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Qiang Zhang
Department of Economics
University of Leicester
Leicester LE1 7RH
U.K.

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Abstract

I study the implications of a consumption and portfolio choice model featuring intertemporal hedging for the relationships between market incompleteness and the optimal consumption, the expected instantaneous consumption growth, and welfare. The model is adapted from Campbell, Chacko, Rodriguez, and Viceira’s (2004) continuous-time formulation of the Campbell and Viceira (1999) setup. The market incompleteness is indexed by $\rho$, the instantaneous correlation of the expected return and the instantaneous return to the risky asset. The degree of market incompleteness has no direct level effect on consumption when either consumption or portfolio choice is rationally myopic. For these two cases, market incompleteness only affects the level of consumption indirectly through wealth (for both types of rational myopicness) or the expected risky return (for rational myopicness in portfolio choice), though for empirically relevant parameter values these indirect effects are quantitively negligible, especially when $\rho \leq 0$. In contrast, when a consumer is rationally forward-looking in both consumption and portfolio decisions, market incompleteness directly impinges on consumption level in a complex nonlinear fashion. On the other hand, regardless of how forward-looking a consumer is, market incompleteness does have a direct growth effect, in the sense that the expected consumption growth explicitly depends on the incompleteness parameter: it declines for $\rho$ values close to $-1$, but mostly increases with it afterwards. And the growth effect of $\rho$ becomes increasingly more nonlinear, when moving from rationally myopic portfolio choice to rationally myopic consumption decision, to fully rational forward-looking in both decisions.
1 Introduction

Markets where some risks are not insurable through purchases of contingent claims are known as incomplete markets. Market incompleteness may affect some important economic decisions such as consumption, saving and portfolio choice. Understanding the effects of incomplete markets on these decisions is crucial for making sense of capital accumulation, asset pricing, the distributions of wealth and consumption, as well as consumption dynamics itself. Therefore, Deaton (1991), Kimball (1990), Leland (1968) and many others have studied precautionary saving caused by an uninsured risk, such as the risk in labor income. These studies have inspired a large literature in economics and finance.

Solving decision problems casted in incomplete markets setups is challenging, however. The closed-form solution to a moderately realistic formulation of the precautionary saving problem remains outstanding after decades of research efforts. As a result, researchers have often had to rely on computational methods to gain some understanding of the problem. Furthermore, it is difficult to disentangle the effect of uninsured labor income risk from that of liquidity constraints on consumption and saving at a conceptual level. See e.g. Deaton (1992) for an explanation.

In the present paper, to avoid these difficulties, I approach the precautionary saving problem from a different angle than the standard practice of directly confronting uninsured labor income risk. I ask how a potentially uninsured risk in a dynamic consumption and portfolio choice problem influences consumption, its expected growth, saving, and welfare. The risk that I investigate is in the expected return to the risky asset, and it can vary continuously from fully insured to completely uninsured. Carroll (2009) points out that any saving due to an insured risk

\footnote{Caballero (1988) provides a solution to the precautionary saving problem under the assumption of exponential utility function. However, as is well known, this specification of preferences have some major undesirable properties, including constant absolute risk aversion.}
is precautionary saving. The advantages of this approach are therefore two-fold. First, it allows for analytical solutions that are thus far impossible in the standard approach to understanding precautionary saving. Some of these solutions are new results, while others are based on exploiting the existing analytical solutions to a class of dynamic consumption and portfolio choice problems. Second, the approach that I follow in this paper demonstrates how varying degrees of market incompleteness impinge on economic decisions, in contrast with the limited analytical results for the extreme cases in the standard approach to precautionary saving.

The setup that I work with is adapted from Campbell, Chacko, Rodriguez, and Viceira (2004), who derive the continuous-time equivalent of the Campbell and Viceira (1999) partial-equilibrium model. And the latter is the first study on intertemporal hedging in portfolio choice that allows for intermediate consumption. Another conceptual advantage of adapting the Campbell et al. (2004) setup is that it disentangles risk aversion from intertemporal substitution in consumption by endowing the consumer with the Duffie and Epstein (1992a,b) stochastic differential utility, a continuous-time formulation of the Epstein and Zin (1989) and Weil (1990) non-expected utility recursive preferences.

In general, intertemporal hedging refers to protecting consumption from shifts in a state variable in multi-period contexts. See Samuelson (1969) and Merton (1969, 1973) for origins of this concept in the early models of consumption and portfolio choice. The insurance of consumption materializes through the wealth change induced by adjusting the composition of asset portfolio to exploit the correlation between the state variable and the investment opportunity set represented by moments of asset returns. When this correlation is absent, the shifts in the state variable become uninsurable. As a result, the rational consumer must save an additional amount out of her financial income to protect her consumption from this risk. The additional saving caused by uninsured shifts in the state variable is in essence no
different from the precautionary saving caused by uninsured labor income risk.

In the particular setup of Campbell et al. (2004) that features two assets, one riskfree and the other risky, the state variable that may be correlated with the investment opportunity set is the stochastic expected return on the risky asset. If the innovations to the risky expected return are uncorrelated with those to the instantaneous return on the risky asset, the risk in the stochastic expected return cannot be insured, because there is no hedge for it in the model. Asset markets are therefore perfectly incomplete in this case. The additional saving, induced by this uninsured risk to cushion consumption against the risky expected return and in excess of what is required by consumption smoothing in the absence of such a risk, is precautionary saving as explained above. As such, even though there is no labor income in the Campbell et al. (2004) model, there is still precautionary saving.

On the other hand, if the innovations to the risky expected return are correlated with those to the instantaneous return, the consumer’s portfolio composition will exploit this correlation to insure her consumption against the risky expected return. When the correlation reaches one in absolute value, the markets are said to be dynamically complete, meaning that the consumer can achieve the same allocation as in the complete markets setup, if she would engage in dynamic trading. In between these two extremes are cases with non-zero instantaneous correlations between the two risks where the consumer can partially insure the risk in the expected return. The extent of insurance increases with the absolute value of this correlation coefficient. I therefore interpret (the absolute value of) the instantaneous correlation coefficient of these two risks as a natural measure of the incompleteness of asset markets, and investigate how the optimal consumption, its expected instantaneous growth, and welfare as measured by the value function vary with this measure.

Giovannini and Weil (1989) find that in dynamic models of consumption and portfolio choice, unit EIS leads to rational myopicness in consumption, and unit
RRA leads to rational myopicness in portfolio choice. To summarize the findings of the present paper, the degree of market incompleteness has no direct level effect on consumption when there is rational myopicness in either consumption or portfolio choice. For these two cases, market incompleteness only affects the level of consumption indirectly through wealth (for both the unit EIS case and the unit RRA case) or the expected risky return (for the unit RRA case). In contrast, when a consumer is rationally forward-looking in both consumption and portfolio decisions (i.e. both RRA and EIS are non-unitary), market incompleteness directly impinges on consumption level in a complex nonlinear fashion.

On the other hand, regardless of how forward-looking a consumer is (i.e. for every possible combination of EIS and RRA), market incompleteness does have a direct growth effect, in the sense that the expected consumption growth explicitly depends on the incompleteness parameter. And the growth effect of that parameter becomes increasingly more nonlinear when we move from unit RRA to unit EIS, and then to non-unit RRA and EIS.

