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# General Properties for Determining Power Loss and Efficiency of Passive Multi-Port Microwave Networks

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**Abstract**

Starting from the scattering matrix formulation, three useful properties are derived that characterize the dissipative loss and the corresponding efficiency of a multiport, passive microwave network. Elementary examples are considered that involve both reciprocal and non-reciprocal networks to demonstrate the utility of the expressions provided. When applied to the equal-split, matched, 3-port resistive divider, they recover the known fact that the device is 50% efficient. The relations yield the new result that the efficiency of a 3-port Wilkinson power divider is 2/3 on the average. Using the results presented it is further shown that the Wilkinson power divider belongs to a class of most efficient, matched, reciprocal 3-port networks that are constrained to provide maximum isolation at the output ports.

**Keywords**: Efficiency, scattering matrix, passive microwave network, dissipation, hypersphere, eigenanalysis.

# 1 Introduction

Multi-port passive microwave devices such as power dividers, circulators, filters, couplers, etc. abound in many RF systems and often times these devices have dissipative losses. Losses are present as a result of employing imperfect components or they may have been introduced deliberately to optimize other performance metrics such as improving bandwidth. An important and commonly used metric for characterizing the dissipative loss in a passive device is its efficiency[1]. For instance, the efficiency of a matched 3-port resistive power divider is 50%, meaning that half of the excited power is wasted as dissipation in the device. While such efficiencies are available for specific input excitations, it is desirable to come up with a single efficiency for the device that is independent of its excitation so that various devices can be directly compared on grounds of dissipation and more deeper questions can be addressed. For instance, what is the most efficient matched, reciprocal, 3-port power divider considering arbitrary excitations? A single loss metric to facilitate such studies is not currently available in any of these standard references [1], [2], [3], [4], [5] and the purpose of this work is to fill this void for an arbitrary passive $n$-port microwave network. Note that our intent is not to propose a single metric that replaces other commonly used metrics such as insertion loss, return loss, isolation, bandwidth *etc.* but rather to propose a single metric that characterizes the dissipative loss of a passive n-port device *independent* of which port it is excited from. In section 2, we arrive at several useful relations for characterizing a lossy microwave network. Several examples involving both reciprocal and non-reciprocal devices are considered in section 3 to show the utility of the expressions derived. Finally, concluding remarks are provided in section 4.

# 2 Theory

Scattering matrices are commonly used to describe input-output relations of passive devices among others [6], [7]. Given the scattering matrix of a multi-port ($n$ ports) passive device, which need not be reciprocal, we arrive in this section at analytical expressions for the average fractional loss and average efficiency while assuming that all input excitations are equally likely. We accomplish this by performing random sampling in an $n$-dimensional space by employing $n-1$ hyperspherical coordinates [8], [9]. After providing some examples to demonstrate the utility of the expressions derived, we show that the Wilkinson power divider belongs to a class of best matched, reciprocal 3-port power dividers.

## 2.1 General Result for $n$-Port Devices

Consider a passive, possibly lossy, $n$-port network characterized by its scattering matrix with entries $S_{ij}$ and satisfying $|S_{ij}| \leq 1$, $i, j = 1, \ldots, n$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the vectors of incoming and outgoing waves at the ports as shown in Figure 1 and related by $\boldsymbol{b} = S\boldsymbol{a}$, where $\boldsymbol{a} = [a_1, a_2, \ldots, a_n]'$ and $\boldsymbol{b} = [b_1, b_2, \ldots, b_n]'$, with primes indicating a matrix transpose.

The power lost due to dissipation in the device is $P_\ell = \boldsymbol{a}^\dagger \boldsymbol{a} - \boldsymbol{b}^\dagger \boldsymbol{b}$, [1], where the subscript $\dagger$ denotes Hermitian conjugate (complex conjugate and transpose). Then the fractional power, $F_\ell$, lost due to dissipation in the network is defined as

$$\begin{aligned} F_\ell &= \frac{P_\ell}{\boldsymbol{a}^\dagger \boldsymbol{a}} = \frac{\boldsymbol{a}^\dagger (I - S^\dagger S)\boldsymbol{a}}{\boldsymbol{a}^\dagger \boldsymbol{a}} =: \frac{\boldsymbol{a}^\dagger H \boldsymbol{a}}{\boldsymbol{a}^\dagger \boldsymbol{a}} \geq 0 \\ &= \boldsymbol{a}^\dagger H \boldsymbol{a}, \text{ subject to } \boldsymbol{a}^\dagger \boldsymbol{a} = 1, \end{aligned} \tag{1}$$

