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A UNIFYING IMPOSSIBILITY THEOREM FOR COMPACT METRIC SOCIAL ALTERNATIVES SPACE

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Abstract. In a companion paper [Man, P., and Takayama S., 2013. "A Unifying Impossibility Theorem". *Economic Theory*, forthcoming], we show that many classical impossibility theorems follow from three simple and intuitive axioms on the social choice correspondence when the set of social alternatives is finite. This supplementary note extends this result to the case where the set of social alternatives is a compact metric space.

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1. Introduction

In Man and Takayama (2013) (henceforth MT) we show that many classical impossibility theorems follow from three simple and intuitive axioms on the social choice correspondence when the set of social alternatives is finite. This note extends the main theorem (Theorem 1) in MT to the case where the set of social alternatives is a compact metric space. We also qualify how versions of Arrow's Impossibility Theorem and the Muller-Satterthwaite Theorem (Muller and Satterthwaite, 1977) can be obtained as corollaries of the extended main theorem. A generalized statement of the Muller-Satterthwaite Theorem for social choice correspondences with weak preferences on a compact metric social alternatives domain under a modified definition of Monotonicity is given. To the best of our knowledge, this is the first paper to document this version of the Muller-Satterthwaite Theorem.

This note is mainly technical. Readers interested in the motivations and discussions of our axioms and main theorem should consult MT.

2. Definitions

Let X be the set of all possible social alternatives, which is a compact metric space with at least three elements. The distance between two points $x, y \in X$ is denoted by $|x - y|$. The

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distance between a point $x \in X$ and a subset $Y \subseteq X$ is defined as $|x - Y| = \inf_{y \in Y} |x - y|$. Given $Y \subseteq X$ and $\varepsilon > 0$, let $B_\varepsilon(Y) = \{x \in X : |x - Y| < \varepsilon\}$ be the open ε-ball around Y.

An *agenda* is a non-empty closed subset $Y \subseteq X$. An *agenda domain*, typically denoted as $\mathscr{Y}$, is a collection of agendas that includes X and all agendas with up to two social alternatives[1]. Several agenda domains of particular interests are the collection of all non-empty closed subsets of X (the *universal agenda domain*), the collection of X and all subsets with up to three alternatives (the *pairs and triplets agenda domain*), and the collection of X and all subsets with up to two alternatives (the *minimal agenda domain*).

We abuse notation to use N as both the set of all individuals and its cardinality, which is finite.

2.1. **Preference Domain.** An weak preference of individual $i \in N$ is a complete, transitive (but not necessarily continuous) binary relation on X, typically denoted by $\succsim_i$. The symbols $\succ_i$ and $\sim_i$ will have their usual derived meanings. Two preferences *agree on* Y if they induce the same preference ordering on the subset of alternatives $Y \subseteq X$. Given a closed subset of alternatives $Y \subseteq X$ and an individual preference $\succsim_i$, let

$$T(Y, \succsim_i) = \{y \in Y : y \succsim_i y' \text{ for all } y' \in Y\}$$

be the set of favorite alternatives ("top set") within Y according to $\succsim_i$. Of course, without restrictions on $\succsim_i$, $T(Y, \succsim_i)$ may be empty. A set of individual preference over X, R_i, is *Choice Normal* if, for each $\succsim_i \in R_i$ and each non-empty closed subset Z of each agenda $Y \in \mathscr{Y}$, $T(Z, \succsim_i)$ is non-empty and closed (in X).[2]

Definition 2.1 (Lifting of Individual Preferences)**.** Given an individual preference $\succsim_i$, a subset $Y \subseteq X$ and an $\varepsilon > 0$, a preference $\succsim_i'$ is a (Y, ε)-*lifting from* $\succsim_i$ if:

(i) $\succsim_i$ and $\succsim_i'$ agree on Y;
(ii) $\succsim$ and $\succsim'$ agree on $X \setminus B_\varepsilon(Y)$[3,4];
(iii) $y \succ_i' x$ whenever $x \notin B_\varepsilon(Y)$ and $y \in Y$; and
(iv) for each $x \notin Y$, there is a $y \in Y$ such that $y \succ_i' x$.

[1]We need all pairwise subsets in order to obtain an implicit "revealed social preference". For the existence of the first dictator, three-element agendas are not needed. To establish the fixed tie-breaking rule for serial dictatorship, however, triplets are required to ensure the existence of a rationalizing preference for the social choice when all individuals are indifferent between all available alternatives (c.f., Arrow, 1959).

[2]We require that the top set be non-empty and closed for each closed subset of each agenda to ensure that an individual can make an optimal choice after previous dictators have restricted the options over which she has influence.

[3]A (Y, ε)-lifting is different from taking Y to the top in MT. When $\succsim_i^Y$ takes Y to the top from $\succsim_i$, $\succsim_i$ and $\succsim_i^Y$ are not required to agree on $X \setminus B_\varepsilon(Y)$.