The rest of the paper is organized into three sections. In the next section, I solve the consumption problem for a unit-RRA consumer and explore its implications for how market incompleteness affects consumption, saving, and welfare. In Section 3, I first consider the case of unit EIS, and then move on to study the non-unit RRA and EIS by extending the results in Campbell et al. (2004). I conclude the present paper in Section 4.

### 2 Optimal Consumption under Unit RRA

One contribution of the present paper is to provide an approximate analytical solution to the optimal consumption and its expected instantaneous growth in a modified version of the Campbell et al. (2004) model, under the additional assumption of
This assumption is worth making, because Giovannini and Weil (1989), using the Euler equation approach, have discovered that consumption and portfolio decisions can be classified into four conceptually distinct categories associated with four configurations of preference parameters. And two of these four configurations involve unit RRA, combined with unit or non-unit EIS. Though they do not solve the decision rules for consumption and portfolio choice, Giovannini and Weil (1989) find that the unit RRA, under the complete markets assumption, makes the portfolio choice rationally myopic by eliminating the intertemporal hedging demand for the risky asset; however, despite the zero hedging demand, the optimal consumption still responds to the expected risky return unless the EIS is equal to one. In this section, I demonstrate that their qualitative results for unit RRA consumers are more robust: they hold regardless of the degree of market incompleteness.

2.1 The Setup

In the setup of Campbell, Chacko, Rodriguez, and Viceira (2004) with two assets, the riskfree asset has sure instantaneous return \( r \). The value of the risky asset \( V(t) \), for a given \( \alpha(t) \), follows a geometric Brownian motion

\[
\frac{dV(t)}{V(t)} = \alpha(t)dt + \sigma dZ(t),
\]

where \( dZ(t) \) is a standard Wiener process, \( \alpha(t) \) is the instantaneous expected risky return, and \( \sigma \), a constant, governs the volatility of the instantaneous return on the risky asset, \( \frac{dV}{V} \). \( \alpha(t) \) follows the mean-reverting AR(1), or the Ornstein-Uhlenbeck, process

\[
d\alpha(t) = \kappa [\mu_\alpha - \alpha(t)] dt + \sigma_{\alpha} dZ_\alpha(t), \quad \kappa > 0.
\]

\(^2\)The derivation in Campbell et al. (2004) implicitly assume non-unit RRA.
Here $\kappa$, $\mu_\alpha$, and $\sigma_{d\alpha}$ are constants, and $dZ_\alpha(t)$ is a Wiener process defined below in eq. (4). $\kappa$ governs the degree of mean reversion in $\alpha(t)$: for given $\alpha(0)$ and $t > 0$,

$$\alpha(t) = e^{-\kappa t} \alpha(0) + \left(1 - e^{-\kappa t}\right) \mu_\alpha + \sigma_{d\alpha} \int_0^t e^{-\kappa(t-q)} dZ_\alpha(q).$$

(3)

Hence the larger $\kappa$ is, the less persistent is the AR(1) process for $\alpha(t)$. In addition, by eq. (3), $\mu_\alpha$ is the "long-run" or unconditional mean of $\alpha(t)$. $\sigma_{d\alpha}$, the standard deviation of $d\alpha(t)$, determines the volatility of $\alpha(t)$ for given $\kappa$ and $\alpha(0)$.

The innovations to $d\alpha(t)$ follow a composite standard Wiener process

$$dZ_\alpha = \rho dZ + \sqrt{1 - \rho^2} dZ'_\alpha,$$

(4)

where $dZ$ and $dZ'_\alpha$ are two mutually independent standard Wiener processes, and $-1 \leq \rho \leq 1$ is a constant. Eq. (4) implies that the instantaneous correlation coefficient of $dZ_\alpha$ (the innovations to expected return) and $dZ$ (the innovations to the instantaneous return) is $\rho$. If $\rho = 0$, there is no hedge for the variation of the expected return $\alpha(t)$ over time. If $|\rho| = 1$, the asset markets are dynamically complete in the sense that the consumer can perfectly hedge against the time-varying $\alpha(t)$ by exploiting its perfect correlation with the instantaneous return to the stock. For other $|\rho|$ values in between, the markets are incomplete but consumers can partially hedge against the time-varying $\alpha(t)$ by exploiting its correlation with the instantaneous return of the stock.

The intuition here is as follows. A positive innovation to the instantaneous return on the risky asset translates (fully if $|\rho| = 1$) into a positive innovation (of the same magnitude if $|\rho| = 1$) to the expected return on the risky asset, $\alpha(t)$. And the AR(1) nature of $\alpha(t)$ means that future expected returns on the stock will also tend to be high. A rational consumer will respond to these instantaneous and intertemporal correlations captured by $\rho$ and $e^{-\kappa}$, respectively, whether they are perfect or not, by
continuously adjusting her portfolio share on the stock accordingly. Such continuous adjustment protects her financial income and wealth from the fluctuations in $\alpha(t)$, thereby insures her consumption against the risky expected return $\alpha(t)$.

The law of motion for wealth $W(t)$ is

$$
\frac{dW(t)}{dt} = \left[ r + \omega(t) (\alpha(t) - r) \right] W(t) dt - C(t) dt + \omega(t) W(t) \sigma dZ(t),
$$

(5)

Here $r$ is the instantaneous sure return to the riskless asset, $\omega(t)$ is portfolio weight on the risky asset, and $C(t)$ is consumption.

The agent maximizes the Duffie and Epstein (1992b) formulation of the stochastic differential utility, originally proposed in Duffie and Epstein (1992a), by choosing $C(t)$ and $\omega(t)$

$$
J(\tau) = E_t \int_{\tau}^{\infty} f(C(s), J(s)) ds.
$$

Here $f(C, J)$ is the aggregator function of the current consumption $C$ and the continuation utility $J$

$$
f(C, J) = \delta \frac{(1 - RRA) J}{1 - \frac{1}{ETS}} \left\{ \left[ \frac{C}{(1 - RRA) J^{1/(1-RRA)}} \right]^{1 - \frac{1}{RRA}} - 1 \right\}.
$$

(6)

Since the goal of this section is to investigate the consumption behavior under unit RRA, it is necessary to obtain the $RRA \to 1$ limit of $f(C, J)$ first. However, the formulation in eq. (6) does not allow for the $RRA \to 1$ limit to be taken if $J(t)$ remains finite for unit $RRA$, as that will produce $0^\infty$ in the denominator within the square bracket above, which is not an indeterminate form. l’Hopital’s rule therefore cannot be invoked in such a case. This is also why the first-order condition for consumption in Campbell et al. (2004, p. 2203) cannot be used to take the same limit to obtain the optimal consumption policy for unit RRA.\(^\text{3}\) On the

\(^3\)Eq. (26) in Campbell et al. (2004) contains the first-order condition for consumption in the...
other hand, if \( J(t) \) is not finite when \( RRA \to 1 \), it is possible that \((1 - RRA) J(t)\) converges to one. In that case, when \( RRA \to 1 \), the denominator within the square bracket above gives \( 1^\infty \), a proper indeterminate form. In the following, I prove that the second case must be true.