where $H = I - S^\dagger S$ is a Hermitian matrix and $I$ is an identity matrix of size $n$. In the literature $H$ is referred to as the *dissipation matrix* [1, p. 271]. The spectral theorem for Hermitian matrices [10, p.171] guarantees that $n$ eigenpairs of $H$ exist. Let $(\lambda_i, \boldsymbol{e}_i)$, $i = 1, \ldots, n$ denote the $i$th eigenpair of the matrix $H$ such that $H\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$, and $\boldsymbol{e}_i^\dagger \boldsymbol{e}_j = \delta_i^j$, where $\delta_i^j$ is the Kronecker's delta. In view of the positive semi-definite property of $H$ suggested by the quadratic form $\boldsymbol{a}^\dagger H \boldsymbol{a} \geq 0$, the eigenvalues $\lambda_i$ are not only real, but also non-negative. We express an arbitrary excitation in terms of the eigenvectors as

$$\boldsymbol{a} = \sum_{i=1}^{n} \mu_i \boldsymbol{e}_i \tag{2}$$

with $\mu_i$ being real and satisfying

$$\sum_{i=1}^{n} \mu_i^2 = 1. \tag{3}$$

[1] Of course a device will have several efficiencies–one relating to dissipation, one relating to mismatch, *etc.* In this paper we are only concerned with the former.

Note that $\mu_i$ can always be made real by appropriately choosing the reference planes at the ports. With these definitions, the fractional dissipated power becomes

$$F_\ell = \sum_{i=1}^{n} \mu_i^2 \lambda_i \tag{4}$$

subject to (3). The efficiency, $\eta_\ell$, of the device is the ratio of the power remaining in the network after dissipation to the excitation power and related to $F_\ell$ via

$$\eta_\ell = 1 - F_\ell = \sum_{i=1}^{n} \mu_i^2 (1 - \lambda_i). \tag{5}$$

If only the $k$th eigenvector is excited, $\mu_k = 1$ and $\mu_i = 0, i \neq k$. In that case $F_\ell = \lambda_k$, i.e., the fractional power lost equals the $k$th eigenvalue and $\eta_\ell = (1 - \lambda_k)$. From the positivity of $F_\ell$ and $\eta_\ell$, it is also clear that $0 \leq \lambda_k \leq 1 \ \forall \ k$. This further implies in view of (3) that $0 \leq F_\ell \leq 1$ and $0 \leq \eta_\ell \leq 1$ as it should.

To determine the average power lost for all possible combinations of the excitations, we choose $\mu_i$ to lie randomly and uniformly[2] on the surface of a unit-radius hypersphere in $n$-dimensional space and take the expectation with respect to the location on the hypersphere. To this end, we select $n-1$ independent hyperangles $\phi_i, \ i = 1, \ldots, n-1$ with $\phi_i \in (0, \pi), i = 1, \ldots, n-2$ and $\phi_{n-1} \in (0, 2\pi)$ and a hyper-radius $r$ and set [8], [9]

$$\mu_i = r \cos\phi_i \sin\phi_{i-1} \cdots \sin\phi_2 \sin\phi_1, i \in [1, n-1] \tag{6}$$

$$\mu_n = r \sin\phi_{n-1} \cdots \sin\phi_2 \sin\phi_1. \tag{7}$$

These equations are generalizations to $n$-dimensional space of the spherical coordinates $\mu_1 = r\cos\theta$, $\mu_2 = r\cos\phi\sin\theta$, $\mu_3 = r\sin\phi\sin\theta$ that are well known in 3D. Note that the $\mu_i$ defined in (6)-(7) automatically satisfy the condition in (3) with $r = 1$. The elemental surface $ds_n$ area on the hypersphere can be obtained from the Jacobian of the above transformation $J_n = \|\partial\mu_i/\partial(r, \phi_j)\|$, where $\|A\|$ denotes the absolute value of the determinant of the matrix $A$. Writing the elemental volume $dv_n = d\mu_1 d\mu_2 \ldots d\mu_n = dr ds_n = J_n dr d\phi_1 \ldots d\phi_{n-1}$, it is straightforward to see that the Jacobian satisfies the recursive relation

$$J_n = r J_{n-1} \prod_{j=1}^{n-2} \sin\phi_j, \qquad n = 2, 3, \ldots \tag{8}$$

with $J_1 = 1$ and the elemental surface $ds_n$ on a unit sphere (i.e., with $r = 1$) equals

$$ds_n = \prod_{j=1}^{n-2} \sin^{n-1-j}(\phi_j) \prod_{j=1}^{n-1} d\phi_j. \tag{9}$$