[4]Conditions (i) and (ii) are jointly different from saying that $\succsim_i$ and $\succsim_i'$ agree on $Y \cup (X \setminus B_\varepsilon(Y))$! The latter requires that the ordering between any $y \in Y$ and any $x \in X \setminus B_\varepsilon(Y)$ to remain unchanged, which violates the idea of a lifting.

We say $\succsim_i'$ is a *strict* (Y,ε)*-lifting from* $\succsim$ if (iv) is strengthened to

(iv') For each $x \notin Y$, $y \succ_i' x$ whenever $y \in \arg\min_{y' \in Y} |x - y'|$.

We will only consider domains of preference profiles which are product sets of individual preference domains[5]. A typical preference profile is denoted as $\succsim = (\succsim_1, \ldots, \succsim_N)$. Two preference profiles $\succsim$ and $\succsim'$ *agree on* Y if $\succsim_i$ and $\succsim_i'$ agree on Y for each i. The lifting of preference profiles is less straight-forward:

Definition 2.2 (Lifting of Preference Profiles)**.** Let $\succsim$ be a preference profile. Given a subset Y, let $S_1, \ldots, S_n$ be a partition of N such that individuals in the same partition have the same preference over Y. For an $\varepsilon > 0$, a preference profile $\succsim'$ is a (Y,ε)*-lifting from preference profile* $\succsim$ if

(i) For each i, $\succsim_i'$ is a (Y,ε)-lifting from $\succsim_i$; and

(ii) For each partition of individuals S_k and each $x \in B_\varepsilon(Y) \setminus Y$, there is a $y \in Y$ such that $y \succ_s' x$ for all $s \in S_k$.

We say $\succsim'$ is a a *strong* (Y,ε)*-lifting from* $\succsim$ if (ii) is strengthened to

(ii') For each $x \in B_\varepsilon(Y) \setminus Y$, there is a $y \in Y$ such that $y \succ_i' x$ for all $i \in N$.

In the rest of this paper, when no confusion over the ε involved in a lifting may arise, we will simply speak of a Y-lifting.

Definition 2.3 (Preference Domain Restrictions)**.** Let $\mathscr{Y}$ be an agenda domain. A preference domain $R = \prod_{i \in N} R_i$ is

Closed under (Strong) Lifting relative to $\mathscr{Y}$ if for each subset $Y \in \mathscr{Y}$, there exists an $\varepsilon_Y > 0$ such that R contains a (strong) (Y,ε)-lifting from $\succsim$ whenever $\succsim \in R$ and $\varepsilon \in (0, \varepsilon_Y)$;

(Strongly) Admissible relative to $\mathscr{Y}$ if it is Closed under (Strong) Lifting relative to $\mathscr{Y}$ and each R_i is Choice Normal;

(Strongly) Comprehensive relative to $\mathscr{Y}$ if it is (Strongly) Admissible relative to $\mathscr{Y}$ and contains the preference profile in which all individuals are indifferent between all alternatives.

If X is finite, the universal preference domain (the collection of all possible weak preference profiles) is comprehensive relative to the universal agenda domain. The set of all strict preference profiles over X is admissible relative to the universal agenda domain. In this sense, admissibility and comprehensiveness can be viewed as generalizations of the universal

[5]The product set assumption is used in the proof when we construct preference profiles by performing different liftings on different individuals.

strict preference domain and the universal preference domain, respectively, to a compact metric space of social alternatives.

The next two examples depict two important preference domains. The proofs of the claims can be found in the appendix.

Example 2.4. The product set of all complete, transitive and continuous[6] individual preferences on X is comprehensive relative to the universal agenda domain.

It is unclear to us whether the product set of all complete, transitive and continuous preferences is strongly comprehensive relative to the universal agenda domain[7]. However, due to the remark after Facts 2.6 and 2.7, this canonical preference domain is strongly comprehensive relative to the pairs and triplets agenda domain and the minimal preference domain.

The next preference domain, on the other hand, is strongly comprehensive even relative to the universal agenda domain.

Example 2.5. The product set of all individual preferences over X with an upper semi-continuous[8] piecewise linear utility function representation is strongly comprehensive relative to the universal agenda domain.

If a preference domain R is (strongly) admissible relative to an agenda domain $\mathscr{Y}$, it is also (strongly) admissible relative to any subset of $\mathscr{Y}$. Hence the above two preference domains are in fact comprehensive relative to *all* agenda domains. They are strongly comprehensive relative to all agenda domains containing, in addition to X, only finite agendas.

Admissibility and comprehensiveness do rule out some preference domains. If $X = [0, 1]$, the product set of all weak but not necessarily continuous preferences on X is not choice normal, hence inadmissible. Meanwhile, the product set of all continuous single-peaked preferences on the same X is not closed under lifting relative to any agenda domain (since there is no lifting of any pairwise subset from the indifferent preference) and is hence inadmissible.

The virtue of admissible domains lies largely on the following two facts. Their proofs are relegate to the Appendix.

