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Abstract: In this note we consider a Merton model for default risk, where the firm’s value is driven by a Brownian motion and a compound Poisson process.

Keywords: Merton model, default risk, default probability, processes with jumps

1 Introduction

Various models of Merton’s type for credit risk have been studied so far (refer [1] to [8]). This paper aims to our recent results, where a model driven by a jump process is studied in [9] and another model governed by a jumps-diffusion is investigated in [10]. Suppose that the asset value \( V_t \) of a company, under a risk neutral measure, is given by the following differential equation

\[
dV_t = (r - \beta \lambda) V_t dt + \sigma V_t dW_t + V_t - dQ_t,
\]

(1.1)

where \( W_t \) is a standard Brownian motion, \( Q_t = \sum_{i=1}^{N(t)} Y_i \) is a compound Poisson process, \( N(t) \) is a Poisson process with intensity \( \lambda > 0 \). \( Y_i \)'s are independent and identically distributed random variables with \( E(Y_i) = \beta \). All of these processes are supposed to be considered under the risk neutral measure. In (1.1), \( r \) is the interest rate, \( \sigma > 0 \) is a constant and \( N_t \) expresses the number of jumps of \( Q_t \) while \( Y_i \) is the \( i \)-th jump size of \( Q(t) \).

The model (1.1) reflects a fact that, the firm’s value can change randomly not only in a continuous way but also in a cumulatively discrete fashion.

We will study on the probability of default of the company when its value \( V_t \) is less than some debts.

2 Case of one debt \( L \)

A bankruptcy situation will occur at some time \( t \) when the company asset value is less than a debt \( L \). And the problem is how to calculate the default probability \( P(V_t < L) \).

It is known that the solution of (1.1) is given by (see [7])

\[
V_t = V_0 \exp[\sigma W_t + (r - \beta \lambda - \frac{\sigma^2}{2}) t] \prod_{i=1}^{N_t} (Y_i + 1).
\]

(2.1)

We see that

\[
\ln V_t = \ln V_0 + \sigma W_t + (r - \beta \lambda - \frac{\sigma^2}{2}) t + \sum_{i=1}^{N_t} \ln(Y_i + 1).
\]

(2.2)

And the event \( \{V_t < L\} \) or \( \{\ln V_t < \ln L\} \) means that

\[
\sigma W_t + Z_t < x_t,
\]

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We calculate first the characteristic function $\Psi_{Z_t}(s)$ of $Z_t$. $Z_t$ is also a compound Poisson process where $U_i$’s are i.i.d. random variables.

We recall first the characteristic function $\Psi_{Z_t}(s)$ of $Z_t$:

$$\Psi_{Z_t}(s) = E(e^{isZ_t}) = \sum_{j=0}^{\infty} E(e^{isZ_t}|N_t = j)P(N_t = j)$$

$$= \sum_{j=0}^{\infty} E(e^{is(U_1+\ldots+U_j)})P(N_t = j)$$

$$= \sum_{j=0}^{\infty} (Ee^{isU_1}\ldots Ee^{isU_j})P(N_t = j)$$

$$= \sum_{j=0}^{\infty} (\psi_U(s))^j \frac{(\lambda t)^j}{j!} e^{-\lambda t} = \exp[\lambda t(\psi_U(s) - 1)]$$  

where $\psi_U(s)$ is the common characteristic function of $U_i$’s.

It is known also that, for a compound Poisson process as $Z_t$ we have $\mu(t) = EZ_t = \lambda t E(U_1) = \lambda t E \ln(1 + Y_i) = \lambda tm$; $\sigma^2(t) = VarZ_t = \lambda t E(U_1^2) = \lambda t E[\ln(1 + Y_i)]^2 = \lambda t \gamma^2$, where $E \ln(1 + Y_i) = m$ and $E[(\ln(1 + Y_i))^2] = \gamma^2$.

Denote by $Z_t$ the normalization of $Z_t$:

$$Z_t = \frac{Z_t - \mu(t)}{\sigma(t)}.$$

And we will show that $Z_t$ has an approximately normal distribution.

Indeed, according to the Taylor expansion for characteristic function

$$\psi_U(s) = \sum_{k=0}^{\infty} \frac{(is)^k}{k!} E|U|^k,$$

we can write

$$\psi_U(s) = 1 + ism - \frac{s^2}{2} + o(s^2).$$  

(6.2)