By integrating the elemental surface area over the space of all angles, the surface area $S_n$ of a unit sphere is obtained as

$$S_n = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \tag{10}$$

where $\Gamma(\cdot)$ is the Gamma function. For instance, $S_2 = 2\pi$ and $S_3 = 4\pi$, corresponding to a circle in 2D and a sphere in 3D, respectively. The average fractional power dissipated is

$$\bar{F}_\ell = \int_{\phi_1=0}^{\pi} \cdots \int_{\phi_{n-2}=0}^{\pi} \int_{\phi_{n-1}=0}^{2\pi} \frac{1}{S_n} F_\ell(\phi_1, \ldots, \phi_{n-1}) \, ds_n, \tag{11}$$

where $ds_n/S_n$ can be interpreted as the probability that the excitation vector lies in the direction $(\phi_1, \phi_2, \ldots, \phi_{n-1})$. Using (4) in (11) leads to

$$\bar{F}_\ell = \frac{1}{S_n} \sum_{i=1}^{n} G_i \lambda_i \tag{12}$$

[2] It is straightforward to include non-uniform port excitations into this formulation by including a weight factor $w_i$ inside the summation in (4).

where

$$G_i = \int_{\phi_1=0}^{\pi} \cdots \int_{\phi_{n-2}=0}^{\pi} \int_{\phi_{n-1}=0}^{2\pi} \prod_{j=1}^{i-1} \sin^{n+1-j}(\phi_j) \sin^{n-1-i}(\phi_i) \cos^2(\phi_i) \prod_{j=i+1}^{n-2} \sin^{n-1-j}(\phi_j) \times \prod_{j=1}^{n-1} d\phi_j, \quad \text{for } 1 \le i \le n-1 \tag{13}$$

and

$$G_i = \int_{\phi_1=0}^{\pi} \cdots \int_{\phi_{n-2}=0}^{\pi} \int_{\phi_{n-1}=0}^{2\pi} \prod_{j=1}^{n-1} \sin^{n+1-j}(\phi_j) d\phi_j, \quad \text{for } i = n. \tag{14}$$

Using the identities [11, 3.621-3,4]

$$\int_0^{\pi} \sin^{\nu} x \, dx = \frac{(\nu-1)!!}{\nu!!} \begin{cases} \pi, & \text{for } \nu \text{ even} \\ 2, & \text{for } \nu \text{ odd} \end{cases} \tag{15}$$

where $\nu!! = \nu(\nu-2)\cdots 2 = 2^{n/2}\left(\frac{n}{2}\right)!$ or $\nu!! = \nu(\nu-2)\cdots 1$ for $\nu$ even or odd, respectively, and some algebraic manipulations, it is straightforward to show that $G_i = S_n/n$ for all $i$. Thus we have the first key property that

**Property-1:** *The average fractional loss, $\bar{F}_\ell$, of an n-port microwave network characterized by the scattering matrix $S$ with the corresponding dissipation matrix $H = I - S^\dagger S$ is equal to*

$$\bar{F}_\ell = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} \operatorname{Tr}(H) \tag{16}$$

*where $\operatorname{Tr}(H)$ denotes the trace (sum of diagonal elements) of the matrix $H$.*

Equation (16) is a general result that is valid for any passive $n$-port network, whether it is reciprocal or not. It can be interpreted to mean that under equally likely port excitations the probability of exciting each eigenvector is $1/n$ and that the average loss is the sum of the loss experienced by each eigenvector and weighted by this probability. It is a single number used to characterize the loss of a passive network, independent of the excitation, and may be employed to compare the efficiency of various networks. An efficiency, $\bar{\eta}_\ell$, may be associated with the average fractional loss via $\bar{\eta}_\ell = 1 - \bar{F}_\ell$ that describes the overall power delivering capacity of the network from port to port. On using (16) we have the second key property that

**Property-2**: *The average efficiency $\bar{\eta}_\ell$ of a passive n-port device is equal to*

$$\bar{\eta}_\ell = \frac{1}{n} \operatorname{Tr}(S^\dagger S) = \frac{1}{n} \sum_{i,j=1}^{n} |S_{ij}|^2 = \frac{1}{n} \|S\|_F^2, \tag{17}$$