Fact 2.6. Let R be a preference domain closed under lifting relative to agenda domain $\mathscr{Y}$ and let Y be a finite subset of X (which need not belong to $\mathscr{Y}$). For any preference profile

[6]A preference $\succsim_i$ on X is continuous if for all $x \in X$, the (weak) upper contour set $\overline{U}(x) = \{x' \in X : x' \succsim_i x\}$ and (weak) lower contour set $\overline{L}(x) = \{x' \in X : x \succsim_i x'\}$ are closed in X.

[7]For instance, given an arbitrary continuous preference profile on $X = [0, 1]$, is it possible to lift the Cantor set in a continuous manner while maintaining the requirement that each alternative outside the Cantor set is Pareto dominated?

[8]A function $f : X \to \mathbb{R}$ is upper semi-continuous if for all real numbers r, the set $\{x \in X : f(x) \geq r\}$ is closed in X.

$\succsim \in R$ where each indifference class in Y for each individual is an element of $\mathscr{Y}$, there exists a strong (Y, ε)-lifting from $\succsim$ in R for all sufficiently small ε.

Fact 2.7. Let R be a preference domain closed under lifting relative to agenda domain $\mathscr{Y}$ and let Y be a finite subset of X (which need not belong to $\mathscr{Y}$). For any strict preference profile $\succ$ on Y, there is a preference profile $\succsim' \in R$ such that $\succ$ and $\succsim'$ agree on Y; and $y \succ'_i x$ for all i whenever $x \notin Y$ and $y \in \arg\min_{y' \in Y} |x - y'|$.

Say an agenda domain $\mathscr{Y}$ is *closed under taking finite subsets* if, for each $Y \in \mathscr{Y}$, $Y \neq X$, each subset of Y is in $\mathscr{Y}$. Consider an agenda domain $\mathscr{Y}$ that contains (except X) only finite subsets and is closed under taking finite subsets. (Examples include the collection of X and all finite agendas, the pairs and triplets agenda domain, and the minimal agenda domain.) Fact 2.6 implies that, relative to $\mathscr{Y}$, closure under lifting and closure under strong lifting are equivalent. This feature is important for implementation (see Section 4.2).

2.2. **Social Choice Correspondence.** Given an agenda domain $\mathscr{Y}$ and a preference domain R, a *social choice correspondence* is a mapping $f : \mathscr{Y} \times R \to 2^X \setminus \emptyset$ such that $f(Y, \succsim) \subseteq Y$ for all $Y \in \mathscr{Y}$ and all $\succsim \in R$.

We impose the following conditions on a social choice correspondence:

Definitions 2.8. A social choice correspondence $f : \mathscr{Y} \times R \to 2^X \setminus \emptyset$ is

Strongly Unanimous (SU) if $f(X, \succsim) = \{x\}$ whenever x is uniquely weakly Pareto dominant at $\succsim$ (i.e., for all $y \in X \setminus \{x\}$, $x \succsim_i y$ for all $i \in N$ with at least one individual having a strict preference);

Independent of Infeasible Alternatives (IIF) if $f(Y, \succsim) = f\left(Y, \succsim'\right)$ whenever $\succsim$ and $\succsim'$ agree on $Y \in \mathscr{Y}$;

Independent of Losing Alternatives (ILA) if $f(Y, \succsim) = f(X, \succsim) \cap Y$ whenever $Y \in \mathscr{Y}$ and the intersection is non-empty; and

Upper Hemi-continuous at indifference for pairs if $\{x_n\} \subset X$ converges to x and $x_n \in f(\{x_n, y\}, \succsim^0)$ for all n imply $x \in f(\{x, y\}, \succsim^0)$, where $\succsim^0$ is the preference profile in which all individuals are indifferent between all alternatives.

The first three conditions are the three main axioms in MT. The last condition deals with technical issues when X is infinite. Since upper hemi-continuity of f at other preference profiles and for other subsets will not be required in this paper, we will omit the qualification "at indifference for pairs" from now on.

Let $\pi = (\pi_1, \ldots, \pi_N)$ be a permutation of the set of all individuals. Write also $\pi^k = (\pi_1, \ldots, \pi_k)$ for any $k \leq N$. Given a preference profile $\succsim$ and π^k, write $\succsim^{\pi^k} = \left(\succsim_{\pi_1}, \ldots, \succsim_{\pi_k}\right)$ as the preferences of individuals in π^k. For all subsets Y, all preference profiles $\succsim$ and all permutations of individuals π, define the kth iteration of the top set operator such that

$$T^0(Y, \succsim^{\pi^0}, \pi^0) = Y$$
$$T^k(Y, \succsim^{\pi^k}, \pi^k) = T(T^{k-1}(Y, \succsim^{\pi^{k-1}}, \pi^{k-1}), \succsim_{\pi_k}).$$

Definitions 2.9. A social choice correspondence $f : \mathscr{Y} \times R \to 2^X \setminus \emptyset$ is

Dictatorial if there exists an individual $i \in N$ such that $f(Y, \succsim) \subseteq T(Y, \succsim_i)$ for all $Y \in \mathscr{Y}$ and all $\succsim \in R$;

Serially Dictatorial if there exists a permutation of individuals π and a complete, transitive and continuous preference ρ on X such that

$$f(Y, \succsim) = T(T^N(Y, \succsim, \pi), \rho) \quad \text{for all } Y \in \mathscr{Y}, \text{ all } \succsim \in R.$$

Again these definitions are essentially the same as their counterparts in MT, with the extra continuity requirement on the tie-breaking preference ρ. However, ρ is not required to belong to any one of the R_i.