Now we compute the characteristic function of $Z_t = \frac{1}{\sigma(t)}Z_t - \frac{\mu(t)}{\sigma(t)}$,

$$\Psi_{Z_t}(s) = e^{-is\frac{\mu(t)}{\sigma(t)}}\Psi_{Z_t}(s/\sigma(t)).$$

Taking account of (2.5) and (2.6) we have

$$\Psi_{Z_t}(s) = e^{-is\frac{\mu(t)}{\sigma(t)}}\exp[\lambda t(\psi_U(s/\sigma(t)) - 1)]$$

$$= e^{-is\frac{\mu(t)}{\sigma(t)}}\exp[i\lambda t m \frac{s}{\sigma(t)} - \frac{\lambda t \gamma^2}{\sigma(t)^2} \frac{s^2}{2} + o\left(\frac{s^2}{t}\right)]$$

$$= e^{-is\frac{\mu(t)}{\sigma(t)}}\exp[is\mu(t)/\sigma(t) - \frac{\sigma^2(t)}{\sigma^2(t)} \frac{s^2}{2} + o\left(\frac{s^2}{t}\right)]$$

$$= \exp\left[-\frac{s^2}{2} + o\left(\frac{s^2}{t}\right)\right], \text{ as } t \to \infty.$$
Then $Z_t \sim \mathcal{N}(0, 1)$ or $Z_t \sim \mathcal{N}(\mu(t), \sigma(t)^2)$, where $\mu(t) = \lambda t E\ln(1 + Y_t)$, $\sigma(t) = \sqrt{\lambda t E[\ln(1 + Y_t)]^2} = \sqrt{\lambda t \gamma}$. Now we can consider $\sigma_W + Z_t$ as a sum of two independent normal random variables for each $t$ large enough, so it has also a normal distribution with mean

$$\mu^*(t) = \mu(t) = \lambda t E\ln(1 + Y_t)$$

and variance

$$\sigma^*(t) = \sigma^2 + \sigma^2(t) = \sigma^2 + \lambda t E[\ln(1 + Y_t)]^2,$$

where $\sigma > 0$ is a known constant as in (1.1). And

$$P(\sigma W_t + Z_t < x_t) \approx \Phi\left(\frac{x_t - \mu^*(t)}{\sigma^*(t)}\right), \quad (2.7)$$

where $\Phi(x)$ is the standard normal distribution function.

We are now in the position to state the following theorem.

**Theorem 2.1** The default probability can be approximated by

$$P_{\text{default}} \approx \frac{1}{\sigma^*(t)\sqrt{2\pi}} \int_{-\infty}^{x_t} e^{-\frac{-(u-\mu^*(t))^2}{2\sigma^2(t)}} du, \quad (2.8)$$

where

$$x_t = \ln L - (r - \beta \lambda - \sigma^2/2)t - \ln V_0$$

$$\mu^*(t) = \lambda t E\ln(1 + Y_t), \quad \sigma^*(t) = \lambda t E[\ln(1 + Y_t)^2].$$

### 3 Case of many liabilities $L_1, L_2, \ldots, L_m$

Now we consider the case where the company faces up numerous debts $L_1, L_2, \ldots, L_m$ that should be paid at times $t_1, t_2, \ldots, t_m$ respectively, with $t_1 < t_2 < \ldots < t_m = T$.

The company will jump into default position before the time $T$ if and only if at one of time $t_i$ ($i = 1, 2, \ldots, m$), it happens that $V_{t_i} < L_i$.

So the probability of default before $T$ is

$$P_{\text{default}}(0, T) = 1 - P(V_{t_i} > L_i, \forall t_i).$$

Denote $L = \max\{L_1, \ldots, L_m\}$ It is easy to see that for all $t_i$ ($i = 1, \ldots, m$) we have

$$V_{t_i} > L_i \supset (V_0 > L).$$

Then

$$P_{\text{default}}(0, T) \leq 1 - P(V_0 > L, \forall t_i). \quad (3.1)$$

Put $X_t = \sigma W_t + Z_t$, where, as before $Z_t = \sum_{i=1}^{N_t} U_i$, $U_i = \ln(1 + Y_i)$. The inequality $V_{t_i} > L$ is equivalent to

$$X_{t_i} = \sigma W_{t_i} + Z_{t_i} > \ln L - \ln V_0 - (r - \beta \lambda - \sigma^2/2)t_i := x_{t_i}.$$  

Consider the event

$$A = \{V_{t_i} > L, \forall t_i\} = \bigcap_{i=1}^{m} \{X_{t_i} > x_{t_i}\}. \quad (3.2)$$

Then

$$P_{\text{default}}(0, T) \leq 1 - P(A).$$

It is known that a compound Poisson process is a process of independent increments. The processes $(W_t)$ and $(Z_t)$ are independent and both are of independent increments, so is the process $X_t = \sigma W_t + Z_t$.

