the study of pulse propagation in a causal dispersive media, such as the Lorentz- or Drude-model medium is discussed in section II-E.

II. Theory

A. Poisson Process and Preliminaries of Stochastic Calculus

A Poisson random process $N(t)$ is constructed from a sequence of independent and identically distributed exponential random variables $\tau_1, \tau_2, \ldots$ known as *inter-arrival times*, each with mean τ [22]. Starting from a value of zero at time $t = 0$, the process $N(t)$ jumps by unit value at the *arrival times* given by

$$t_n = \sum_{i=1}^{n} \tau_i, \quad n = 1, 2, \ldots. \tag{1}$$

The function $N(t)$ is assumed to be *right-continuous* in that $N(t) = \lim_{s\downarrow t} N(s)$ ($s \downarrow t$ means $s \to t$ from $s > t$; likewise $s \uparrow t$ means $s \to t$ from $s < t$) and can be expressed as

$$N(t) = \sum_{n=1}^{\infty} \Theta(t - t_n) \tag{2}$$

where $\Theta(t)$ is the unit-step function equal to 0 for $t < 0$ and equal to 1 for $t > 0$. Fig. 1(a) shows one sample path of a Poisson process. Because the expected time between jumps is τ, the jumps arrive at an average rate of $\lambda := 1/\tau$ and the process itself has the distribution

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, \ldots, \tag{3}$$

where $\mathbb{P}$ denotes probability. This last result can be derived from the fact that t_n given in (1) has a gamma distribution [21]. In addition, $N(t)$ satisfies the following properties:

P1: Independence of Increments–for all $0 \leq t_0 < t_1 < t_2 < \cdots < t_n, n \geq 1$, the random variables $N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ are independent of each other;

P2: Stationarity of Increments–$N(t+h) - N(s+h)$ has the same distribution as $N(t) - N(s)$ for all $h > 0$ and $0 \leq s < t$, *i.e.,* $\mathbb{P}\{N(t+h) - N(s+h) = k\}$ is independent of $h > 0$ and equals $e^{-\lambda(t-s)}[\lambda(t-s)]^k/k!$.

Using Property P2 it is easy to establish that the expected increment, which is equal to the average number of jumps between times s and t for $t \geq s$ is

$$\mathbb{E}[N(t) - N(s)] = \lambda(t - s), \tag{4}$$

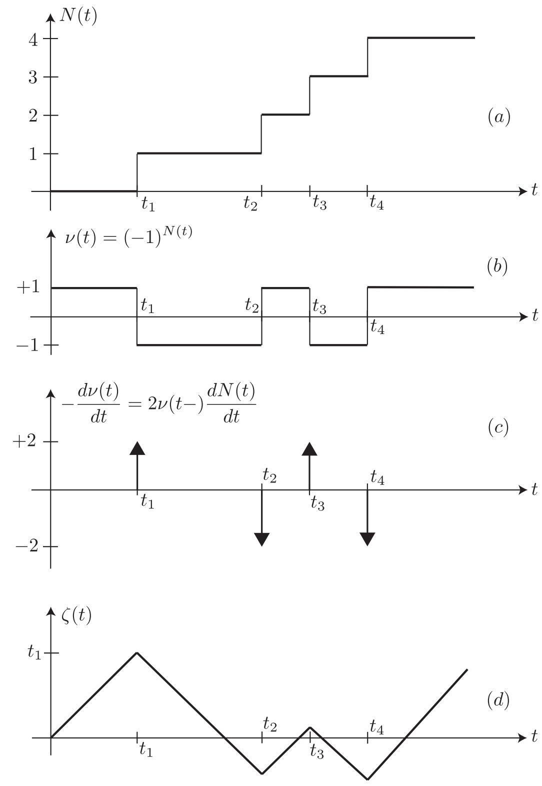

Fig. 1. Poisson processes and the functions generated from it. (a) Sample Poisson process, (b) the Telegraph signal, (c) increment of the Telegraph signal, and (d) Random time.

where $\mathbb{E}$ denotes expectation operator and it is assumed that the right-hand side of (4) will be rounded to the nearest integer. Hence the expected number of jumps between s and t is only dependent on the elapsed time $(t - s)$, but not on the epoch s. A stochastic integral equation of the form

$$X(t) = X(0) + \int_0^t f[X(\sigma); \sigma]\, d\sigma + \int_0^t g[X(\sigma); \sigma]\, dN(\sigma) \tag{5}$$

will admit a solution in the Itô sense if, on an interval where $N(t)$ is constant, X satisfies $\dot{X} = f[X(t), t]$ and if $N(t)$ jumps at t_n, X behaves in the neighborhood of t_n according to the rule [23]

$$X(t_n+) = X(t_n-) + g[\lim_{t\uparrow t_n} X(t), t_n] \tag{6}$$

where $\dot{X} = dX/dt$. Note that the jump of $X(t)$ at time t_n depends on the value of $g(X(t); t)$ evaluated at a time just *before* the jump of $N(t)$. In such a case we say that the coefficient function $g[X(t); t]$ in (5) is *left continuous* in contrast to the process $N(t)$ that is right continuous. The increment $dN(\sigma)$ is to be interpreted as being equal to $N(\sigma + d\sigma) - N(\sigma)$. With this understanding, the stochastic differential equation (SDE) corresponding to (5) is then written as

$$dX(t) = f(X(t); t)dt + g(X(t); t)dN(t), \tag{7}$$

where $dX(t)$ represents the change in $X(t)$ when t changes infinitesimally, *i.e.,* $dX(t) = X(t + dt) - X(t)$.

Given (7) and a function $\psi(x,t)$ with continuous first derivatives in both x and t, we have the following Itô-Doeblin rule for the jump process $\psi[X(t);t]$ [21, Ch. 11],[23]

$$\begin{aligned} d\psi(X(t);t) &= \left[\frac{\partial\psi(X(t);t)}{\partial t} + \frac{\partial\psi(X(t);t)}{\partial x} f(X(t);t)\right] dt \\ &\quad + \Big\{\psi\left(X(t-) + g\left[X(t-);t\right]\right) \\ &\quad - \psi\left(X(t-);t\right)\Big\} dN(t). \end{aligned} \tag{8}$$

When the Poisson process jumps at $t = t_n$, there will be jump in the function $X(t)$ according to (6). We would expect $\psi(X(t);t)$ to jump too and the coefficient of dN in (8) governs the size of this jump. Again, it is important to observe that the jump size is governed entirely by the values of $X(t)$ and $g[X(t);t]$ evaluated at the left limit t_n-. Thus, there is statistical independence between the jump in $N(t)$ and the jump it causes in its multiplying factor. At times when there is no jump in $N(t)$, the term involving dN would vanish and one obtains the usual chain rule formula from calculus given by the term proportional to dt in (8). Keeping these in mind and using (4), the expectation of an incremental process of the type $dY(t) = h(X(t);t)dN(t)$ is interpreted as

$$\begin{aligned} \mathbb{E}[dY(t)] &= \mathbb{E}\left[h(X(t);t)dN(t)\right] \\ &= \mathbb{E}\left[h(X(t);t)\right]\mathbb{E}\left[dN(t)\right] \\ &= \lambda\mathbb{E}\left[h(X(t);t)\right]dt, \end{aligned} \tag{9}$$

because the function $h(X(t);t)$ is assumed to be left continuous and dN is right continuous. As already stated, the jump in $N(t)$ is assumed to take place first independent of $h(X(t);t)$ and the corresponding jump in $h(X(t);t)$ occurs *momentarily* later. Some sort of causality and finite speed of signal propagation is thus inherently assumed in processes defined by $dY(t) = h(X(t);t)dN(t)$. This has very important implications in further analysis as we will show in Section II-C.