3. Main Theorem

Theorem 3.1. *Let $\mathscr{Y}$ be an agenda domain that contains the pairs and triplets domain and let R be a comprehensive preference domain relative to $\mathscr{Y}$. Any upper hemi-continuous social choice correspondence defined on $\mathscr{Y} \times R$ that satisfies Strong Unanimity, Independence of Infeasible Alternatives and Independence of Losing Alternatives is Serially Dictatorial.*

Note that only comprehensiveness, rather strong comprehensiveness, is needed for this theorem.

The proof follows the same approach as that in MT. The key insight is that the crux of the proof in MT only requires strong lifting of *finite* subsets, the existence of which is guaranteed by Fact 2.6.

We outline the proof of Theorem 3.1 in the rest of this section. To illustrate the use of Facts 2.6 and 2.7, we give the full proof of the parallel of Lemmas 2 and 3 in MT. However, we will only demonstrate how one can obtain the preference profiles for making the same argument as in MT regarding the proof of the theorem. Issues specific to an infinite social alternatives space are highlighted.

3.1. **Preliminary Results.**

The next two lemmas are analogous to Lemmas 2 and 3 in MT.

Lemma 3.2. *Let f be a social choice correspondence defined on a preference domain that is admissible relative to its agenda domain. If f satisfies SU, IIF and ILA, then $x \notin f(X, \succsim)$ whenever x is weakly Pareto dominated at $\succsim$.*

Table 1. Construction of Preference Profiles for the Proof of Lemma 3.3

Individual	Preference under $\succsim'$	Preference under $\succsim''$
Type x	A strict $(\{x\},\varepsilon)$-lifting on a strict $(\{y,z\},\varepsilon)$-lifting from $\succsim_i$	A strict $(\{x,y\},\varepsilon)$-lifting on $\succsim_i$
Type y	A strict $(\{x,y\},\varepsilon)$-lifting on $\succsim_i$	A strict $(\{y\},\varepsilon)$-lifting on a strict $(\{x,z\},\varepsilon)$-lifting from $\succsim_i$
Type xy	A strict $(\{x,y\},\varepsilon)$-lifting on $\succsim_i$	A strict $(\{x,y\},\varepsilon)$-lifting on $\succsim_i$
Not in S	Same as $\succsim_i$	Same as $\succsim_i$

Proof. Suppose x is weakly dominated by y at $\succsim$. By Fact 2.6 we can find a strong $(\{x,y\},\varepsilon)$-lifting $\succsim' \in R$ from $\succsim$ for some sufficiently small ε. For each $z \neq x, y$, there exists a $y^* \in \{x,y\}$ such that $y^* \succ'_i z$ for all $i \in N$. Thus y strictly dominates all $z \neq x, y$ at $\succsim'$ since $y \succsim'_i y^* \succ'_i z$ for all $i \in N$. As y also weakly dominates x, it is weakly dominant at $\succsim'$. SU requires $f(X,\succsim') = \{y\}$. By IIF and ILA, $f(\{x,y\},\succsim) = f(\{x,y\},\succsim') = \{y\}$. Thus $x \notin f(X,\succsim)$ or else ILA would be violated. □

Lemma 3.3. *Let f be an admissible, SU, IIF and ILA social choice correspondence and let $\succsim$ be a preference profile. Let $S \subseteq N$ be the set of individuals who are not indifferent between all alternatives at $\succsim$. Then*

$$f(X,\succsim) \subseteq \bigcup_{i\in S} T(X,\succsim_i)$$

whenever the union contains fewer than two elements.

Proof. If the union contains only one element the statement follows from SU. So let $\bigcup_{i\in S} T(X,\succsim_i) = \{x,y\}$ and suppose by contradiction that $z \in f(X,\succsim_i)$ for some $z \neq x, y$. There are three types of individuals in S: 1. Those whose unique favorite is x (Type x); 2. Those whose unique favorite is y (Type y); and 3. Those whose favorites are exactly x and y (Type xy).

Pick an ε such that the ε-balls around x, y and z are pairwise disjoint. Construct preference profiles $\succsim'$ and $\succsim''$ as in Table 1. Using the same argument as in the proof of Fact 2.6, $\succsim'$ and $\succsim''$ belongs to the preference domain. Observe that

(1) $\succsim$, $\succsim'$ and $\succsim''$ agree on $\{x,y\}$;
(2) $\succsim$ and $\succsim'$ agree on $\{y,z\}$; and
(3) $\succsim$ and $\succsim''$ agree on $\{x,z\}$.