Denoting by $A_i$ the event $\{X_{t_i} > x_{t_i}\}, i = 1, 2, \ldots, m$ we can see that

$$A_1 = \{X_{t_1} > x_{t_1}\} = \{X_{t_1} - X_0 > x_{t_1}\},$$

$$A_2 = \{X_{t_2} > x_{t_2}\} = \{X_{t_2} - X_{t_1} > x_{t_2}\},$$

$$\ldots$$

$$A_m = \{X_{t_m} > x_{t_m}\} = \{X_{t_m} - X_{t_{m-1}} > x_{t_m}\}.$$
\[ A_2 = \{X_{t_2} > x_{t_2}\} = \{X_{t_2} - X_{t_1} > x_{t_2} - x_{t_1}\} \supset \{X_{t_2} - X_{t_1} > x_{t_2} - x_{t_1}\}, \]

if \( A_1 \) occurs.

\[ \ldots \]

\[ A_m = \{X_{t_m} > x_{t_m}\} = \{X_{t_m} - X_{t_{m-1}} > x_{t_m} - x_{t_{m-1}}\} \supset \{X_{t_m} - X_{t_{m-1}} > x_{t_m} - x_{t_{m-1}}\}, \]

if \( A_1, \ldots A_{m-1} \) occur.

Put \( B_i = \{X_{t_i} - X_{t_{i-1}} > x_{t_i} - x_{t_{i-1}}\} \) for \( i = 1, 2, \ldots, m \) and \( x_0 = 0 \) by convention. It follows that

\[ \bigcap_{i=1}^{m} B_i \subset \bigcap_{i=1}^{m} A_i = A. \]

Because of the independence of increments we have

\[ P(A) \geq P\left( \bigcap_{i=1}^{m} B_i \right) = \prod_{i=1}^{m} P(B_i), \tag{3.3} \]

And by definition of \( B_i \),

\[ P(B_i) = P(X_{t_i} - X_{t_{i-1}} > x_{t_i} - x_{t_{i-1}}) = P(\sigma(W_{t_i} - W_{t_{i-1}}) + (Z_{t_i} - Z_{t_{i-1}}) > x_{t_i} - x_{t_{i-1}}). \tag{3.4} \]

Put \( X_i = X_{t_i} - X_{t_{i-1}}, W_i = \sigma(W_{t_i} - W_{t_{i-1}}) \) and \( Z_i = Z_{t_i} - Z_{t_{i-1}} \), where \( Z_i \) is defined as in (2.4). The random variable \( W_i \) has normal distribution \( \mathcal{N}(0, \sigma^2(t_i - t_{i-1})) \). The random variable \( Z_i = \sum_{k=1}^{N_i} U_k \) has the same distribution with that of \( \sum_{k=1}^{N_i} U_k \) since \( U_i \)'s are i.i.d and \( N_i \) is a process of stationary and independent increments.

We can see that the distribution of \( Z_i \) is given by

\[ F_{Z_i}(z) = P(Z_i \leq z) = \sum_{n=0}^{\infty} P(N_i = n)P(Z_i \leq z/N_i = n) \]

\[ = \sum_{n=0}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(Z_i \leq z/N_i = n) \]

\[ = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(\sum_{k=1}^{n} U_k \leq z) \]

\[ = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} F_{U^n}^{(n)}(z), \tag{3.5} \]

where \( F_{U^n}^{(n)} \) is the \( n \) fold convolution of common distribution of \( U_i \)'s.

Suppose now that \( U_i \)'s are continuous random variables, so are \( Z_i \)'s and \( Z_i \)'s. Then the density function of \( X_i = W_i + Z_i \) is

\[ f_{X_i}(x) = f_{W_i} * f_{Z_i}(x) = \int_{-\infty}^{\infty} f_{W_i}(x-z)f_{Z_i}(z)dz \]

\[ = \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})}\right] f_{Z_i}(z)dz, \tag{3.6} \]

where \( f_{Z_i}(z) = \frac{d}{dz} F_{Z_i}(z) \) is the density function of \( Z_i \).

Now we have

\[ P(B_i) = 1 - \int_{-\infty}^{x_i-x_{i-1}} f_{X_i}(x)dx, \]

where \( f_{X_i}(x) \) is defined by (3.6).

And so, the following assertion is ready to be stated:
**Theorem 3.1** If $U_k$’s are continuous random variables then the probability of default before $T$ is estimated by

\[
P_{\text{default}}(0, T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \int_{-\infty}^{x_i - x_{i-1}} \left[ \frac{1}{\sigma \sqrt{2\pi(t_i - t_{i-1})}} \times \right] \right. \\
\times \int_{-\infty}^{\infty} \exp \left[ - \frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} \right] f_{Z_i}(z) dz \left. dx \right),
\]

where

\[
x_i = \ln L - \ln V_0 - (r - \beta \lambda - \frac{\sigma^2}{2}) t_i
\]

and

\[
f_{Z_i}(z) = \sum_{n=0}^{\infty} \frac{d^z}{d z^n} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} p(Z_i \leq z/N_{t_i-t_{i-1}} = n).
\]

**4 Particular cases of Theorem 3.1**

We consider some particular cases for distribution of $U_k$’s.