B. Random Time

Consider a random *velocity*[1] $\nu(t)$ (unitless) and a corresponding 'position' $\zeta(t)$ (with units of time) defined with respect to the Poisson process $N(t)$ as

$$\nu(t) = (-1)^{N(t)}; \tag{10}$$

$$\zeta(t) = \int_0^t \nu(\sigma)\, d\sigma. \tag{11}$$

Note that $\nu(0) = 1$ and $\zeta(0) = 0$. Even though it is physically appealing to think of $\zeta(t)$ as the position of a particle whose velocity flips randomly between -1 and $+1$, the fact that it has units of time render it to be termed more appropriately as *random time*. Fig. 1(b) and (d) respectively show sample waveforms of $\nu(t)$ and $\zeta(t)$. The velocity jumps at the same instants t_n when $N(t)$ jumps and takes values of ± 1. The random time $\zeta(t)$ is continuos, but has slope discontinuities at t_n. If there are no jumps in the interval $(0,t)$, then $\zeta(t) = t$, otherwise $\zeta(t) \in (-t,t]$. It is clear that in an infinitesimal interval dt

$$\begin{aligned} d\nu &= \nu(t+dt) - \nu(t) \\ &= (-1)^{N(t)}\left[(-1)^{dN(t)} - 1\right] = -2\nu(t)dN(t). \end{aligned} \tag{12}$$

It is evident from Fig. 1(c) that the coefficient of $dN(t)$, *viz.*, $\nu(t)$ is to be evaluated at $t-$, which confirms the rules laid out in Section II-A in relation to (7). Furthermore,

$$\begin{aligned} d\zeta &= \zeta(t+dt) - \zeta(t) = \int_t^{t+dt} \nu(\sigma)\, d\sigma = \int_0^{dt} \nu(t+\sigma)\, d\sigma \\ &= (-1)^{N(t)} \int_0^{dt} (-1)^{N(t+\sigma)-N(t)}\, d\sigma = \pm\nu(t)dt, \end{aligned} \tag{13}$$

where the $+$ sign in (13) pertains to the case when no collisions occur in the time interval dt and the $-$ sign corresponds to the case when there is exactly one collision in the infinitesimal time interval dt. The relation between $d\zeta$ and $\nu(t)$ is also clear from Figs. 1(b) and (d). The following properties of the telegraph signal can be established from the definitions of the corresponding quantities and the properties of the Poisson process [24]:

$$\mathbb{E}\left[\nu(t)\right] = e^{-2\lambda t}, \tag{14}$$

$$\mathbb{E}\left[\nu(t+\Delta t)\nu(t)\right] = e^{-2\lambda|\Delta t|}. \tag{15}$$

Using these we can show that

$$\gamma_1(t) = \mathbb{E}\left[\zeta(t)\right] = e^{-\lambda t}\sinh(\lambda t)/\lambda, \tag{16}$$

$$\lambda\mathbb{E}\left[\zeta(t+\Delta t)\zeta(t)\right] = t - e^{-\lambda\Delta t}\cosh(\lambda\Delta t)\mathbb{E}[\zeta(t)]. \tag{17}$$

Both $\nu(t)$ and $\zeta(t)$ are therefore non-stationary random processes. For large times, $\gamma_1(t) \sim 0.5/\lambda$ and $\lambda\mathbb{E}\left[\zeta^2(t)\right] \sim t$. Hence the standard deviation of random time increases as $\sqrt{t/\lambda}$ for large times.

C. Relation Between Wave and Telegrapher's Equations

Consider now a function $\psi(t)$ and define the deterministic function $\varphi(t) = \mathbb{E}\left[\psi(\zeta(t))\right]$, *i.e.*, $\varphi(t)$ is obtained from $\psi(t)$ by replacing the actual time variable t with the random time $\zeta(t)$ and then performing an averaging operation with respect to the underlying jump process. Using (8), (9), (12), (13) and the commutation of the expectation and differential operators

$$\begin{aligned} d\varphi(t) &= \mathbb{E}\left[d\psi(\zeta)\right] = \mathbb{E}\left[\nu(t)\psi'(\zeta)\right]dt \qquad (18) \\ d^2\varphi(t) &= \mathbb{E}\left[d\nu(t)\psi'(\zeta) + \nu(t)d(\psi'(\zeta))\right]dt \\ &= -2\mathbb{E}\left[\nu(t)\psi'(\zeta)dN\right]dt + \mathbb{E}\left[\nu^2(t)\psi''(\zeta)\right](dt)^2 \\ &= \left\{-2\lambda\mathbb{E}\left[\nu(t)\psi'(\zeta)\right] + \mathbb{E}\left[\psi''(\zeta)\right]\right\}(dt)^2 \\ &= -2\lambda d\varphi(t)dt + \mathbb{E}\left[\psi''(\zeta)\right](dt)^2. \end{aligned}$$

In other words,

$$\frac{d^2\varphi(t)}{dt^2} + 2\lambda\frac{d\varphi(t)}{dt} = \mathbb{E}\left[\psi''(\zeta)\right] = \mathbb{E}\left[\left.\frac{d^2\psi(t)}{dt^2}\right|_{t=\zeta(t)}\right], \tag{19}$$

[1]In the literature $\nu(t)$ is referred to as the *semi-random telegraph signal* [24].

where $\psi'(\zeta) = d\psi(t)/dt|_{t=\zeta(t)}$ denotes the derivative with respect to its argument. The first line in the expansion for $d^2\varphi$ above is valid for terms up to order dt^2; higher order terms vanish in the limit of $dt \to 0$ in the eventual quantity of interest $d^2\varphi/dt^2$. If ψ and φ depend additionally on the spatial coordinates $\mathbf{r} = (x, y, z)$, *i.e.,* if $\psi = \psi(\mathbf{r}; t)$ and ψ is a solution of the partial differential equation (PDE)

$$\frac{\partial^2 \psi}{\partial t^2} = L_{\mathbf{r}} \psi, \tag{20}$$

where $L_{\mathbf{r}}$ is some linear operator with respect to the spatial coordinates, subject to the initial conditions $\psi(\mathbf{r}; 0) = f(\mathbf{r})$ (this can be thought of as the specification of an *initial position*) and $\partial\psi(\mathbf{r}; t = 0)/\partial t = g(\mathbf{r})$ (this can be thought of as the specification of an *initial velocity*), then the term inside the expectation operator on the right-hand-side of (19) will be of the form $\partial^2\psi/\partial t^2|_{t=\zeta(t)}$, which is equal to $L_{\mathbf{r}}\psi|_{t=\zeta(t)}$. Hence

$$\varphi(\mathbf{r}; t) = \mathbb{E}\left[\psi(\mathbf{r}; t = \zeta(t))\right] \tag{21}$$

with $\psi(\mathbf{r}; t)$ given by (20) satisfies the PDE

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\lambda \frac{\partial \varphi}{\partial t} = L_{\mathbf{r}} \varphi \tag{22}$$