Since $z \in f(X,\succsim)$, ILA requires $z \in f(\{y,z\},\succsim)$. By observation (2) and IIF, $z \in f(\{y,z\},\succsim')$ as well. We claim that all alternatives other than x and y are weakly Pareto dominated at $\succsim'$. To see this, let $x' \neq x, y$ and let y^* be the closest element to x' among x,

Type x	Type y	Type z
x	y	z
y	z	x
z	x	y

FIGURE 1. A Condorcet Cycle

y and z. If $y^* = x$ or y, then strict lifting implies $y^* \succ_i' x'$ for all $i \in S$. If instead $y^* = z$, $x \succ_i' z = y^* \succsim_i' x'$ for all Type x individuals; while $x \succ_i' x'$ for all other individuals in S since x' is outside $B_\varepsilon(\{x, y\})$. Thus $f(X, \succsim') \subseteq \{x, y\}$ by Lemma 3.2. This means $y \notin f(X, \succsim')$, otherwise $z \in f(\{y, z\}, \succsim') \neq f(X, \succsim') \cap \{y, z\}$, violating ILA. Therefore $f(X, \succsim') = \{x\}$. By a similar argument using the subset $\{x, z\}$ and observation (3), $f(X, \succsim'') = \{y\}$.

Now ILA requires $f(\{x, y\}, \succsim') = \{x\}$ and $f(\{x, y\}, \succsim'') = \{y\}$. This contradicts IIF in light of observation (1). □

3.2. **Proof of the Main Theorem.** Given a SU, IIF and ILA social choice correspondence f, we construct the sequence of serial dictators π such that for all $k \geq 0$,

$$f(Y, \succsim) \subseteq T^k(Y, \succsim^{\pi^k}, \pi^k) \quad \text{for all } Y \in \mathscr{Y}, \text{ all } \succsim \in R. \tag{1}$$

The case for $k = 0$ follows by the feasibility of the social choice correspondence. Now suppose π^{k-1} is defined and Equation (1) holds for $k - 1$. We construct π_k that satisfies Equation (1) for k. The proof is essentially the same as in MT, so we will only sketch its outline here.

Step 1. If $N < 3$ or $k > N - 2$ proceed directly to Step 4. Otherwise, pick three distinct alternatives $x, y, z \in X$ and construct the Condorcet cycle in Figure 1. By Fact 2.7, there is a preference profile $\succsim^* \in R$ such that:

(1) All previous dictators, $\pi_1, \ldots, \pi_{k-1}$, are indifferent between all alternatives;
(2) Each remaining individual's preference agree with one of the Types x, y and z preference in the Condorcet cycle on $\{x, y, z\}$ (with at least one individual agreeing with each type), and $y^* \succ_i^* x'$ whenever $x' \neq x, y, z$ and $y^* \in \arg\min_{y' \in \{x,y,z\}} |x' - y'|$.

Under $\succsim^*$, all alternatives other than x, y and z are weakly Pareto dominated by the closest element in $\{x, y, z\}$. Lemma 3.2 requires $f(X, \succsim^*) \subseteq \{x, y, z\}$. Without loss suppose $x \in f(X, \succsim^*)$. Let S be the subset of individuals with the Type x preference at $\succsim^*$.

Step 2. As in MT, Step 2 is divided into three smaller steps. Using Fact 2.6, Steps 2.1 and 2.2 can be completed by "pretending" that the alternative space is finite.

Step 2.1. Choose an $\varepsilon > 0$ sufficiently small that the ε-balls around x, y and z are pairwise disjoint. Construct a preference profile $\succsim^1$ by performing a $(\{z\}, \varepsilon)$-lifting on $\succsim_i^*$ for all individuals $i \notin S \cup \{\pi_1, \ldots, \pi_{k-1}\}$ and keeping all other individuals' preferences the same as in $\succsim^*$. Using this $\succsim^1$, proceed with the same argument as in MT to conclude that

$f(\{x,y\},\succsim) = \{x\}$ whenever all previous dictators are indifferent between x and y and all individuals in S strictly prefer x to y.

The preference profile $\succsim^2$ is constructed by performing a $(\{y\},\varepsilon)$-lifting on $\succsim_i^1$ for all individuals $i \in S$ while keeping all other individuals' preferences unchanged. Proceed as in MT to conclude that $f(\{x,y\},\succsim) = \{y\}$ whenever all previous dictators are indifferent between x and y and all individuals in S strictly prefer y to x.