First, transform the aggregator function above by introducing a continuation utility \( K(t) \) that is increasing in \( J(t) \)

\[
K = -\frac{1}{1 - RRA} + J. \tag{7}
\]

This transformation does not change the preference ordering of different \( \{J(t)\} \) streams. Second, use eq. (7) to rewrite eq. (6) as

\[
f(C, J) = f\left[C, \frac{1 + (1 - RRA) K}{1 - RRA}\right] = \delta \frac{[1 + (1 - RRA) K]}{1 - \frac{1}{ETS}} \left\{ \frac{C}{(1 + (1 - RRA) K)^{1/(1 - RRA)}} \right\}^{1 - \frac{1}{ETS}} - 1 \\
\equiv g(C, K). \tag{8}
\]

Now if the \( \lim_{RRA \to 1} K(t) \) is finite, \((1 + (1 - RRA) K)^{1/(1 - RRA)}\) converges to \( 1^\infty \), a proper indeterminate form that can be handled by l’Hopital’s rule. That \( \lim_{RRA \to 1} K(t) \) is finite will be verified later.

The \( RRA \to 1 \) limit of \( f(C, J) \) is therefore

\[
p(C, K) \equiv \lim_{RRA \to 1} g(C, K) = \frac{\delta}{1 - \frac{1}{ETS}} \left[ \left( \frac{C}{e^K} \right)^{1 - \frac{1}{ETS}} - 1 \right]. \tag{9}
\]

utility maximization problem. It is obvious from inspecting that equation that the \( RRA \to 1 \) limit for consumption does not exist, because the second term in the right-hand side becomes \( 0^\infty \) as \( RRA \to 1 \). However, \( 0^\infty \) is, again, not an indeterminate form so l’Hopital’s rule cannot be applied.
2.2 Solving for the Continuation Utility

In this subsection, I solve the continuation utility $K(t)$ for EIS values that are in the neighborhood of one. This restriction on EIS is dictated by the need to linearize a second-order nonlinear ordinary differential equation wherein the unknown is one additive component of the conjectured $K(t)$. The linearization allows an approximate analytical solution to $K(t)$ to be obtained. Such an analytical solution in turn makes it possible to solve the optimal consumption policy under unit RRA for the first time.

With the aggregator function $p(C, K)$ in eq. (9), the Bellman equation is

$$0 = \sup_{\{C(t), \omega(t)\}} \{ p(C(t), K(t)) + K_W ([\omega(t)\alpha(t) + (1 - \omega(t)) r] W(t) - C(t))$$

$$+ \kappa K_\alpha [\mu_\alpha - \alpha(t)] + 0.5 K_{WW} W(t)^2 \omega(t)^2 \sigma^2$$

$$+ K_{W\alpha} W(t) \alpha(t) \rho \sigma d_\alpha + 0.5 K_{\alpha\alpha} \sigma^2 d_\alpha \}.$$  \hspace{1cm} (10)

Here $K_W$ and $K_{WW}$ are the first and second partial derivatives of $K(t)$ with respect to $W(t)$, and likewise for $K_\alpha$ and $K_{\alpha\alpha}$. $K_{W\alpha}$ is the cross partial derivative. The first-order conditions are therefore

$$C(t) = \delta^{EIS} K_W^{EIS} e^{(1-EIS) K(t)}$$  \hspace{1cm} (11)

$$\omega(t) = -\frac{K_W}{W(t) K_{WW}} \cdot \frac{\alpha(t) - r}{\sigma^2} - \frac{K_{W\alpha}}{W(t) K_{WW}} \cdot \frac{\rho \sigma d_\alpha}{\sigma}$$  \hspace{1cm} (12)

Here the second term on the right-hand side of eq. (12) captures the hedging demand for the risky asset. Giovannini and Weil (1989, p. 32) point out that the hedging demand for the unit-RRA consumer should vanish. In the present setup, this requires $K_{W\alpha} = 0$. Hence I conjecture that the two state variables $W$ and $\alpha$ are additively separable in the aggregator function $K$, but $W$ enters with a natural
logarithmic transformation

\[ K(t) \equiv K(W(t), \alpha(t)) = \ln W(t) + \Phi(\alpha(t)). \] (13)

Unlike in Giovannini and Weil’s (1989) framework where it is hard to explicitly solve \( \Phi(\alpha) \) and therefore the optimal consumption, the formulation above allows both of them to be solved.

With the conjecture in eq. (13), eqs. (11) and (12) may be used to derive the expressions for the optimal \( C \) and \( \omega \).

\[ C(t) = \delta EIS e^{(1-EIS)\Phi(\alpha(t))} W(t), \] (14)

\[ \omega(t) = \frac{\alpha(t) - r}{\sigma^2}. \] (15)

Although \( \omega \) can already be solved explicitly, \( C \) is only solved up to the unknown function \( \Phi(\alpha) \). Substituting the last two equations into the Bellman equation (10) yields

\[ 0 = \delta EIS e^{(1-EIS)\Phi(\alpha(t))} + (EIS - 1) \kappa [\mu - \alpha(t)] \Phi'(\alpha(t)) + \frac{\sigma^2 d\alpha}{2} (EIS - 1) \Phi''(\alpha(t)) + \frac{1}{2} (EIS - 1) \frac{[\alpha(t) - r]^2}{\sigma^2} + (EIS - 1) r - \delta EIS. \] (16)

Here \( \Phi'(\cdot) \) and \( \Phi''(\cdot) \) are the first and second derivatives of \( \Phi(\cdot) \). The Bellman equation now involves only one state variable, \( \alpha(t) \), because of the logarithmic functional form for \( W(t) \) and the separability of the two state variables in the conjectured eq. (13).

Due to the exponential term \( e^{(1-EIS)\Phi(\alpha(t))} \), the equation above is a second-order
nonlinear ordinary differential equation that does not have an exact analytical solution. However, if \((1 - EIS)\Phi (\alpha(t))\) is close to zero for any finite values of \(\alpha(t)\) and non-unitary values of \(EIS\), the exponential term can be approximated by its first-order Taylor series expansion. Such a linearization allows an approximate analytic solution to eq. (16) for a certain range of \(EIS\) values for a given time preference rate \(\delta\). See the Appendix for details.

Under the assumed \(AR(1)\) process for \(\alpha(t)\), there is an infinitesimal probability for infinite values of \(\alpha(t)\). However, they do not seem to have been observed in human history and do not seem empirically relevant. So I assume finite values of \(\alpha(t)\) in the present paper.

The definition of \(P(C(t), K(t))\) above requires non-unitary \(EIS\). After the linearization described above, dividing both sides of eq. (16) by \(1 - EIS\) yields a second-order linear ordinary differential equation as follows

\[
0 = \delta^{EIS} \Phi (\alpha(t)) - \kappa [\mu - \alpha(t)] \Phi' (\alpha(t)) - \frac{\sigma_{\alpha}^2}{2} \Phi'' (\alpha(t)) - \frac{[\alpha(t) - r]^2}{2\sigma^2} - r + \frac{\delta^{EIS} - \delta EIS}{1 - EIS}.
\]