with the same initial conditions, *viz.,*

$$\varphi(\mathbf{r}; t = 0) = f(\mathbf{r}) \text{ and } \frac{\partial \varphi(\mathbf{r}; t = 0)}{\partial t} = g(\mathbf{r}), \tag{23}$$

where we have used $\zeta(0) = 0$ and $\nu(0) = 1$ in $\varphi(\mathbf{r}; 0) = \mathbb{E}[\psi(\mathbf{r}; \zeta(0)] = f(\mathbf{r})$ and $\partial\varphi(\mathbf{r}; t = 0)/\partial t = \mathbb{E}[d\zeta/dt\, \partial\psi(\mathbf{r}; \zeta)/\partial\zeta|_{t=0}] = \mathbb{E}[\nu(0)\partial\psi(\mathbf{r}; t = 0)/\partial t] = g(\mathbf{r})$ in establishing (23). It is seen that when a field quantity satisfies an operator equation that involves a second order time derivative term, the effect of replacing the time variable in the original field quantity with the random time, followed by averaging with respect to the underlying Poisson process, has the effect of introducing a new field variable that satisfies a modified operator equation containing an additional first-order time derivative term. The coefficient of the first-derivative term is proportional to the average arrival rate of the Poisson process. The initial conditions for the modified field remain the same as the initial conditions for the original field. As a special case, if the original field ψ satisfies the scalar wave equation with $L_{\mathbf{r}} = v^2\nabla^2$, where v is the propagation speed, then the new field φ satisfies the *reduced Telegrapher's equation* (or the lossy wave equation)

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\lambda \frac{\partial \varphi}{\partial t} = v^2 \nabla^2 \varphi. \tag{24}$$

This remarkable result was first demonstrated in 1D by Kac in [18] using an entirely different mathematical machinery. His conjecture that it continues to be valid for higher dimensions has been formally proven here using concepts from stochastic calculus. An elementary application of Kac's result is that the solution for the current in a series linear RLC circuit can be obtained in terms of the current of a the lossless LC circuit when the time variable is replaced with random time having parameter $\lambda = R/2L$.

What is even more interesting is that this kind of correspondence is not limited to wave equation alone, as is clearly evident from (22). In most literature (24) is itself referred to as the telegrapher's equation although a clarification is in order. The telegrapher's equation was originally derived for a lossy transmission line having parameters $R, G, L,$ and C; respectively, the series resistance, the shunt conductance, the series inductance, and the shunt capacitance per unit length, all being non-negative. With $\alpha = G/C, \beta = R/L$ and $v = 1/\sqrt{LC}$, the *telegrapher's equation* for the voltage $V(z; t)$ along the line reads

$$\frac{\partial^2 V}{\partial t^2} + (\alpha + \beta) \frac{\partial V}{\partial t} + \alpha\beta V = v^2 \frac{\partial^2 V}{\partial z^2}, \tag{25}$$

where z is the space variable. With the change of variable $V = e^{-kt}\varphi$, where $k = \min(\alpha, \beta)$, one gets the equation

$$\frac{\partial^2 \varphi}{\partial t^2} + |\alpha - \beta| \frac{\partial \varphi}{\partial t} = v^2 \frac{\partial^2 \varphi}{\partial z^2}, \tag{26}$$

which is seen to be the reduced telegrapher's equation in 1D with the parameter $2\lambda = |\alpha - \beta|$. In the case of $\alpha = \beta$, (26) reduces to the wave equation. Equation (26) is also the equation satisfied by a wave propagating in a linear, homogeneous conducting medium characterized by the usual electrical parameters (ϵ, μ, σ) [25]. The quantity $v = 1/\sqrt{\mu\epsilon}$ is the speed of light in a lossless medium with parameters (ϵ, μ), and $T = 1/2\lambda = \epsilon/\sigma$ is the relaxation time of charges in the conducting medium.

Using the d'Alembert's formula for the solution of a wave equation in 1D [25], [26], the solution of the corresponding lossy wave equation in an unbounded medium with the initial conditions given in (23) can be expressed from the stochastic representation (21) as

$$\begin{aligned} \varphi(z; t) \quad &= \quad \mathbb{E}\left[\frac{f(z + v\zeta(t)) + f(z - v\zeta(t))}{2} + \right. \\ &\left. + \frac{1}{2v} \int_{z - v\zeta(t)}^{z + v\zeta(t)} g(\xi)\, d\xi \right]. \end{aligned} \tag{27}$$

Equation (27), with the random time $\zeta(t)$ replaced by the deterministic time variable t and the expectation operator removed, is simply the d'Alembert's formula given in [25], [26] for the solution of a wave equation in 1D. Kac [18] only considered the case with zero velocity (i.e., with $g(x) \equiv 0$), but (27) generalizes his result. We may use (21) and (27) to establish some useful stochastic identities. Equation (27) can be put in a more suitable form if the initial data are expressed in terms of their Fourier transforms. The function $f(z)$ and its Fourier transform $F(k_z)$ are related by the usual transform relationship

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_z) e^{ik_z z}\, dk_z. \tag{28}$$

Using a similar definition for $g(z)$ and substituting into (27) we arrive at

$$\begin{aligned} 2\pi\varphi(z; t) \quad &= \quad \int_{-\infty}^{\infty} F(k_z) e^{ik_z z} \mathbb{E}\left[\cos(k_z v\zeta(t))\right] dk_z + \\ &+ \int_{-\infty}^{\infty} G(k_z) \frac{\mathbb{E}\left[\sin(k_z v\zeta(t))\right]}{k_z v} e^{ik_z z}\, dk_z. \end{aligned} \tag{29}$$

This alternate form is sometimes advantageous because the averaging operation has been transferred to known functions.

The explicit solution for the lossy wave equation in 1D is given in [26]

$$\begin{aligned}\varphi(z;t) &= e^{-\lambda t}\Bigg\{\frac{f(z+vt)+f(z-vt)}{2} \\ &+ \frac{1}{2v}\int\limits_{z-vt}^{z+vt}\left[f(\xi)\left(\lambda+\frac{\partial}{\partial t}\right)+g(\xi)\right] \\ &\cdot I_0\left(\frac{\lambda}{v}\sqrt{v^2t^2-(z-\xi)^2}\right)\Bigg\}\,d\xi. \qquad (30)\end{aligned}$$

Equations (27), (29) and (30) may be equated to each other to yield a number of interesting results. For instance using $z=0$ and $f(z)\equiv 0$, we get

$$\mathbb{E}\left[\int\limits_{-v\zeta(t)}^{v\zeta(t)} g(\xi)\,d\xi\right] = e^{-\lambda t}\int\limits_{-vt}^{vt} g(\xi) I_0\left(\lambda t\sqrt{1-\frac{\xi^2}{v^2t^2}}\right)d\xi. \qquad (31)$$

Note as already indicated that the quantity $v\zeta(t)$ could be positive or negative even if t is positive. If $g(\xi)$ is chosen such that the left-hand side and the right-hand side integrals in (31) could be evaluated in a closed form, certain useful relations could be obtained. As an example, with $g(\xi)=e^{-a\xi/vt}$, we can show that [27]

$$\begin{aligned}\mathbb{E}\left[\sinh\left(\frac{a\zeta(t)}{t}\right)\right] &= \frac{ae^{-\lambda t}}{\sqrt{a^2+\lambda^2t^2}}\sinh\left(\sqrt{a^2+\lambda^2t^2}\right) \\ &= e^{-\lambda t}\frac{d}{da}\left[\cosh\left(\sqrt{a^2+\lambda^2t^2}\right)\right] \qquad (32)\end{aligned}$$

This identity could be used to find the nth moment $\gamma_n(t)=\mathbb{E}[\zeta^n]$ for any n. For example, differentiating both sides of (32) with respect to a and evaluating at $a=0$ yields $\gamma_1(t)=\exp(-\lambda t)\sinh(\lambda t)/\lambda$ for the first moment of $\zeta(t)$, which agrees with that given in (16). The 2D version of the main result in (21) was also used in [27] to derived some useful integral identities encountered in wave propagation area.