Step 2.2. Fix x, y as in the Condorcet cycle used above. Take any pairwise subset $\{w,z\} \subset X$ (the z need not be the same z as in the Condorcet cycle) that does not contain y. Pick $\varepsilon > 0$ such that the ε-balls around each of x, y, z and w are pairwise disjoint. Let $\succ_S^3$ be an individual strict preference over $\{x,y,z,w\}$ such that $w \succ_S^3 x \succ_S^3 y$ and $w \succ_S^3 z$. Let $\succ_{-S}^3$ be another individual strict preference over $\{x,y,z,w\}$ such that $y \succ_{-S}^3 x'$ for every $x' \in \{x,z,w\}$. By Fact 2.7, there exist a preference profile $\succsim^3$ such that

(1) All previous dictators, $\pi_1,\ldots,\pi_{k-1}$, are indifferent between all alternatives;
(2) For each individual $i \in S$, $\succsim_i^3$ agrees with $\succsim_S^3$ on $\{x,y,z,w\}$, and each $x' \notin \{x,y,z,w\}$ is strictly worse than the closest element in the four-element set; and
(3) For each remaining individual, $\succsim_i^3$ agrees with $\succsim_{-S}^3$ on $\{x,y,z,w\}$, and each $x' \notin \{x,y,z,w\}$ is strictly worse than the closest element in the four-element set.

Proceed as in MT using this $\succsim^3$.

Step 2.3. Given $Y \in \mathscr{Y}$, define as in the main text $Y_{k-1} = T^{k-1}(Y, \succsim^{\pi^{k-1}}, \pi^{k-1})$. Since R is choice normal, both Y_{k-1} and $T(Y_{k-1},\succsim_i)$ are non-empty and closed. Take a preference profile $\succsim$ at which all individuals in S have the same preference over Y. Fix an $\varepsilon > 0$. Let $\succsim^Y$ be a (Y,ε)-lifting from $\succsim$.

We first claim that $f(X,\succsim^Y) \subseteq Y_{k-1}$. If $k = 1$, consider an $x \notin Y = Y_0$. Since individuals in S have the same preference over Y at $\succsim$, there is a $y \in Y$ that all individuals in S strictly prefer to x. Yet group S has dictatorial power over any pairwise subset. Thus $x \notin f(\{x,y\},\succsim^Y)$. By ILA, $x \notin f(X,\succsim^Y)$. If instead $k > 1$, notice that for any individual preference $\succsim_i$, $T(Y,\succsim_i) = T(Y,\succsim_i^Y)$. The induction hypothesis (Equation (1)) then implies $f(Y,\succsim^Y) \subseteq Y_{k-1}$. ILA requires $f(X,\succsim^Y) \subseteq Y_{k-1}$.

Next we show that $f(X,\succsim^Y) \subseteq T(Y_{k-1},\succsim_S^Y)$. First note that the previous dictators $\pi_1,\ldots,\pi_{k-1}$ are indifferent between all alternatives in Y_{k-1}. Now if y is a favorite of group S in Y_{k-1} and y' is not, group S's k-th order dictatorship over pairwise subsets dictates $y' \notin f(\{y,y'\},\succsim^Y)$. ILA implies $y' \notin f(X,\succsim^Y)$. Thus $f(X,\succsim^Y) \subseteq T(Y_{k-1},\succsim_S^Y)$.

Since $f(X,\succsim^Y) \subseteq T(Y_{k-1},\succsim_S^Y) \subseteq Y$, ILA implies $f(Y,\succsim^Y) \subseteq T(Y_{k-1},\succsim_S^Y)$. As $f(Y,\succsim) = f(Y,\succsim^Y)$ (IIF) and $T(Y_{k-1},\succsim_S^Y) = T(Y_{k-1},\succsim_S)$, $f(Y,\succsim) \subseteq T(Y_{k-1},\succsim_S)$ for all $Y \in \mathscr{Y}$.

Remark on Step 2. The remark on Step 2 in MT — that the choice at the Condorcet profile is a singleton — is unaffected by having an infinite X.

Step 3. If S is a singleton, letting $\pi_k = S$ completes the induction step. Otherwise, Fact 2.7 guarantees the existence of a preference profile $\succsim^{**}$ such that

(1) All previous dictators, $\pi_1, \ldots, \pi_{k-1}$, are indifferent between all alternatives;

(2) For each individual $i \in S$, $\succsim_i^{**}$ agrees with either the Type x or Type y preference (with at least one individual assigned to each type) in the Condorcet cycle (Figure 1) on $\{x, y, z\}$, and $y^* \succ_i^* x'$ whenever $x' \neq x, y, z$ and $y^* \in \arg\min_{y' \in \{x,y,z\}} |x' - y'|$; and

(3) For each remaining individual, $\succsim_i^{**}$ agrees with the Type z Condorcet preference on $\{x, y, z\}$, and $y^* \succ_i^{**} x'$ whenever $x' \neq x, y, z$ and $y^* \in \arg\min_{y' \in \{x,y,z\}} |x' - y'|$.

The dictatorship of group S implies that $z \notin f(X, \succsim^{**})$. Proceed as in MT to shrink S down to a singleton.

Step 4. For the case when there are only two individuals left, construct $\succsim^*$ as in Step 1 above, without using the Type z preference. Lemma 3.3 requires $f(X, \succsim^*) \subseteq \{x, y\}$. Proceed as above.