Consider the following conjecture for the solution to this equation

\[
\Phi (\alpha(t)) = b_0 + b_1 \alpha(t) + b_2 \alpha(t)^2,
\]  

where the \(b\)'s are constants to be determined. With this conjecture, the differential equation (17) turns into the following quadratic equation in \(\alpha(t)\)

\[
0 = \left[ (\delta^{EIS} + 2\kappa) b_2 - \frac{1}{2\sigma^2} \right] \alpha(t)^2 + \left[ (\delta^{EIS} + \kappa) b_1 - 2b_2\kappa \mu + \frac{r}{\sigma^2} \right] \alpha(t) + \left[ \delta^{EIS} b_0 - \kappa \mu b_1 - \sigma_{\alpha}^2 b_2 - \frac{r^2}{2\sigma^2} - r + \frac{\delta^{EIS} - \delta EIS}{1 - EIS} \right].
\]
For this equation to hold for any finite values of \( \alpha(t) \), each of the three square-bracketed term on the right-hand side must be zero. This fact allows the underdetermined coefficients in eq. (18) to be solved recursively as follows

\[
b_2 = \frac{1}{2\sigma^2 (\delta^{EIS} + 2\kappa)} > 0, \tag{19}\]

\[
b_1 = \frac{2\kappa \mu_a b_2 - \frac{r}{\sigma_a^2}}{\delta^{EIS} + \kappa} > 0, \tag{20}\]

\[
b_0 = \frac{c^2 + r - \frac{\delta^{EIS} - \delta^{EIS}}{1 - \delta^{EIS}} + \kappa \mu_a b_1 + \sigma_{da}^2 b_2}{\delta^{EIS}} < 0. \tag{21}\]

Here \( b_2 > 0 \) holds because all the parameters entering the right-hand side of eq. (19) are positive. The sign of \( b_1 \) is for the parameter values estimated from the U.S. data by Campbell et al. (2004). And the sign of \( b_0 \) is based on restricting \( EIS \) to be not larger than 1.18872. There is currently no consensus on the size of \( EIS \). Empirical estimates based on aggregate data point to small \( EIS \) values, often closer to zero than to one. However, estimates based on household-level data produce \( EIS \) values around one.

It is noteworthy that the solutions in eqs. (19)-(21) are independent of \( \rho \), the index of market incompleteness. Mathematically, this is because \( K(t) \) is additively separable between the two state variables \( W(t) \) and \( \alpha(t) \), so that \( K_{W\alpha} = 0 \) in eq. (12), resulting in vanishing hedging demand. This mathematical convenience, on the other hand, is driven by the unit RRA assumption. In addition, the riskiness of the expected return affect these three coefficients through \( \sigma_{da} \) and \( \kappa \). \( \sigma_{da} \) only influences \( b_0 \). In contrast, the less persistent \( \alpha(t) \) is, i.e. the larger \( \kappa \) is, the smaller

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4The quarterly time discount rate \( \delta \) is not estimated from the data in their paper. Rather they assume it to be 0.0102. Here I follow them and use that value for \( \delta \).

5For \( EIS > 1.12545 \), \( b_0 \) becomes positive.

6However, unit RRA is not necessary for the conditional independence of the MPC from \( \rho \), as will be clear in the next section on unit \( EIS \).
Now substitute eqs. (19)-(21) into eq. (18), and use the resulting $\Phi(\alpha(t))$ in eq. (13) to solve for $K(t)$

$$K(t) = \ln W(t) + b_0 + b_1\alpha(t) + b_2\alpha(t)^2,$$

where the $b$'s are, again, given in eqs. (19) to (21). This solution to $K(t)$, derived for $RRA \to 1$, does not tend to infinity for finite $\alpha(t)$, contradicting the infinite $\lim_{RRA \to 1} K(t)$ obtained from eq. (7) if $\lim_{RRA \to 1} J(t)$ therein is finite. This contradiction implies that $\lim_{RRA \to 1} J(t)$ cannot be finite.

With infinite $\lim_{RRA \to 1} J(t)$, the question of whether the consumer's optimization problem as outlined above remains well-defined emerges. The answer to this question is affirmative, because in the aggregator in eq. (6), it is the limiting behavior of $(1 - RRA)J(t)$ that really matters for the continuation utility $f(C, J)$ to be well-defined. And by eq. (7), $(1 - RRA)J(t)$ remains finite when $RRA$ is driven to one.

### 2.3 Consumption Level and Market Incompleteness

The solution to $K(t)$ above can be substituted into eq. (11) to solve for optimal consumption

$$C(t) = \delta^{EIS} \left\{ 1 + (1 - EIS) \left[ b_0 + b_1\alpha(t) + b_2\alpha(t)^2 \right] \right\} W(t).$$

And to the best of my knowledge, this is the first time that an explicit analytical solution is obtained for the unit RRA case. Eq. (23) cannot be deduced directly from Campbell and Viceira's (1999) solution to consumption because the latter is derived under the assumption of non-unit $RRA$. 

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Several properties of this solution are noteworthy. First, the average propensity to consume (APC)

\[
APC \equiv \frac{C(t)}{W(t)} = \delta^{EIS} \{1 + (1 - EIS) \left[ b_0 + b_1 \alpha(t) + b_2 \alpha(t)^2 \right]\}
\]  

(24)
is a quadratic function of the expected risky return \(\alpha(t)\), reflecting the forward-looking nature of the consumption decision noted by Giovannini and Weil (1989), despite that the portfolio decision is rationally myopic. See Figures 1 and 2 for how the APC varies with \(\alpha(t)\) based on 10,000 simulated data points with \(EIS = 0.95\) and 1.05, respectively.

Second, since \(W(t)\) is endogenous, whether the APC can be interpreted as the marginal propensity to consume (MPC) depends on the source of the variation in \(W(t)\). This is because the MPC may be regarded as

\[
\frac{\partial C}{\partial x} = \frac{\partial APC}{\partial W} W + APC,
\]

and it differs from the APC as long as the first term on the right-hand side is not zero. Here \(x\) stands for an exogenous variable that may trigger a change in \(W(t)\), such as a change in the initial wealth \(W(0)\), in a preference parameter, or in an attribute in asset returns. Since the APC in eq. (24) does not depend on \(W(0)\), the APC is the MPC when the change in wealth is induced by a change in \(W(0)\). But for any other source of the change in \(W(t)\), this is no longer the case because then \(\partial APC / \partial x \neq 0\) and \(\partial W / \partial x\) is finite. For example, when the riskiness of the expected return \(\sigma_{\alpha}^2\) changes [See eq. (3)], the MPC becomes

\[
MPC_{\sigma_{\alpha}^2} \equiv \frac{\partial C}{\partial \sigma_{\alpha}^2} \frac{\partial \sigma_{\alpha}^2}{\partial W} = -1 + APC.
\]

Hence how consumption varies with wealth depends on what triggers the latter to
change.

Third, recall that the b’s in this equation, as presented in eqs. (19) to (21), are independent of ρ, the instantaneous correlation of the expected return and the instantaneous return on the risky asset that indexes market incompleteness. It therefore follows that conditional on the two state variables W(t) and α(t), the optimal consumption decision for a unit-RRA consumer does not depend on the degree of market incompleteness ρ. Equivalently, the consumption function is isomorphic over the entire range of market incompleteness.

Fourth, unconditionally, the optimal consumption in eq. (23) changes with ρ, because α(t), as given in eq. (3), is a function of ρ after dZα = ρdZ + \sqrt{1 - ρ^2}dZ′ α is substituted into it. In addition, eq. (5) implies that W(t) is also a function of ρ due to the presence of α(t). Therefore, market incompleteness indirectly influences the consumption and saving choice through the two state variables α(t) and W(t). As a result, the APC, which is a function of α(t), varies with ρ. So qualitatively speaking, the isomorphism of the consumption rule does not imply constant APC over the degree of dynamic market incompleteness.