D. First Order Statistics of the Random Time

In view of the central role played by random time in the stochastic representation of the solution of the telegrapher's equation such as in (27), it is of interest to determine the distribution of the random variable $\zeta(t)$ for a fixed t on the probability space $(\Omega,\mathbb{P})$, where Ω denotes sample space. Consequently, we wish to determine the probability density function (PDF) $p(r;t)$ defined by $\mathbb{P}[\omega\in\Omega;\zeta(t;\omega)\in(r,r+dr)]=p(r;t)dr,\ r\in(-t,t]$. It will be demonstrated later in this section that the PDF $p(r;t)$ itself satisfies the reduced telegrapher's equation in 1D with r taking the place of the spatial variable. As it is more convenient to first generate the transform of $p(r;t)$, we define the transform function $P(\kappa;s)$ which is obtained from $p(r;t)$ by taking the Laplace transform, $\mathcal{L}$, with respect to t and Fourier transform, $\mathcal{F}$, with respect to r:

$$\begin{aligned}P(\kappa;s) &= \int\limits_{t=0}^{\infty} e^{-st}\int\limits_{r=-\infty}^{\infty} p(r;t)e^{-i\kappa r}\,drdt \\ &= \mathcal{L}\left(\mathcal{F}\left[p(r;t)\right]\right)=\mathcal{L}\left(\mathbb{E}\left[e^{-i\kappa\zeta(t)}\right]\right), \qquad (33)\end{aligned}$$

where $i=\sqrt{-1}$. The use of Fourier-Laplace transform in the solution of linear PDEs with constant coefficients is a standard procedure and documented in a number of books, including [28]. From (8), (12), (13), and using the fact that $\nu^2(t)=1$, we get

$$d\left(e^{-i\kappa\zeta(t)}\right)=-i\kappa\nu(t)e^{-i\kappa\zeta(t)}dt, \qquad (34)$$

and

$$d\left(\nu(t)e^{-i\kappa\zeta(t)}\right)=-\left[i\kappa dt+2\nu(t)dN(t)\right]e^{-i\kappa\zeta(t)}. \qquad (35)$$

To facilitate further analysis we let $u_1(\kappa;t)=\mathbb{E}[e^{-i\kappa\zeta(t)}]$ and $u_2(\kappa;t)=\mathbb{E}[\nu(t)e^{-i\kappa\zeta(t)}]$. Then taking expectation on both sides of (34) we have

$$\frac{du_1(\kappa;t)}{dt}=-i\kappa u_2(\kappa;t). \qquad (36)$$

Similarly, taking the expectation on both sides of (35) and making use of (9) we get

$$-\frac{du_2(\kappa;t)}{dt}=i\kappa u_1(\kappa;t)+2\lambda u_2(\kappa;t). \qquad (37)$$

Equations (36) and (37) constitute a system of coupled, linear first order ordinary differential equations. Using $\mathbb{E}[\nu(0)]=1$ and $\kappa=0$ in (37) and solving for $u_2(0;t)$ recovers (14). If we now take Laplace transform with respect to t on both sides of (36) and (37) and use the usual properties of the Laplace transform of the derivative of a function, we arrive at

$$\begin{aligned}sP &= -i\kappa Q+u_1(\kappa;0) \qquad (38)\\ sQ &= -i\kappa P-2\lambda Q+u_2(\kappa;0), \qquad (39)\end{aligned}$$

where $Q(\kappa;s)=\mathcal{L}[u_2(\kappa;t)]$ and $P(\kappa;s)$ is defined in (33). Using $u_1(\kappa;0)=\mathbb{E}[1]=1$ and $u_2(\kappa;0)=\mathbb{E}[\nu(0)\cdot 1]=\mathbb{E}[1]=1$ and solving the simultaneous equations (38) and (39) we finally get

$$\begin{aligned}P(\kappa;s) &= \frac{(s+2\lambda)-i\kappa}{s^2+2\lambda s+\kappa^2} \qquad (40)\\ Q(\kappa;s) &= \frac{s-i\kappa}{s^2+2\lambda s+\kappa^2}. \qquad (41)\end{aligned}$$

Equation (40) is also derived in [29], but using the expressions of the Laplace transform of the various moments provided in [18]. However, the approach taken in this paper is more straightforward and as a bonus one can derive expressions for other quantities of interest such as $Q(\kappa;s)$. If $P(\kappa;s)$ is evaluated at $\kappa=0$ and the inverse Laplace transform taken, one gets

$$\int\limits_{-\infty}^{\infty} p(r;t)\,dr=\Theta(t)=1, \quad t>0, \qquad (42)$$

verifying that $p(r;t)$ is a valid candidate for PDF for $t>0$. It is also interesting to note from (40) that the PDF $p(r;t)$ itself satisfies the reduced telegrapher's equation. Indeed, for a function $w(r;t)$ satisfying the reduced telegrapher's equation $\partial^2 w/\partial t^2 + 2\lambda \partial w/\partial t = v^2 \partial^2 w/\partial r^2$ with initial conditions $p(r;0) = p_0(r)$ and $\partial p(r;t)/\partial t|_{t=0} = v_0(r)$, the Fourier-Laplace transform $W(\kappa;s)$ satisfies

$$W(\kappa;s) = \frac{(s+2\lambda)\tilde{p}_0(\kappa) + \tilde{v}_0(\kappa)}{s^2 + 2\lambda s + \kappa^2 v^2} \tag{43}$$

where $\tilde{p}_0(\kappa) = \mathfrak{F}[p_0(r)]$ and $\tilde{v}_0(\kappa) = \mathfrak{F}[v_0(r)]$. Comparing (43) with (40), we see that $p(r;t)$ satisfies the reduced telegrapher's equation in 1D with the speed $v = 1$, $\tilde{p}_0(\kappa) = 1$, $\tilde{v}_0(\kappa) = -i\kappa$, the latter translating to the initial conditions $p_0(r) = \delta(r)$ and $v_0(r) = \delta'(r) = dp_0(r)/dr$. These initial conditions make perfect sense in view of the fact that $\zeta(0) = 0$ deterministically. Likewise, the inverse transform of $Q(\kappa;s)$ satisfies the same reduced telegrapher's equation, but with initial conditions $q(r;0) = \delta(r)$ and $\partial q(r;t)/\partial t|_{t=0} = \delta'(r) - 2\lambda\delta(r)$.