When there is only one individual left, consider a preference profile $\succsim \in R$ and an agenda $Y \in \mathscr{Y}$. Let $\succsim^Y$ be a (Y, ε)-lifting from $\succsim$. By the argument in Step 2.3, $f(X, \succsim^Y) \subseteq Y_{N-1}$. If y is a favorite of π_N in Y_{N-1} and y' is not, y' is weakly Pareto dominated by y. Lemma 3.2 requires $f(X, \succsim^Y) \subseteq T(Y_{N-1}, \succsim^Y)$. Since $f(Y, \succsim) = f(Y, \succsim^Y)$ (IIF) and $T(Y_{N-1}, \succsim^Y_{\pi_N}) = T(Y_{N-1}, \succsim_{\pi_N})$, $f(Y, \succsim) \subseteq T(Y_{N-1}, \succsim_{\pi_N})$.

Notice that we need neither the indifference profile nor upper hemi-continuity to establish the existence of the first dictator. Also, the proof above can be completed without any three-element agenda (unless the agenda itself is a $Y \in \mathscr{Y}$). Hence we have the following result:

Theorem 3.4. *Let $\mathscr{Y}$ be an agenda domain and let R be an admissible preference domain relative to $\mathscr{Y}$. Any social choice correspondence defined on $\mathscr{Y} \times R$ that satisfies Strong Unanimity, Independence of Infeasible Alternatives and Independence of Losing Alternatives is Dictatorial.*

When the preference domain is comprehensive and the agenda domain contains all triplets, we can continue to construct the tie-breaking preference ρ by defining the binary relation on X such that for all $x, y \in X$,

$$x \succsim_\rho y \quad \text{if and only if} \quad f(\{x, y\}, \succsim^0)$$

where $\succsim^0$ denotes the preference profile at which all individuals are indifferent between all alternatives in X.

The next Lemma is analogous to Lemma 4 in MT, with the additional condition that ρ is continuous.

Lemma 3.5. *The binary relation ρ is complete, transitive and continuous.*

Proof. Completeness: Same as in MT.

Transitivity: Take $x, y, z \in X$ and suppose $x \succsim_\rho y$ and $y \succsim_\rho z$. Choose an $\varepsilon > 0$ such that the ε-balls around x, y and z are pairwise disjoint. Construct $\succsim'$ by taking a strong $(\{x, y, z\}, \varepsilon)$-lifting from $\succsim^0$. This construction is feasible by Fact 2.7. All alternatives other than x, y and z are Pareto dominated by each of them at $\succsim'$. Proceed with the same argument as in MT to establish the transitivity of ρ.

Continuity: Let $\{x_n\} \subset X$ be a sequence of alternatives converging to x and suppose $x_n \succsim_\rho y$ for each n. This means $x_n \in f(\{x_n, y\}, \succsim^0)$ for each n. Upper hemi-continuity of f implies $x \in f(\{x, y\}, \succsim^0)$. In other words, $x \succsim_\rho y$. □

The rest of the proof of Theorem 3.1 is the same as in MT, using a (Y, ε)-lifting whenever Y is being taken to the top in the original proof.

4. Extensions

4.1. **Arrow's Impossibility Theorem.** Let $R = \prod_{i \in N} R_i$ be a preference domain. Let R^* be the set of complete, transitive and continuous preferences over X. A *social welfare function* is a mapping $F : R \to R^*$. Given a preference profile $\succsim \in R$, a permutation of individuals $\pi = (\pi_1, \ldots, \pi_N)$ and a tie-breaking preference $\rho \in R^*$, write $\succsim_{N+1} = \rho$ and define the lexicographic preference $L(\succsim, \pi, \rho)$ over X by: $x \; L(\succsim, \pi, \rho) \; y$ if and only if whenever $y \succ_{\pi_k} x$, there exists a $l < k$ such that $x \succ_{\pi_l} y$.

Definitions 4.1. A social welfare function $F : R \to R^*$ is

- *Strongly Pareto* if x is strictly preferred to y according to $F(\succsim)$ whenever x weakly Pareto dominates y under $\succsim$;
- *Independent of Irrelevant Alternatives* if $F(\succsim)$ and $F(\succsim')$ agree on $\{x, y\}$ whenever $\succsim$ and $\succsim'$ agree on the same set;
- *Dictatorial* if there exists an individual $i \in N$ such that x is strictly preferred to y according to $F(\succsim)$ whenever $x \succ_i y$;
- *Serially Dictatorial* if there exists a permutation of individuals π and a tie-breaking preference $\rho \in R^*$ such that $F(\succsim) = L(\succsim, \pi, \rho)$ for all $\succsim \in R$.

Given an agenda domain $\mathscr{Y}$ and social welfare function $F : R \to R^*$, define

$$f(Y, \succsim) = T(Y, F(\succsim)) \quad \text{for all } Y \in \mathscr{Y}, \text{ all } \succsim \in R. \tag{2}$$

Notice that $f(Y, \succsim)$ is always non-empty and closed since all agenda are non-empty and compact while $F(\succsim)$ is continuous.