*where $\|S\|_F$ denotes the Frobenius norm of the matrix $S$.*

For a non-dissipative network, $S^\dagger S = I$, giving $\bar{\eta}_\ell = 1$. For the more general case, recall that the eigenvalues of $H$ lie in the range 0 and 1. Now for $(\lambda, \mathbf{e})$ to be an eigenpair of $H$ we require that $(H - \lambda I)\boldsymbol{e} = (I - S^\dagger S - \lambda I)\boldsymbol{e} = \mathbf{0}$ which implies that $(S^\dagger S)\mathbf{e} = \mathbf{0}$ when $\lambda = 1$. For a non-trivial solution of this equation we need $\det(S^\dagger S) = 0 = \det(S^\dagger)\det(S)$ or $\det(S) = 0$. Thus we have the additional property that

**Property-3:** *If the scattering matrix of a device with n-available ports is singular, then $\lambda = 1$ will be an eigenvalue of the dissipation matrix $H$, implying that $\bar{F}_\ell \ge 1/n$ and the corresponding average efficiency $\bar{\eta}_\ell \le (n-1)/n$.*

# 3 Some Lossy 3-Port Devices

In this section we will apply the formulas derived in the previous section to some practical multiport devices and recover some of the standard results on loss and efficiency as well as arrive at some new ones.

### 3.1 Resistive Power Divider

For a matched, reciprocal, equal-split resistive power divider, the scattering matrix $S$ [5, p. 327] and the corresponding dissipation matrix $H$ are

$$S = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}; \quad H = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{18}$$

For this device, equations (16) and (17) suggest $\bar{F}_\ell = 0.5$ and $\bar{\eta}_\ell = 0.5$, a well known fact that the matched resistive power divider is only 50% efficient [5, p. 328].

### 3.2 Non-Ideal Circulator

We now apply the results to a non-reciprocal microwave network. Consider a non-ideal, but symmetric 3-port circulator (signal flow in the direction of Port 1 to Port 2 to Port 3) which has a return loss of $-20\log\alpha$ dB, insertion loss of $-20\log\gamma$ dB, and an isolation of $-20\log\beta$ dB. Accordingly, we set $S_{11} = S_{22} = S_{33} = \alpha e^{j\psi}$, $S_{12} = S_{23} = S_{31} = \beta e^{j\theta}$, $S_{21} = S_{13} = S_{32} = \gamma e^{j\phi}$, where $\psi$, $\theta$, and $\phi$ are the phases and $\alpha, \beta, \gamma$ are the moduli of the respective S-parameters. For this case, it is easy to see that $\mathrm{Tr}(H) = 3[1 - (\alpha^2 + \beta^2 + \gamma^2)]$ and thus

$$\bar{\eta}_\ell = \left(\alpha^2 + \beta^2 + \gamma^2\right) \leq 1, \tag{19}$$

where the last inequality is due to the constraint $\alpha^2 + \beta^2 + \gamma^2 \leq 1$ imposed by the positive semidefinite character of $H$. As a numerical example, for a circulator designed with a return loss of 10 dB (VSWR $\simeq 2$), an insertion loss of 3 dB, and an isolation[3] of 20 dB, $\alpha = 1/\sqrt{10}$, $\beta = 1/10$, $\gamma = 1/\sqrt{2}$ resulting in an average efficiency of $\bar{\eta}_\ell = 0.61$. This is greater than the result $\eta = 0.5$ obtained from insertion loss calculation alone because insertion loss includes both mismatch loss and dissipative loss.

### 3.3 Wilkinson Power Divider

For the equal-split, 3-port Wilkinson power divider, with port-1 regarded as the input port, the scattering matrix $S$ [5, p. 330] and the dissipation matrix $H$ are

$$S = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix}; \quad H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{20}$$

The device is matched at all of its input ports and the average fractional power loss and average efficiency are $\bar{F}_\ell = 1/3$, $\bar{\eta}_\ell = 2/3$, respectively, indicating that the Wilkinson power divider is more efficient than the resistive divider. Even though this general belief is known, equation (17) provides for the first time a quantitative measure to establish that fact. It is worth noting that the Wilkinson power divider approaches the upper limit suggested by Property-3.