The PDF $p(r;t)$ is obtained from the inverse relation

$$p(r;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{\Gamma} \frac{s + 2\lambda - i\kappa}{s^2 + 2\lambda s + \kappa^2} e^{st} e^{i\kappa r}\, ds d\kappa, \tag{44}$$

where the contour Γ is a straight line parallel to imaginary axis in the complex s-plane such that all singularities lie to the left of it. The integrand has simple poles at $s_{\frac{1}{2}} = -\lambda \pm i\sqrt{\kappa^2 - \lambda^2}$ in the complex s-plane. The Laplace inversion may be carried out by elementary means and the result is

$$\begin{aligned} p(r;t) \quad &= \quad \frac{e^{-\lambda t}}{2\pi} \int_{-\infty}^{\infty} \Bigg[\cos\left(\sqrt{\kappa^2 - \lambda^2}\, t\right) + \\ &\quad + (\lambda - i\kappa) \frac{\sin(\sqrt{\kappa^2 - \lambda^2}\, t)}{\sqrt{\kappa^2 - \lambda^2}} \Bigg] e^{i\kappa r}\, d\kappa. \end{aligned} \tag{45}$$

Along the same lines, the functions $u_{1,2}(\kappa;t)$, which relate to the mean of the stochastic processes $e^{-i\kappa\zeta(t)}$ and $\nu(t)e^{-i\kappa\zeta(t)}$ can be shown to be

$$u_{\frac{1}{2}}(\kappa;t) = e^{-\lambda t} \left[\cos\sqrt{\kappa^2 - \lambda^2} t \pm \frac{\lambda \mp i\kappa}{\sqrt{\kappa^2 - \lambda^2}} \sin\sqrt{\kappa^2 - \lambda^2} t \right]. \tag{46}$$

Defining

$$p_1(r;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(\sqrt{\kappa^2 - \lambda^2}\, t)}{\sqrt{\kappa^2 - \lambda^2}} e^{i\kappa r}\, d\kappa, \tag{47}$$

the PDF in (45) could be expressed as

$$e^{\lambda t} p(r;t) = \left(\lambda - \frac{\partial}{\partial r} + \frac{\partial}{\partial t} \right) p_1(r;t). \tag{48}$$

To evaluate the integral in (47) we use the identity [26, p. 303]

$$\frac{\sin(\sqrt{\kappa^2 - \lambda^2}\, t)}{\sqrt{\kappa^2 - \lambda^2}} = \frac{1}{2} \int_{-t}^{t} J_0(\lambda\sqrt{r^2 - t^2}) e^{\pm i\kappa r}\, dr, \tag{49}$$

where $J_0(\cdot)$ is the Bessel function of the first kind of order 0. This equation demonstrates a Fourier transform relationship between the function $\sin(\cdot)/(\cdot)$ and the Bessel function $J_0(\cdot)$. The Fourier inverse desired in (47) is then equal to

$$p_1(r;t) = \frac{1}{2} I_0\left(\lambda\sqrt{t^2 - r^2}\right) R_t(r), \tag{50}$$

where $R_t(r)$ is a unit-valued rectangular pulse function centered at the origin and of support $r \in (-t, t]$, $I_0(\cdot)$ is the modified Bessel function of order 0, and we have used the fact that $I_0(x) = J_0(ix)$. Using this in (48), the PDF is finally obtained as

$$\begin{aligned} p(r;t) \quad &= \quad e^{-\lambda t} \Bigg\{ \delta(t - r) + \frac{\lambda}{2} \Bigg[I_0\left(\lambda\sqrt{t^2 - r^2}\right) + \\ &\quad + (r + t) \frac{I_1\left(\lambda\sqrt{t^2 - r^2}\right)}{\sqrt{t^2 - r^2}} \Bigg] \Bigg\}, \\ &\quad \text{for } r \in (-t, t],\ t \geq 0. \end{aligned} \tag{51}$$

This is a new result for the PDF of the random time and has not been derived before, although an expression for the even part of it, *viz.*, $h(r;t) = p(r;t) + p(-r;t)$ without the delta function term is available in [29]. It is also different from the expression given in [13] and [19], because these works assume different initial conditions that do not pertain to the random time $\zeta(t)$ that we are considering in the paper. It is seen that the PDF in (51) is asymmetric in r as it should be because $\zeta(t) = +t$ until the time first jump occurs, indicating that the random time is skewed towards positive times. Furthermore, the mean value, $\gamma_1(t)$, of $\zeta(t)$ per equation (16) is $\gamma_1(t) = \exp(-\lambda t)\sinh(\lambda t)/\lambda$, implying that the PDF cannot be symmetric about $r = 0$. The expression given in [19] is symmetric and even though it is a solution to the reduced telegrapher's equation, the PDF does not correspond to $\zeta(t)$. The delta function contribution in (51) arises from the event when there are no jumps in the interval $(0, t)$ and this, we know, happens for the Poisson process with a probability $e^{-\lambda t}$. The remaining terms are due to various jumps happening between $(0, t)$. Fig. 2 shows a plot of $p(r;t)$ for $\lambda = 0.1\,\text{sec}^{-1}$, $t = 60$ secs, compared with numerical simulations generated with $N_r = 10^6$ realizations for the random time $\zeta(t)$. For the parameters chosen, the expected number of jumps in the interval $(0, t)$ is $\lambda t = 6$. The delta function has been excluded in the analytical data shown in Fig. 2. The asymmetry in r and the spike emerging at $r = t$ in the numerical results are clearly seen from the plot.

E. Application to Pulse Propagation in Dispersive Media

In this subsection, we will demonstrate how the stochastic model developed in the foregoing sections can be applied to derive some useful field relations in pulse propagation in causal dispersive media. We will consider 2D Maxwell's equations, together with the commonly used dielectric dispersion models of Lorentz and Drude [30]. Extension to 3D and other dispersion models can be proceeded along the same lines. Recall that the permittivity expression for a Lorentz model is derived by treating the interaction of a bound electron, itself represented as a damped harmonic oscillator, under the influence of a harmonic electric field [31]. Assuming an $e^{-i\omega t}$ time convention, where ω is the radian frequency, the complex dielectric permittivity of

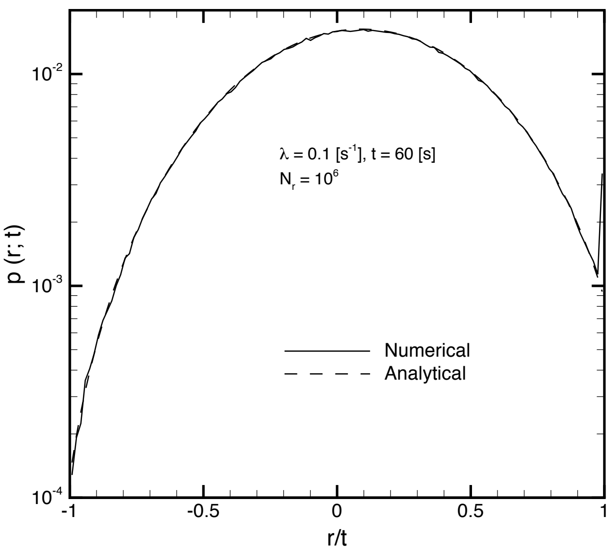

Fig. 2. PDF $p(r;t)$ of the random time for $t = 60\,\mathrm{sec}$.

a single resonance Lorentz model is

$$\epsilon_{\mathrm{rc}} = 1 - \frac{\omega_p^2}{(\omega^2 + i\gamma\omega - \omega_0^2)} := 1 + \chi_e(\omega), \tag{52}$$

where ω_p is the plasma frequency, γ is the damping constant resulting from collisions of electrons in the medium, ω_0 is the resonant frequency of the oscillator, and $\chi_e(\omega)$ is the electric susceptibility. When $\omega_0 = 0$, the model is known as the Drude model and is used in optical propagation in conductors as well as in radiowave propagation in the ionosphere [32]. For many practical media, including ionosphere and sea-water, $\gamma < \omega_p$.