However, with simulation it is straightforward to quantitatively depict how the APC indirectly varies with ρ through α(t). Figure 3 plots the median values of the simulated α(t) and APC sequences of 10,000 observations against alternative values of ρ, for three EIS values 0.82, 0.95 and 1.10 under the assumption δ = 0.0102.

Perhaps surprisingly, despite the upward tendency and the volatilities in W(t), the

\[ W(t) = \mu α, \sigma α, ρ and the history of innovations (i.e. the two Brownian paths of dZ and dZ′ α), are given. Similarly, given the dependence of W(t) on α(t), conditional on W(t) means that in addition to given α(t), the initial wealth W(0) along with the other parameters δ, EIS, r, and σ are given. Therefore, for given α(t), a change in W(t) can only come from W(0) or any of the latter four parameters. However, any change in the four parameters just mentioned also varies the coefficient of W(t) in eq. (23) through either the term δEIS or the b’s. The interpretation of this coefficient of W(t) as the marginal propensity to consume (MPC) out of wealth is valid only when the change in W(t) is induced by a change in W(0). wealth is endogenous is therefore not well-defined. In spite of this problem, in the consumption and portfolio choice literature and the precautionary saving literature, MPC has been used extensively. Here I follow this convention for the ease of exposition only.
median APCs appear to be fairly constant over the entire range of $\rho$ for the latter two EIS values. For $EIS = 0.82$, the APC tracks the upward tendency in $\alpha(t)$ slightly more for the high-end of $\rho$ value. The reason for the lack of conspicuous comovement in the APCs is two-fold: the $\Phi(\alpha(t))$ component in the APC is scaled down by $1 - EIS = 0.05$, and the 5% of $\Phi(\alpha(t))$ that is transmitted to the APC is further scaled down by the coefficient $\delta^{EIS}$ (slightly over 1%) in eq. (23). When $\delta = 0.005$, for the same EIS values, the median APCs change even less over $\rho$, as shown in Figure 4.

It is possible that for smaller EIS values the APC may indirectly vary more with $\rho$ than shown in Figures 3 and 4. However, since the analytical solution in eq. (23) is obtained by restricting $EIS$ to be sufficiently close to 1, it is out of the scope of the present paper to look into that scenario. The tentative conclusion from these figures is therefore that for empirically relevant parameter values, the consumption decision may be regarded as both directly and indirectly independent of market dynamic incompleteness.

Fifth, conditional on the two state variables, the riskiness of the expected return $\sigma_{da}$ affects $C(t)$ only through $b_0$. Unconditionally, $\sigma_{da}$ also changes $C(t)$ through the two state variables.

Sixth, the separation of portfolio decision $\omega(t)$ from the past consumption or saving decisions holds in the following sense: $\omega(t)$ does not depend on wealth $W(t)$, a quantity that is affected by past consumption or saving decisions.

Seventh, in terms of the other state variable $\alpha(t)$, there is no separation of portfolio decision $\omega(t)$ from the current consumption/saving decision, as both decisions are driven by $\alpha(t)$. This link between portfolio and consumption/saving decisions is absent in the standard I.I.D. returns model.

Last but not least, how $C(t)$ responds to $\alpha(t)$ in this model depends on if $EIS > 1$, but not necessarily as what the standard theory of intertemporal choice predicts.
Differentiating with respect to $\alpha(t)$ in eq. (23) yields

$$\frac{\partial C}{\partial \alpha} = APC \frac{\partial W}{\partial \alpha} + W \frac{\partial APC}{\partial \alpha} = APC \frac{\partial W}{\partial \alpha} + W \delta^{EIS} [(1 - EIS) b_1 + 2b_2 \alpha], \quad (25)$$

where every term on the right-hand side except $(1 - EIS)$ is unambiguously positive [even for EIS values larger than one but not sufficiently large to make APC negative]. Therefore, if $EIS < 1$, $\frac{\partial C}{\partial \alpha} > 0$ must hold. This is intuitive and conforms to a well-known result in the theory of intertemporal choice embodied in the overlapping-generations model of Diamond (1965), because $EIS < 1$ in the standard theory implies that the income effect on $C(t)$ of a change in $\alpha(t)$ is larger than the substitution effect. Hence, when $\alpha(t)$ increases, current consumption rises if $EIS < 1$.

However, what is surprising is that when $EIS = 1$ (so that $APC = \delta$), the right-hand side of eq. (25) is still positive

$$\left. \frac{\partial C}{\partial \alpha} \right|_{EIS=1} = \delta \frac{\partial W}{\partial \alpha} + \frac{W \delta}{\sigma^2 (\delta + 2\alpha)} \alpha > 0.$$ 

This is surprising because $\left. \frac{\partial C}{\partial \alpha} \right|_{EIS=1} = 0$ is supposed to hold according to the standard theory of intertemporal choice. The standard theory maintains that the income and substitution effects on consumption cancel each other out when $EIS = 1$, leaving $C(t)$ unchanged.

When $EIS > 1$, the consumption response to $\alpha(t)$ is ambiguous, because the second term on the right-hand side of eq. (25) is ambiguous. This also contrasts with the prediction of the standard theory, which with $EIS > 1$ implies that the substitution effect is larger than the income effect and therefore $\frac{\partial C}{\partial \alpha}$ is expected to be negative in this case.
The differences from the standard theory of intertemporal choice are due to the fact that \( \alpha(t) \) enters the APC quadratically with a positive coefficient.

### 2.4 Consumption Growth and Market Incompleteness

In order to understand if the expected instantaneous consumption growth responds to market incompleteness in a way different from how \( C(t) \) does, I now apply Ito’s lemma to eq. (23) to obtain

\[
g(t) \big|_{RRA=1} = \frac{E_t dC(t)}{C(t) \, dt} = \omega(t)\alpha(t) + [1 - \omega(t)] \, r - \delta^{EIS} \{ 1 + (1 - EIS) \left[ b_0 + b_1 \alpha(t) + b_2 \alpha(t)^2 \right] \} + (1 - EIS) \left\{ \kappa [\mu_\alpha - \alpha(t)] + b_1 \rho \sigma \sigma_{da} \omega(t) - 0.5 \left[ (1 - EIS) b_1^2 + 2 b_2 \alpha(t) \right] \sigma_{da}^2 \right\}.
\]

(26)

Note that the market incompleteness parameter \( \rho \) appears only once in eq. (26). That \( g(t) \big|_{RRA=1} \) depends on \( \rho \) is dictated by Ito’s lemma, which says that the covariation of the expected return and the instantaneous return to the risky asset must have first-order effect. Therefore, despite that \( \rho \) does not directly impinge on the level of optimal consumption, it does affect the expected instantaneous growth of consumption. Indirectly, \( \rho \) also affects \( g(t) \big|_{RRA=1} \) through \( \alpha(t) \) as well as \( W(t) \).

How \( \rho \) affects growth \( g(t) \big|_{RRA=1} \) is determined by the term \( (1 - EIS) b_1 \rho \sigma \sigma_{da} \omega(t) \) in the last curly bracket in eq. (26). Consider the case of \( \omega(t) > 0 \) first. Since \( b_1 > 0 \) for empirically relevant parameter values, the expected instantaneous growth of consumption increases with \( \rho \) (i.e. the less dynamically incomplete the asset markets are) if \( EIS < 1 \), i.e. the consumer is not very willing to substitute consumption over time in response to changes in asset returns. If \( EIS > 1 \) so that the consumer
is very willing to shift consumption over time, an increase in $\rho$ lowers the expected instantaneous growth. On the other hand, in times of low $\alpha(t)$ so that $\omega(t) < 0$, the converse occurs. These results enrich the familiar conclusion based on the expected utility iso-elastic preferences that restrict the RRA and the EIS to be the opposite of each other, because with that restriction, market incompleteness only increases expected consumption growth.