**4.1. Case of normal random variables**

Suppose that $U_k \sim \mathcal{N}(0, 1)$ then we have $\sum_{k=1}^{m} U_k \sim \mathcal{N}(0, n)$ with density function $\frac{1}{\sqrt{2\pi n}} e^{-z^2/2n}$ and the density of $Z_i$ is

\[
f_{Z_i}(z) = \frac{1}{\sqrt{2\pi n}} \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} e^{-z^2/2n}
\]

(4.1)

From (3.8) and (4.1) we have

\[
P_{\text{default}}(0, T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \right) \frac{1}{2\sigma \sqrt{n(t_i - t_{i-1})}} \times \right. \\
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ - \frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} - \frac{z^2}{2n} \right] dz dx.
\]

**4.2. Case of exponential random variable $U_k$ with parameter $\nu > 0$**

We know that if $U_k \sim \exp(\nu)$ then $\sum_{k=1}^{m} U_k \sim \text{Gamma}(n, \nu)$ with the density function

\[
\frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)}
\]

where $\Gamma$ is Gamma function. Then

\[
f_{Z_i}(z) = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)}
\]

We can see the estimation in (3.8):

\[
P_{\text{default}}(0, T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \right) \frac{1}{\sigma \sqrt{2\pi(t_i - t_{i-1})}} \times \right. \\
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ - \frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} - \frac{z^2}{\nu} \right] \frac{z^{n-1}}{\nu^n \Gamma(n)} dz dx.
\]

(4.3)
5 When \( U = U_k \)'s are general discrete random variables

In this case we have

\[
P(Z_t = z) = P\left( \sum_{k=1}^{N_{t-t_{t-1}}} U_k = z \right) = \sum_{n=1}^{\infty} P(N_{t-t_{t-1}} = n) P\left( \sum_{k=1}^{n} U_k = z/N_{t-t_{t-1}} = n \right) = \sum_{n=1}^{\infty} \frac{\lambda^n (t_{i-1})^n}{n!} e^{-\lambda (t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right) = \sum_{n=1}^{\infty} \frac{\lambda^n (t_{i-1})^n}{n!} e^{-\lambda (t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right).
\]

(5.1)

Denote by \( \mathcal{L} \) the set of all possible values of \( Z_t \equiv \sum_{k=1}^{N_{t-t_{t-1}}} U_k \). So that

\[
P(\mathcal{X}_t < x) = P(\sigma W_t + Z_t < x) = \sum_{z \in \mathcal{L}} P(\sigma W_t < x - z) P(Z_t = z)
\]

\[
= \sum_{z \in \mathcal{L}} \sum_{n=1}^{\infty} \int_{-\infty}^{x-z} \frac{1}{\sigma \sqrt{2\pi (t_{i-1})}} \exp \left[ -\frac{(u^2)}{2\sigma^2(t_{i-1})} \right] \times \frac{\lambda^n (t_{i-1})^n}{n!} e^{-\lambda (t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right) du.
\]

(5.2)

The default probability in this case is estimated by

\[
P_{\text{default}}(0, T) \leq 1 - \prod_{t=1}^{m} \left( 1 - \sum_{z=0}^{\infty} \frac{1}{\sigma \sqrt{2\pi(t_{i-1})}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(x-z)^2}{2\sigma^2(t_{i-1})} \right] dx \times \frac{\lambda^n (t_{i-1})^n}{n!} e^{-\lambda (t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right) \right).
\]

(5.3)

6 \( U \) is Poisson random variable with parameter \( \beta > 0 \)

If \( U = U_k \sim \text{Poisson} (\beta) \) then

\[
\sum_{k=1}^{n} U_k \sim \text{Poisson} (n\beta)
\]

with mass probability

\[
p_z = P\left( \sum_{k=1}^{n} U_k = z \right) = e^{-n\beta (n\beta)^z} \frac{1}{z!}, \ z = 0, 1, 2, ...
\]

Then

\[
P_{\text{default}}(0, T) \leq 1 - \prod_{t=1}^{m} \left( 1 - \sum_{z=0}^{\infty} \frac{1}{\sigma \sqrt{2\pi(t_{i-1})}} \int_{-\infty}^{\infty} \exp \left[ -\frac{x^2}{2\sigma^2(t_{i-1})} \right] dx \times \frac{\lambda^n (t_{i-1})^n}{n!} e^{-\lambda (t_{i-1})} (n\beta)^z \right) \times \sum_{z=0}^{\infty} \frac{1}{z!} \frac{\lambda^n (t_{i-1})^n (n\beta)^z}{\sigma \sqrt{2\pi(t_{i-1})}} \int_{-\infty}^{\infty} \exp \left[ -\frac{x^2}{2\sigma^2(t_{i-1})} \right] dx.
\]

(6.1)

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