Consider 2D ($\partial/\partial y = 0$) electromagnetic fields (E_z, H_y) that evolve in a semi-infinite Lorentz or Drude medium ($x > 0$). The field components are related through Maxwell's equations by

$$\frac{\partial E_z}{\partial x} = \mu_0 \frac{\partial H_y}{\partial t}, \tag{53}$$

$$\frac{\partial H_y}{\partial x} = \epsilon_0 \frac{\partial E_z}{\partial t} + \frac{\partial P_z}{\partial t}, \tag{54}$$

where the electric polarization P_z is related to the electric field E_z via

$$\frac{\partial^2 P_z}{\partial t^2} + \gamma \frac{\partial P_z}{\partial t} = \epsilon_0 \omega_p^2 E_z - \omega_0^2 P_z. \tag{55}$$

The left hand side of (55) is similar to that of (26), thus permitting stochastic formulation. It may be interesting to note that the average rate, λ, of the Poisson process determining the random time $\zeta(t)$ has a physical interpretation here in that it is directly proportional to the collision frequency, γ, of electrons within the medium comprising the model. Indeed $2\lambda = \gamma$ for the Lorentz/Drude media. By considering the particular solution

$$Q_z(x;t) = \epsilon_0 \omega_p^2 \int_{-\infty}^{t} E_z(x;\tau_1) \frac{\sin \omega_0 (t - \tau_1)}{\omega_0}\, d\tau_1 \tag{56}$$

of the inhomogeneous wave equation corresponding to (55):

$$\frac{\partial^2 Q_z}{\partial t^2} + \omega_0^2 Q_z = \epsilon_0 \omega_p^2 E_z, \tag{57}$$

the stochastic representation of P_z can be obtained as

$$P_z(x;t) = \epsilon_0 \omega_p^2\, \mathbb{E}\left[\int_{-\infty}^{\zeta(t)} E_z(x;\tau_1) \frac{\sin \omega_0 (\zeta(t) - \tau_1)}{\omega_0}\, d\tau_1 \right], \tag{58}$$

which could be substituted back into (54) to yield:

$$\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial x} = \frac{\partial E_z}{\partial t} + \omega_p^2 F(x;t), \tag{59}$$

where

$$F(x;t) = \Re \left\{ \mathbb{E} \left[\nu(t) e^{i\omega_0 \zeta(t)} \int_{-\infty}^{\zeta(t)} E_z(x;\tau_1) e^{-i\omega_0 \tau_1}\, d\tau_1 \right] \right\}, \tag{60}$$

and $\Re(\cdot)$ denotes real part of. The system comprised of (53) and (59) can then be solved for (E_z, H_y) subject to boundary conditions specified at $x = 0$. The system may also be implemented using numerical schemes such as FDTD [33] to update (E_z, H_y) in time and space. In numerical implementations it would be necessary to examine how the quantity $F(x;t)$ evolves in time. Using properties (P1) and (P2) of the Poisson process, we can establish the following exact relation valid for any $\Delta t > 0$:

$$\begin{aligned} F(x;t+\Delta t) = \Re\Big\{ & g(t,\Delta t) F(x;t) + \\ & + \mathbb{E}\Big[\nu(t+\Delta t) e^{i\omega_0 \zeta(t+\Delta t)} \times \\ & \times \int_{\zeta(t)}^{\zeta(t+\Delta t)} E_z(x;\tau_1) e^{-i\omega_0 \tau_1}\, d\tau_1 \Big] \Big\}, \end{aligned} \tag{61}$$

where

$$g(t,\Delta t) = \mathbb{E}\left(\nu(t)\nu(t+\Delta t) e^{i\omega_0 [\zeta(t+\Delta t) - \zeta(t)]} \right). \tag{62}$$

Hence it is seen that a knowledge of the correlation function of the complex-valued random process $\nu(t) \exp[i\omega_0 \zeta(t)]$ is required to update the field $F(x;t)$. The mean of this random process is given by the function $u_2(\kappa = -\omega_0; t)$ considered in subsection II-D and its correlation function is given by (77) in the Appendix.

Owing to simplicity, we will henceforth consider the Drude medium ($\omega_0 = 0$) and derive some useful relations for the evolution of the field $F(x;t)$. However, the more general case with $\omega_0 \neq 0$ can be treated in a similar manner. From the second order statistics given in (15), we see that the exact evolution equation for the Drude medium is

$$\begin{aligned} F(x;t+\Delta t) = \; & e^{-\gamma \Delta t} F(x;t) + \mathbb{E}\Big[\nu(t+\Delta t) \times \\ & \int_{\zeta(t)}^{\zeta(t+\Delta t)} E_z(x;\tau_1)\, d\tau_1 \Big]. \end{aligned} \tag{63}$$

It is seen from (63) that the future value of $F(x;t)$ depends on its present value, on the statistics of the jump process in the time interval $(t, t+\Delta t)$, on the telegraph signal $\nu(t+\Delta t)$ and on the electric field values in the interval $(\zeta(t), \zeta(t+\Delta t))$. Equation (59) together with (63) may be compared to the corresponding equation that appears in the recursive convolution method of FDTD formulations that is used to treat propagation in linear, dispersive media [33]. The convolutional integral there involving $d\chi_e(t)/dt$ and $E_z(x;t)$ is replaced in the stochastic formulation with the last term on the right hand side of (59). The benefit of not having a convolutional integral is somewhat offset by the appearance of the expectation operator and the random upper limit of the inner integral in (60) in the stochastic formulation. If $\gamma\Delta t \ll 1$ as used in many FDTD calculations [34], [35] (Reference [35] uses $\gamma/\omega_p = 6.732\times10^{-3}$, $\gamma\Delta t = 2.61\times10^{-4}$, reference [33] uses $\gamma\Delta t = 2.564\times10^{-2}$), the expected number of collisions in an interval Δt as given by (4) is equal to 0. In such a case $N(t+\Delta t) - N(t) \approx 0$, $\nu(t+\Delta t) \approx \nu(t)$, $\zeta(t+\Delta t) - \zeta(t) \approx \nu(t)\Delta t$ (see eqn. (13), and we get from (63) on using $\nu^2(t) = 1$ the approximate evolution equation

$$F(x;t+\Delta t) \approx e^{-\gamma\Delta t}F(x;t) + \Delta t\,\mathbb{E}\left[E_z(x;\zeta(t))\right], \tag{64}$$

on assuming the electric field to remain constant in the interval $|\zeta(t+\Delta t) - \zeta(t)| \leq \Delta t$. Thus the updated field $F(x;t+\Delta t)$ depends on the current field $F(x;t)$ and the expected value of $E_z(x;t)$ at the random time $t = \zeta(t)$. It is safe to say that equation (64) will continue to hold for larger Δt such that $\gamma\Delta t \leq 0.1$. In fact, the probability of having no collisions in an interval Δt from (3) is $e^{-\lambda\Delta t} > 0.95$ for $\gamma\Delta t = 0.1$. In numerical computations, the field in normally evaluated over a uniform temporal grid. So, some sort of interpolation will be involved in computing the last term in (64). Furthermore, since the standard deviation of $\zeta(t)$ increases as $\sqrt{t}$ for large t (see eqs. (16) and (17)), more computations will be required at longer times to implement the expectation operator in (64) and the computation time involving (54) and (64) will approximately increase as $O(\sqrt{t})$ in time. This increase in computations at long times is also true for numerical schemes employing the traditional convolutional integral. It remains to be seen whether the representation (59) together with the approximate evolution equation (64) will result in more accurate or more stable numerical schemes than the ones currently used in FDTD. In this regard, one has to weigh in the operation count of the entire algorithm as well.