That the market dynamic incompleteness $\rho$ only has growth effect but not level effect on consumption is somewhat akin to the different roles played by the saving rate in the Solow growth model. In that model, while the steady-state level of income is determined by the saving rate, the steady-state growth of income does not depend on it.

In Figures 5 and 6, for two time preference rates, $\delta = 0.0102$ and $0.005$, respectively, I display how the median of the expected consumption growth sequence varies with $\rho$, given the sequences of $dZ$ and $dZ'_{\alpha}$. Overall, for $\rho$ values close to $-1$, the growth effect is negative. However, it becomes positive for most of other $\rho$ values, though the relationship is not monotone. The median expected growth rises to a level that is about 1.2~2 percentage points higher at $\rho = 1$ than at $\rho = -1$.

2.5 The Effect of Precautionary Saving on the Expected Consumption Growth

If the expected return is constant so that $\kappa = \sigma_{\Delta \alpha} = 0$, the last term in eq. (26), which is led by the common factor $(1 - EIS)$, vanishes. Since the last two components in this term are proportional to $\sigma_{\Delta \alpha}$, it should not be controversial to attribute them to precautionary saving due to the uncertainty in $\alpha(t)$, the expected risky return. The first component $\kappa [\mu_{\alpha} - \alpha(t)]$ within the sharp brackets of the last term in eq. (26) apparently depends on the persistence parameter $\kappa$, but not on the vari-
ance parameter $\sigma_{da}$, of the expected risky return, so it may not appear reasonable to relate it to precautionary saving.

Such a cursory view, however, ignores the fact that the volatility, and therefore the riskiness, of $\alpha(t)$ in fact also depends negatively on $\kappa$, in addition to $\sigma_{da}$. By eq. (3), the unconditional and conditional variances of the expected return on the risky asset, $\alpha(t)$, are

$$
\text{var}[\alpha(t)|\alpha(0)] = \frac{1 - e^{-2\kappa t}}{2\kappa} \sigma_{da}^2,
$$

$$
\text{var}[\alpha(t)] = \lim_{t \to +\infty} \text{var}[\alpha(t)|\alpha(0)]
= \frac{1}{2\kappa} \sigma_{da}^2
> \text{var}[\alpha(t)|\alpha(0)].
$$

So the riskiness of the expected return, whether conditional or unconditional, has two sources: $\kappa$ and $\sigma_{da}^2$. Both coefficients in the last two equations decrease with $\kappa$, and a larger $\kappa$ implies less persistent expected risky return. As a result, the less persistent the expected return (i.e. larger $\kappa$) is, the less risky it is, holding constant $\sigma_{da}^2$. Therefore, the first component $\kappa [\mu_{\alpha} - \alpha(t)]$ within the curly brackets of the last term in eq. (26) must be ascribed to precautionary saving as well. This first component only vanishes when the expected return $\alpha(t)$ follows the most persistent non-explosive process, the continuous-time random walk that arises when $\kappa = 0$.

In the labor income risk literature, there is the distinction between permanent and transitory shocks. In the setup here, since $\kappa = 0$ makes $\alpha(t)$ a continuous-time random walk, the innovations $dZ_\alpha$ become permanent shocks in that event. Hence the precautionary saving due to permanent shocks to labor income can be mimicked in the present setup by the parameter combination $\kappa = 0$ and $\sigma_{da}^2 > 0$. This suggests that the first component $\kappa [\mu_{\alpha} - \alpha(t)]$ within the sharp brackets of the last term in eq. (26) must be treated as the precautionary saving due to transitory shocks, along
with the other two components in the last term on the right-hand side of eq. (21). 

[With the combination of \( \kappa = 0 \) and \( \sigma^2_{\alpha} > 0 \), \( \text{var}[\alpha(t)|\alpha(0)] = \sigma^2_{\alpha} \) while \( \text{var}[\alpha(t)] \) is infinite.] To have both permanent and transitory shocks is impossible, and this is a shortcoming of the present setup as compared with the standard precautionary saving setup. However, when \( \kappa \) is slightly larger than 0, the innovations \( dZ_\alpha \) are persistent transitory shocks. So I can present a set of results demonstrating how the expected consumption growth varies with \( \kappa \), i.e. the persistence of shocks, to demonstrate how \( C(t) \) responds to the persistence of transitory shocks.

However, when \( \kappa > 0 \), this extra precautionary saving component due to \( \kappa \) must in the long run (or unconditionally, or asymptotically) be balanced out by the mean-reverting behavior of \( \alpha(t) \). This is because the sign of \( \kappa [\mu_\alpha - \alpha(t)] \) is determined by whether \( \alpha(t) \) is smaller or larger than its long-run mean \( \mu_\alpha \), thereby stochastically turning this component of the precautionary saving positive or negative. So for example, for a consumer with an \( EIS \) between zero and one, a decrease in the persistence of the expected return (i.e. a larger \( \kappa \)), ceteris paribus, raises his or her conditional expected consumption growth when \( \alpha(t) \) is smaller than \( \mu_\alpha \), but lowers it when \( \alpha(t) \) becomes larger than \( \mu_\alpha \).

Naturally, the long-run (or unconditional) mean value of \( \kappa [\mu_\alpha - \alpha(t)] \) is zero, suggesting that it contributes to precautionary saving only in the short-run. For this reason, it seems useful to distinguish between long-run (or unconditional) precautionary saving correction to the mean consumption growth from the short-run (or conditional) precautionary saving correction to the conditional expected consumption growth. And the difference between them is \((1 - EIS) \kappa [\mu_\alpha - \alpha(t)] \) (plus the difference between the last two correction terms in eq. (26) and their unconditional means). How large this difference is depends on how far away the instantaneous expected return is from the long run expected return \( \mu_\alpha \).

In addition, there is an indirect effect of the riskiness of the expected risky
return on the expected consumption growth through \( b_0, b_1, \) and \( b_2 \) that appear in the marginal propensity to consume [i.e. the third term in the right-hand side of eq. (26)]. This indirect effect arises because by eqs. (19) to (21), \( b_0 \) depends positively on \( \kappa \) and \( \sigma^2_{\alpha} \), and both \( b_1 \) and \( b_2 \) depends negatively on \( \kappa \). For this reason, the precautionary saving motive should include both the third and the fourth terms on the right-hand side of eq. (26). Furthermore, the precautionary saving terms change with \( \alpha(t) \) itself.