The *average* efficiency given in (17) should not take away from the fact that better efficiencies can be obtained with specific excitations. For instance, if only port-1 of a Wilkinson power divider is excited, the device operates at 100% efficiency when the output ports 2 and 3 are perfectly matched [5, p. 331]. Conversely, when used as a power combiner with inputs appearing at ports 2 and 3 and output appearing at port-1, the device operates at 100% efficiency. But these specific cases can be easily handled by the original, non-averaged quantities defined in (4) and (5) by representing the specific excitation as linear combinations of the eigenvectors of $H$. For the Wilkinson power divider, the eigenvalues and eigenvectors are $\lambda_1 = 0, \boldsymbol{e}_1 = [1,\ 0,\ 0]', \lambda_2 = 0, \boldsymbol{e}_2 = [0,\ 1/\sqrt{2},\ 1/\sqrt{2}]', \lambda_3 = 1, \boldsymbol{e}_3 = [0,\ -1/\sqrt{2},\ 1/\sqrt{2}]'$. If only port-1 is excited we use $\mu_1 = 1, \mu_2 = 0 = \mu_3$ and equations (4) and (5) give the well known results that $F_\ell = 0$ and $\eta_\ell = 1$. When used as a power combiner, we choose $\mu_2 = 1, \mu_1 = 0 = \mu_3$ for in-phase excitation. Equations (4) and (5) once again give $F_\ell = 0$ and $\eta_\ell = 1$.

### 3.4 Most Efficient Equal Split 3-Port Power Divider

Using (17), we now show that the Wilkinson power divider belongs to a class of most efficient reciprocal power dividers that provide isolation between ports-2 and 3 and remain simultaneously matched at all the 3-ports. Indeed, the

[3] Note that for a three-port circulator insertion loss, return loss and isolation cannot be chosen independently [12].

scattering matrix $S$ and the dissipation matrix $H$ of a reciprocal device that is simultaneously matched at all the 3-ports and provides maximum isolation between ports 2 and 3, when fed at port-1 are of the form

$$S = \begin{bmatrix} 0 & s & s \\ s & 0 & 0 \\ s & 0 & 0 \end{bmatrix}; \quad H = \begin{bmatrix} 1-2|s|^2 & 0 & 0 \\ 0 & 1-|s|^2 & -|s|^2 \\ 0 & -|s|^2 & 1-|s|^2 \end{bmatrix}, \tag{21}$$

where $s$ is a complex number with modulus less than 1. The scattering matrix is clearly singular and Property-3 predicts that $\bar{\eta}_\ell \leq 2/3$. The matrix $H$ has eigenvalues $1-2|s|^2, 1-2|s|^2, 1$. Non-negativity of the eigenvalues of $H$ implies that

$$1-2|s|^2 \geq 0 \implies |s| \leq 1/\sqrt{2}. \tag{22}$$

The efficiency of this network is $\bar{\eta}_\ell = 4|s|^2/3$. Thus, the most efficient device is the one that has the highest $|s|$ subject to (22). In other words, it is the network that has $|s| = 1/\sqrt{2}$, thus yielding $\bar{\eta}_\ell = 2/3$, which is also the upper limit predicted by Property-3. Wilkinson power divider, or a quadrature or $180^o$ hybrid with one of the four ports match loaded are devices that share this property.

# 4 Conclusions

Equations (16) and (17) have been established that characterize the average fractional loss and average efficiency of any $n$-port passive network, independent of the excitation. Furthermore Property-3 has been deduced that provides an upper bound to the efficiency of a passive device that has a singular scattering matrix. Several examples are shown to demonstrate the utility of these expressions in making device comparisons. Examples include a resistive power divider, a non-ideal circulator and the Wilkinson power divider. It has further been shown by means of these expressions that the Wilkinson power divider belongs to the class of most efficient, 3-port reciprocal devices that has the requirement of being matched at all the ports while providing maximum isolation between the output ports.

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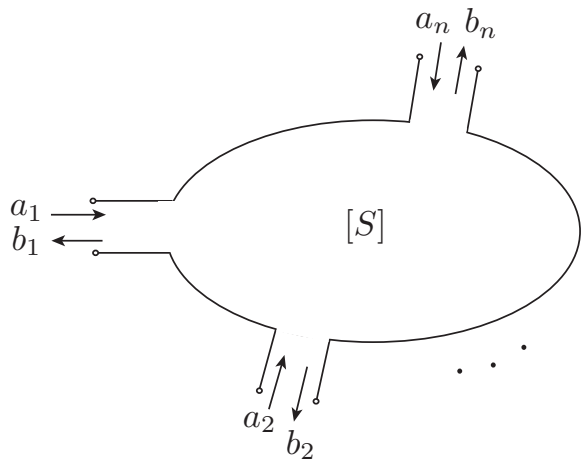


Figure 1: Passive $n$-port lossy RF network.

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