There is an alternative viewpoint in which the random time manifests in frequency-dependent dispersive media. The second order equation satisfied by a cartesian component of the electromagnetic field, E, in a homogeneous, isotropic, locally linear, temporally dispersive medium with no externally supplied sources is the Helmholtz equation $v^2\nabla^2E + \omega^2\epsilon_{rc}E = 0$, where $v = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light in free-space. The exact (pseudo) differential equation for the time-dependent field component $\mathcal{E}$, keeping in mind that a factor $-i\omega$ in the frequency domain translates to the operator $\partial/\partial t$ in time-domain, for the Drude medium is

$$v^2\nabla^2\mathcal{E} - \frac{\partial^2\mathcal{E}}{\partial t^2} - \frac{\omega_p^2\frac{\partial}{\partial t}}{(\gamma + \frac{\partial}{\partial t})}\mathcal{E} = 0. \tag{65}$$

This equation is not directly amenable to treatment by the stochastic model presented in this paper. However, in the low frequency regime where $\omega \ll \gamma < \omega_p$, the complex dielectric constant approximates to $\epsilon_{\rm rc}\omega^2 \sim \omega^2 + i\omega/\tau_0$, where $\tau_0 = \gamma/\omega_p^2$ and the exact equation (65) approximates to

$$v^2\nabla^2\mathcal{E} - \frac{\partial^2\mathcal{E}}{\partial t^2} - \frac{1}{\tau_0}\frac{\partial\mathcal{E}}{\partial t} = 0, \tag{66}$$

which is identical to the modified Telegraphers's equation (24) considered in this paper. At these low frequencies, the statistics of the Poisson process in the stochastic model are governed by the damping constant as well as the plasma frequency. On the other hand, at high frequencies where $\omega \gg \gamma$, the dielectric constant approximates to $\epsilon_{\rm rc}\omega^2 \sim \omega^2 - \omega_p^2$ and (65) reduces to the *Klein-Gordon* equation [25], [36] (which is a relativistic version of the time-dependent Schrödinger equation)

$$v^2\nabla^2\mathcal{E} - \frac{\partial^2\mathcal{E}}{\partial t^2} - \omega_p^2\mathcal{E} = 0. \tag{67}$$

It is not uncommon in electromagnetic pulse propagation studies to consider limiting cases such as these to facilitate asymptotic analysis [37]. Notwithstanding this, it is still of interest to develop a stochastic model for the exact equation (65) valid at all frequencies.

III. Conclusions

Using stochastic calculus pertaining to jump processes, we have in this paper, established Kac's conjecture that the solution of a modified telegrapher's equation can be expressed in terms of the solution of a wave equation by simply replacing the time variable with random time generated from a Poisson process. We are able to demonstrate this for arbitrary initial conditions and for linear PDEs more general than the wave equation that contain second order time derivatives. An expression has been derived for the probability density function of the random time and it is shown that the PDF itself satisfies the modified telegrapher's equation. This expression is more general than the currently available expression in the literature that is applicable only to the even part of the random time.

The theory presented in the paper is applicable to waves propagating in a conducting medium, including the case where the conductivity (or more generally where the complex relative permittivity) is frequency dependent. Accordingly, the concept of random time was applied to pulse propagation in dispersive media in subsection II-E and new update equations for the field components have been derived that should find utility in numerical implementations such as FDTD. In particular Lorentz and Drude media were considered in the paper and we have shown that the computation time for field evaluation goes up in time as $O(\sqrt{t})$. Application to more complicated media such as those characterized by multiple resonant Lorentz model is underway and will be reported in a future paper. It would also be interesting to derive higher order evolution approximations to the field $F(x;t)$ by equating the probability of having fewer that k collisions in an interval Δt to a given number and then determining k from it. This should facilitate development of numerical schemes with some sort of error control. For instance approximation (64) has been derived under the assumption of having no collisions in an interval Δt. But the probability of no collisions in an interval Δt is $e^{-\lambda\Delta t}$. By setting this probability to a given number less than unity, one should be able to determine the maximum permissible Δt for this approximation. Such studies will be taken up in the future.

The concept of random could be utilized to extend the analytical solution of wave propagation problems that are available only for lossless media. For instance, the Cagniard-de Hoop method [38] has been applied in the past to treat transient radiation arising in certain non-conducting, boundary value problems [39], [40]. It should be possible to extend the expressions provided in these works to the corresponding lossy media by simply replacing the time variable with the random time and performing an expectation with respect to the Poisson process, thereby increasing the utility of the Cagniard-de Hoop method.

APPENDIX
CORRELATION FUNCTION OF $\nu(t)e^{i\omega_0\zeta(t)}$

Define

$$g(\eta,\xi) = \mathbb{E}\left\{\nu(\eta)\nu(\eta+\xi)e^{i\omega_0[\zeta(\eta+\xi)-\zeta(\eta)]}\right\}, \quad (68)$$

and

$$h_1(\eta,\xi) = \mathbb{E}\left\{\nu(\eta)e^{i\omega_0[\zeta(\eta+\xi)-\zeta(\eta)]}\right\}. \quad (69)$$

$$(70)$$

Then it is clear from using (14) and (15) that

$$g(\eta,0) = 1, \quad (71)$$

$$h_1(\eta,0) = e^{-2\lambda\eta}. \quad (72)$$

Using (9), (12), and (13), it can be established that for $\xi > 0$

$$\frac{\partial g}{\partial \xi} = -2\lambda g + i\omega_0 h_1, \quad (73)$$

and

$$\frac{\partial h_1}{\partial \xi} = i\omega_0 g. \quad (74)$$

From (71), (72) and (73), we see that

$$\left.\frac{\partial g}{\partial \xi}\right|_{\xi=0} = -2\lambda + i\omega_0 e^{-2\lambda\eta}. \quad (75)$$

A second order equation for g can be obtained on combining (73) and (74):

$$\frac{\partial^2 g}{\partial \xi^2} + 2\lambda\frac{\partial g}{\partial \xi} + \omega_0^2 g = 0, \quad (76)$$

which can be solved by standard means for the initial conditions specified in (71) and (75) to yield

$$g(\eta,\xi) = \left[\cosh(p\xi) - \frac{\lambda - i\omega_0 e^{-2\lambda\eta}}{p}\sinh(p\xi)\right]e^{-\lambda\xi}, \quad (77)$$

where $p = \sqrt{\lambda^2 - \omega_0^2}$. As a quick check, $g(\eta,\xi) = e^{-2\lambda\xi}$ when $\omega_0 = 0$, a result that agrees with (15).

ACKNOWLEDGEMENT

The above work was supported by the Army Research Office under grant ARO W911NF-10-1-0305.

REFERENCES

[1] K. K. Sabelfeld, *Monte Carlo Methods in Boundary Value Problems*. New York, NY: Springer-Verlag, 1991.