Since the solution in eq. (23) is obtained under the assumption of non-unit \( EIS \), it is natural to wonder what the limit of the consumption solution is when \( EIS \to 1 \). With a unit \( RRA \), the unit \( EIS \) assumption further restricts the utility function down to the log utility in the expected utility framework. It is well known that for the log utility, the optimal consumption is a constant proportion (given by \( \delta \)) of wealth. It is reassuring that as \( EIS \to 1 \), the solution to consumption given by eq. (23) converges to that well-known solution.$^8$

In addition, substituting eqs. (11)-(14) into eq. (9), the aggregator function for a unit \( RRA \) consumer, yields

\[
p(C(t), K(t)) = \frac{1}{1 - \frac{1}{EIS}} \left[ \frac{C(t)}{W(t)} - \delta \right] = \frac{\delta}{1 - \frac{1}{EIS}} \left\{ \delta^{EIS-1} \left[ 1 + (1 - EIS) \left[ b_0 + b_1 \alpha(t) + b_2 \alpha(t)^2 \right] \right] - 1 \right\}.
\]

This first equality (27) above implies that once the state variable \( W \) appears in the aggregator function, the other state variable \( \alpha \) does not. In contrast, the second equality, obtained by substituting in eq. (23), demonstrates that on once \( \alpha \) appears in the aggregator function, \( W \) does not. Hence, for a unit \( RRA \) consumer, only one of the two state variables may appear in the aggregator function and thus the

$^8$It is straightforward to verify that \( b_0 \) tends to a finite constant as \( EIS \) tends to one.
stochastic differential utility function. Since $\alpha$ and $W$ are functions of $\rho$, it follows that $p(C, K)$ is dependent on it as well. However, conditional on either of the two state variables, $p(C, K)$ does not vary with $\rho$ independently. Therefore, the welfare of a unit-RRA consumer is affected by $\rho$ only indirectly through either state variable. Since $p(C, K)$ is a linear function of the APC, its dependence on $\rho$ is already captured in Figures 3 and 4, which again show little variations in APC over the entire range of $\rho$. Therefore, it seems that market dynamic incompleteness does not seem to have a discernible effect on the welfare of a unit-RRA agent for most of the $EIS$ values considered in the present paper.

3 Optimal Consumption under Non-Unit RRA

I now turn to the more general case of non-unit RRA (for $EIS$ values close to one) in the same model setup as in Section 2. The even more general case of unrestricted $EIS$ is, however, an unsolved problem in the literature.

Giovannini and Weil (1990) find that when non-unit RRA is combined with unit $EIS$, the optimal consumption becomes rationally myopic, whereas the portfolio demand includes an intertemporal hedging component. When non-unit RRA is combined with non-unit $EIS$, both the consumption and the portfolio decisions are forward-looking. In the setup studied in the present paper, this means that both decisions depend on the correlation of the instantaneous and the expected returns on the risky asset, as is already shown in Campbell et al. (2004). What is new in the present paper for this case is to figure how consumption and its expected instantaneous growth vary with market incompleteness, both directly and indirectly through the state variables.

Depending on whether $EIS$ is unitary, there are two conceptually distinct cases on how market incompleteness influences consumption and its expected growth.
Under unit EIS, market incompleteness plays a role in determining the level and the expected growth of consumption that is somewhat similar to the case of unit RRA, i.e. $\rho$ has no explicit level effect but an explicit growth effect that are highly nonlinear. However, with non-unit EIS, the effects of market incompleteness on consumption and saving turn out to be highly nonlinear and complex.

### 3.1 Unit EIS

Giovannini and Weil (1989) uncovered that unit EIS makes consumers rationally myopic in choosing consumption but not myopic in portfolio choice. Campbell et al. (2004) for the first time obtain the explicit solution to the optimal consumption and demand for the risky asset

\[
C_{h,1}(t) = \delta W(t) ,
\]

\[
\omega_{h,1}(t) = \frac{\alpha(t) - r}{RR\alpha} + \frac{\rho\sigma\alpha}{RR\sigma} [B_0 + C_0\alpha(t)] ,
\]

where $B_0$ and $C_0$ are nonlinear functions of $\rho$ and other parameters of the model. In the consumption solution above, $\rho$ only impinges on consumption indirectly through $\alpha(t)$, which works through $W(t)$. The second component of $\omega_{h,1}(t)$ above captures the hedging demand for the risky asset. This component disappears if $\rho = 0$, i.e. the market is entirely dynamically incomplete.

Conditional on $W(t)$, the optimal consumption above is not directly influenced by $\rho$, similar to the unit RRA case. The consumption solution implies the following expected instantaneous consumption growth

\[
g_{h,1}(t) \equiv \frac{E_t dC(t)}{C(t) \, dt} = \frac{E_t dW(t)}{W(t) \, dt} = r + \omega_{h,1}(t) [\alpha(t) - r] - \delta . \tag{29}
\]

It is clear that the degree of market incompleteness $\rho$ directly impinges on $g_{h,1}(t)$.
linearly through the $\rho \sigma_\alpha$ term in $\omega_{h,1}$, and nonlinearly through the effects of $B_0$ and $C_0$ on the hedging demand for the risky asset in $\omega_{h,1}(t)$, both of which appear in the second term on the right-hand side of eq. (29)

$$\omega_{h,1}(t) [\alpha(t) - r] = \left[ \frac{\alpha(t) - r}{RRA\sigma^2} + \frac{\rho \sigma_\alpha}{RRA\sigma} [B_0 + C_0 \alpha(t)] [\alpha(t) - r] \right].$$

This direct nonlinear effect contrasts with the direct linear effect in the unit RRA case. And the precautionary saving effect is captured by the second term in the right-hand side of eq. (29) displayed above.

### 3.2 Non-Unit EIS

The optimal consumption and portfolio choice for the case of non-unit EIS and RRA provided by Campbell et al. (2004) are

$$C_h(t) = \delta^{EIS} e^{(A_1 - B_1 \alpha(t) - 0.5C_1\alpha(t)^2)} W(t), \quad (30)$$

$$\omega_h(t) = \frac{\alpha(t) - r}{RRA\sigma^2} + \left( 1 - \frac{1}{RRA} \right) \frac{\sigma_\alpha}{\sigma} \rho \frac{B_1 + C_1 \alpha(t)}{EIS - 1}. \quad (31)$$

In these two equations, $A_1$, $B_1$ and $C_1$ are constants determined by $\rho$ and all of the other model parameters. Therefore, unlike the unit-RRA case, here market incompleteness $\rho$ directly impinges on both the consumption and the portfolio choices in nonlinear fashions.

Applying Ito’s lemma to eq. (30) yields the expected instantaneous consumption growth $g_h(t)$ as a highly nonlinear function of $\alpha(t)$, $r$, and model parameters

$$g_h(t) \approx r + \omega_h(t) [\alpha(t) - r] - \delta^{EIS} e^{-(A_1 - B_1 \alpha(t) - 0.5C_1\alpha(t)^2)}$$

$$- \kappa \left[ \mu_\alpha - \alpha(t) \right] [B_1 + C_1 \alpha(t)]$$

$$- 0.5 \left[ C_1 - (B_1 + C_1 \alpha(t)) \right] \sigma^2_{\alpha} - \left[ B_1 + C_1 \alpha(t) \right] \omega_h(t) \rho \sigma \sigma_{\alpha}. \quad (32)$$
Since $\omega_h(t)$ is a nonlinear function of $\rho$ as shown in eq. (31), and $g_h(t)$ is increasing in $\omega_h(t)$, the degree of market incompleteness $\rho$ influences the expected consumption growth through four very different channels. First, as in the unit RRA case, there is the Ito's lemma term at the end of the right-hand side of eq. (32) where $\rho$ enters linearly. Second, there are the highly nonlinear effects of $\rho$ through $e^{-A_1-B_1\alpha(t)-0.5C_1\alpha(t)^2}$ in the third term on the right-hand side of eq. (32). Third, there are also the nonlinear effects of $\rho$ through $\omega_h(t)$. And lastly, $B_1$ and $C_1$ are both nonlinear functions of $\rho$.

4 Discussion and Conclusions

In this paper, I have demonstrated that market dynamic incompleteness, as indexed by the instantaneous correlation of the expected return and the instantaneous return to the risky asset, plays different roles in determining consumption and its expected growth. The degree of market incompleteness has no direct effect on consumption level when there is rational myopicness in either consumption or portfolio choice. For these two cases, market incompleteness only affects the level of consumption indirectly through wealth or the expected risky return. And the indirect effect is quantitatively small for a consumer with unit risk aversion. In contrast, when a consumer is rationally forward-looking in both consumption and portfolio decisions, market incompleteness directly impinges on consumption level in a complex nonlinear fashion.

On the other hand, regardless of how forward-looking a consumer is, market incompleteness does have a direct growth effect, in the sense that the expected consumption growth explicitly depends on the incompleteness parameter. Quantitatively speaking, the growth effect is large, and becomes more so when this parameter moves closer to one. In addition, the growth effect of this parameter becomes in-
creasingly more nonlinear, when moving from rationally myopic portfolio choice to rationally myopic consumption decision, to fully rational forward-looking in both decisions. This pattern of complexity suggests that a model of precautionary saving that directly confronts labor income risk may be easier to solve when it features rational myopic decision in either consumption or portfolio choice.

While the finance literature is more interested in the effect of intertemporal hedging on dynamic portfolio choice, the precautionary saving literature has been exploring how uninsured idiosyncratic labor income risk changes saving and consumption dynamics. For example, Campbell and Viceira (1999) and Chacko and Viceira (2005) measure the additional holdings of an asset owing to the correlation of its current return and a state variable realized in the future than to study how consumption dynamics is altered, even though they do solve the optimal consumption policy. Others typically assume away consumption by postulating that investors derive utility from terminal wealth.

In contrast, the precautionary saving literature has almost always abstracted away from the portfolio choice problem by assuming that there is only one, riskfree, asset in the investment opportunity set. In such setups, though Deaton (1991), Carroll and Kimball (1996), and Carroll (2010) have characterized the consumption function in the presence of labor income risk, its exact analytical form is still unknown. And all that is known about consumption dynamics seems to be that one result based on the second-order Taylor series expansion of the Euler equation. There is therefore the potential to extend the two seemingly separate literatures at the same time by asking how uninsured risk in a portfolio choice context changes consumption and its dynamics, so that the modeling of consumption and saving can benefit from the progress in modeling intertemporal hedging, while the latter, being an important topic in the optimal consumption and portfolio choice literature, is enriched by the new understandings gained on the consumption side. The present
paper fills some of this gap between the consumption and portfolio choice literature in finance and the precautionary saving literature in economics.
5 Appendix A Restrictions on EIS

In this appendix, I show that for what range of EIS the analytical solution to the unit RRA case presented in Section 2.2 is valid depends on the time preference rate $\delta$, the tolerance level of the approximation error, and the highest approximation error percentile that is used to judge approximation. Since the solution requires $(1 - EIS)\Phi(\alpha(t))$ to be close to zero for any finite values of $\alpha(t)$, any EIS values in the neighborhood of one is obviously sufficient for it to hold. In particular, when EIS converges to one, the $b$’s in eq. (18) all converge to finite constants. For EIS values in the neighborhood of one, the log consumption-wealth ratio as implied in eq. (23) is approximately $EIS \ln \delta + (1 - EIS) [b_0 + b_1 \alpha(t) + b_2 \alpha(t)^2]$, similar to Campbell and Viceira’s (1999) solution for the non-unit RRA case.\(^9\)

However, the restriction of EIS values being in the neighborhood of one is not always necessary, depending on what value is assumed for $\delta$ and on how much approximation error may be tolerated. To narrow down the range of EIS for which the solution to the unit RRA case is valid, I first simulate the expected risky return series using the discretized version of eq. (3)

$$t = \mu + e^{\kappa} (t-1 - \mu) + \sigma_{da} \sqrt{1 - e^{-2\kappa}} N_t,$$

where $N_t$ is a standard normal random variable. The values of $\mu$, $\kappa$, and $\sigma_{da}$ are drawn from Campbell et al. (2004), who estimate these and other parameters from the quarterly U.S. data between 1959 and 1995. See their Table 1 for details. The initial value $\alpha_0$ is drawn from the unconditional distribution of $\alpha(t)$, i.e. the normal distribution with mean $\mu$ and variance $\sigma^2_{\alpha}$.

Then I compute the $\Phi(\alpha_t)$ series corresponding to the simulated $\alpha_t$ series, using

\(^9\)Their solution is a quadratic function of the equity risk premium. Here since $b_0$ is a quadratic function of the riskfree rate, the log consumption-wealth is also a quadratic function of the equity risk premium.
eqs. (18)-(21) and the parameter values referenced above. Equipped with the $\Phi(\alpha_t)$ series, I compute the largest relative approximation error, i.e. the maximum of the percentage approximation errors

$$e^{(1-E\text{IS})\Phi(\alpha_t)}\left[1 + (1 - E\text{IS})\Phi(\alpha_t)\right]$$

for 40,000 draws of $\alpha(t)$ according to eq. (33) for every $E\text{IS}$ value from $[0.001, 1.15]$ at increments of 0.004. For $\delta = 0.0102$, the relative approximation errors at different percentiles for alternative $E\text{IS}$ values are given in the figure above.

What this figure shows is that if one adopts the 99th percentile of relative approximation error and 20% as the tolerance level for such error, then the range of $E\text{IS}$ that supports the linearization in Section 2.2 is $[0.76, 1.16]$. For the 10% tolerance level, the supporting $E\text{IS}$ range is $[0.82, 1.13]$ by the 99th percentile approximation error.

On the other hand, for $\delta = 0.005$, which is equivalent to the annual discount rate
of 2%, the relative approximation errors at different percentiles for alternative $EIS$ values are presented in the figure below.

Hence if the 20% tolerance level is chosen, by the 99th percentile of the relative approximation error the solution to the unit $RRA$ case in Section 2.2 is valid for the $EIS$ range of $[0.67, 1.07]$. On the other hand, if the threshold for the 99th percentile of the relative approximation error is lowered to 10%, then the relevant $EIS$ range changes to $[0.73, 1.05]$. 

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Figure 1: The Average Propensity to Consume (APC) plotted against $\alpha(t)$, for $EIS = 0.95$, $\delta = 0.0102$ and $\rho = -0.9626$.

Figure 2: The Average Propensity to Consume (APC) plotted against $\alpha(t)$, for $EIS = 1.05$, $\delta = 0.0102$ and $\rho = -0.9626$. 

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Figure 3: The median values of the Average Propensity to Consume (APC) and $\alpha(t)$ plotted against $\rho$ for alternative EIS values and $\delta = 0.005$.

Figure 4: The median values of the Average Propensity to Consume (APC) and $\alpha(t)$ plotted against $\rho$ for alternative EIS values and $\delta = 0.005$. 
Figure 5: The Expected Consumption Growth (E.C.G.) plotted against $\rho$, for three alternative EIS values and $\delta = 0.0102$.

Figure 6: The Expected Consumption Growth (E.C.G.) plotted against $\rho$, for three alternative EIS values and $\delta = 0.005$. 
References