[2] R. C. Garcia and M. N. O. Sadiku, "Monte Carlo fixed-radius floating random walk solution for potential problems," no. DOI: 10.1109/SECON.1996.510032. Tampa, FL: IEEE Southeastcon '96: Bringing Together Education, Science and Technology, 1996, pp. 88–91.

[3] Y. L. L. Coz, H. J. Greub, and R. B. Iverson, "Performance of random walk capacitance extractors for IC interconnects: A numerical study," *Solid-State Electronics*, vol. 42, pp. 581–588, 1998.

[4] M. K. Chati, M. D. Grigoriu, S. S. Kulkarni, and S. Mukherjee, "Random walk method for the two- and three-dimensional Laplace, Poisson and Helmholtz's equations," *International Journal for Numerical Methods in Engineering*, vol. 51, no. DOI: 10.1002/nme.178, pp. 1133–1156, 2001.

[5] K. Chatterjee, P. Matos, and Y. L. L. Coz, "A novel Dirichlet-Neumann random walk algorithm for the solution of time-harmonic Helmholtz equation at multiple wavelength length scales: 1D aand 2D verifications," *Applied Computational Electromagnetic Society Journal*, vol. 19, no. 16, pp. 127–132, March 2004.

[6] K. Chatterjee, C. Yu, H. Yue, and J. Poggie, "A parallelized Monte Carlo algorithm for the one-dimensional wave equation: validation with analytical benchmark solutions," *WSEAS Transactions on Mathematics*, vol. 6, no. 3, pp. 487–493, March 2007.

[7] B. V. Budaev and D. B. Bogy, "A probabilitic approach to wave propagation and scattering," *Radio Science*, vol. 40, no. RS6S07, p. doi: 10.1029/2004RS003176, 2005.

[8] ——, "Two-dimensional diffraction by a wedge with impedance boundary conditions," *IEEE Trans. Antennas Propagat.*, vol. 53, no. 6, pp. 2073–2080, June 2005.

[9] ——, "Diffraction of plane skew electromagnetic wave by a wedge with general anisotropic impedance boundary conditions," *IEEE Trans. Antennas Propagat.*, vol. 54, no. 5, pp. 1559–1567, May 2006.

[10] G. I. Taylor, "Diffusion by continuous movements," *Proc. Lond. Math. Soc. (2)*, vol. 20, pp. 196–212, 1921/2.

[11] R. Courant, K. Friedrichs, and H. Lewy, "On the partial difference equations of mathematical physics," *IBM Journal*, pp. 215–234, March 1967. (This is a republication of the origianl paper that appeared in *Mathematische Annalen* 100, 32-74, 1928).

[12] M. Kac, "Random walk and the theory of Brownian motion," *The Amer. Math. Month.*, vol. 54, no. 7, Pt. 1, pp. 369–391, Aug.-Sep. 1947.

[13] S. Goldstein, "On diffusion by discontinuous movements, and on the Telegraph equation," *Quart. J. Mech. and Appl. Math.*, vol. IV, Pt. 2, pp. 129–156, 1951.

[14] J. L. Doob, *Stochastic Processes*, ser. Wiley Classic Library. New York: John Wiley & Sons, 1953.

[15] E. B. Dynkin and A. A. Yuskevich, *Markov Processes: Theorems and Problems*. New York: Plenum, 1969.

[16] M. Freidlin, *Functional Integration and Partial Differential Equations*, ser. Annals of Mathematical Studies. Princeton, NJ: Princeton University Press, 1985, no. 109.

[17] G. H. Weiss, *Aspects and Applications of the Random Walk*. New York: North-Holland, 1994.

[18] M. Kac, "A stochastic model related to the telegrapher's equation," *Rocky Mountain Journal of Mathematics*, vol. 43, no. 3, pp. 479–509, Summer 1974.

[19] G. H. Weiss, "Some applications of persistent random walks and the telegrapher's equation," *Physica A*, vol. 311, pp. 381–410, 2002.

[20] A. V. Plyukhin, "Stochastic processes leading to wave equations in dimensions higher than one," *Physical Review E*, vol. 81, no. DOI: 10.1103/PhysRevE.81.021113, pp. 021 113–1 to 021 113–5, 2010.

[21] S. E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*. New York: Springer, 2004.

[22] W. Feller, *An Introduction to Probability Theory and Its Applications, vol. 1*, 3rd, Ed. New York: John Wiley & Sons, 1957.

[23] R. W. Brockett, W. Gong, and Y. Guo, "Stochastic analysis for fluid queueing systems." Phoenix, Arizona: 38th IEEE Conference on Decision & Control, December 1999, pp. 3077–3082.

[24] A. Papoulis and S. Unnikrishnan, *Probablity, Random Variables and Stochastic Processes*. New York: McGraw-Hill, 2002.

[25] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, vol. I*. New York: McGraw-Hill, 1953.

[26] J. A. Stratton, *Electromagnetic Theory*. New York: McGraw-Hill, 1941.

[27] R. Janaswamy, "Evaluation of certain integrals using stochastic formulation of the lossy wave equation," in *IEEE Antennas and Propagation Society International Symposium*. Chicago, IL: Paper No. 563.7, July 8-14, 2012.

[28] A. Debnath, *Integral Transforms and Their Applications*. New York: CRC Press, 2000.

[29] C. DeWitt-Morette and S. K. Foong, "Path-integral solutions of wave equations with dissipation," *Physical Review Letters*, vol. 62, no. 19, pp. 2201–2204, 8 May 1989.

[30] K. E. Oughstun, *Electromagnetic and Optical Pulse Propagation-1: Spectral Representations in Temporally Dispersive Media*. New York: Springer, 2006.

[31] R. E. Hummel, *Electronic Properties of Materials: An Introduction to Engineers*. New York: Springer-Verlag, 1985.

[32] K. G. Budden, *The Propagation of Radio Waves: The Theory of Radio Waves of Low Power in the Ionosphere and Magnetosphere*. New York: Cambridge University Press, 1985.

[33] A. Taflove and S. Hagness, *Computational Electrodynamics: The Finite Difference Time Domain Method*, 3rd ed. Boston: Artech House, 2005.

[34] Y. Takayama and W. Klaus, "Reinterpretation of the auxiliary differential equation method for FDTD," *IEEE Microw. Wirel. Compon. Lett.*, vol. 12, no. 3, March 2002.

[35] M. Okoniewski and E. Okoniewska, "Drude dispersion in ADE FDTD revisited," *IEE Electronics Letters*, vol. 42, no. 9, pp. 503–504, 27 April 2006.

[36] G. B. Whitham, *Linear and Nonlinear Waves*, ser. Wiley Interscience Publication. New York: John-Wiley & Sons, 1974.

[37] K. E. Oughstun, *Electromagnetic and Optical Pulse Propagation-2: Temporal Pulse Dynamics in Dispersive, Attenuative Media*. New York: Springer, 2009.

[38] A. T. de Hoop, "A modification of Cagniard's method for solving seismic pulse problems," *Appl. Sci. Res.*, vol. 8, no. Sec. B, pp. 349–356, 1960.

[39] ——, "Pulsed electromagnetic field radiation from a line source in a two-media configuration," *Radio Science*, vol. 14, no. 2, pp. 253–268, March-April 1979.

[40] A. T. de Hoop, M. Stumpf, and I. E. Lager, "Pulsed electromagnetic field radiation from a wide slot antenna with a dielectric layer," *IEEE Trans. Antennas Propagat.*, vol. 59, no. 8, pp. 2789–2798, August 2011.