Proposition 4.2. *If a social welfare function $F : R \to R^*$ is Strongly Pareto and Independent of Irrelevant Alternatives, then the social choice correspondence f defined in Equation* (2) *is SU, IIF, ILA and upper hemi-continuous.*

Proof. The proofs for SU, IIF and ILA are the same as in MT. Upper Hemi-continuity is satisfied trivially if $\succsim^0$, the indifference profile, does not belong to R. Otherwise, let $\{x_n\} \to x$ and suppose $x_n \in f(\{x_n, y\}, \succsim^0)$ for all n. Equation (2) implies $x_n F(\succsim^0) y$ for all n. Continuity of $F(\succsim^0)$ requires $xF(\succsim^0)y$, which in turns means $x \in f(\{x, y\}, \succsim^0)$. $\square$

If R is comprehensive relative to the pairs and triplets agenda domain, then f defined in Equation (2) (using the pairs and triplets agenda domain as $\mathscr{Y}$) is serially dictatorial by Theorem 3.1. Thus there exists a permutation of individuals π and a tie-breaking preference ρ such that $f(\{x, y\}, \succsim) = T(T^N(\{x, y\}, \succsim, \pi), \rho)$ for all pairs of $x, y \in X$. By Equation (2), $x \in f(\{x, y\}, \succsim)$ is equivalent to $x\, F(\succsim)\, y$. Meanwhile, $x \in T(T^N(\{x, y\}, \succsim, \pi), \rho)$ is equivalent to $x\, L(\succsim, \pi, \rho)\, y$. This gives us a serially dictatorial version of Arrow's Impossibility Theorem (Arrow, 1963):

Corollary 4.3. *Let R be a comprehensive preference domain relative to the pairs and triplets agenda domain. Any social welfare function $F : R \to R^*$ that is Strongly Pareto and Independent of Irrelevant Alternatives is Serially Dictatorial.*

If R is admissible relative to the minimal agenda domain, then f defined in Equation (2) (using the minimal agenda domain as $\mathscr{Y}$) is dictatorial by Theorem 3.4. Thus there exists an individual i such that $f(Y, \succsim) \subseteq T(Y, \succsim_i)$ for all $\succsim \in R$ and all $Y \in \mathscr{Y}$. If $x \succ_i y$, then $f(\{x, y\}, \succsim) = \{x\}$, which is equivalent to x being strictly preferred to y according to $F(\succsim)$.

Corollary 4.4. *Let R be an admissible preference domain relative to the minimal agenda domain. Any social welfare function $F : R \to R^*$ that is Strongly Pareto and Independent of Irrelevant Alternatives is Dictatorial.*[9]

4.2. **Implementation.** Let R be a preference domain. A social choice correspondence defined only on the set of all social alternatives is called an *overall social choice correspondence*, which is a mapping $f^* : R \to 2^X \setminus \emptyset$. Given a preference profile $\succsim$ and a subset $Y \subseteq X$, a preference profile $\succsim'$ is a Y*-advancement from* $\succsim$ if $y \succsim_i x$ implies $y \succsim'_i x$ (with $y \succ_i x$ implying $y \succ'_i x$) whenever $x \in X$ is not Pareto dominated at $\succsim'$ and $y \in Y$.

Definitions 4.5. An overall social choice correspondence $f^* : R \to 2^X \setminus \emptyset$ is

Strongly Unanimous if $f^*(\succsim) = \{x\}$ whenever x is uniquely weakly Pareto dominant at $\succsim$;

[9]Actually, the range of F can be expanded to the set of all complete, transitive preferences over X that admits a maximal element in X, even if such preferences may not be continuous.

Pareto Monotonic if $Y \subseteq f^*(\succsim') \subseteq f^*(\succsim)$ whenever $Y \subseteq f^*(\succsim)$ and $\succsim'$ is a Y-advancement from $\succsim$;[10]

Dictatorial if there exists an individual $i \in N$ such that $f^*(\succsim) \subseteq T(X, \succsim_i)$ for all $\succsim \in R$;

Serially Dictatorial if there exists a permutation of individuals π and a tie-breaking preference $\rho \in R^*$ such that $f^*(\succsim) = T(T^N(X, \succsim, \pi), \rho)$ for all $\succsim \in R$;

Upper Hemi-continuous at Indifference for Pairs if $\{x_n\} \subset X$ converges to x and $x_n \in f^*(\succsim^{\{x_n,y\}})$ for all n imply $x \in f^*(\succsim^{\{x,y\}})$, where for any subset Y, $\succsim^Y$ is a strong lifting of Y from the preference profile in which all individuals are indifferent between all alternatives.

As in other instances, we will omit the qualification "at indifference for pairs" regarding upper hemi-continuity.

Let $\mathscr{Y}$ be an agenda domain and R be a strongly admissible preference domain with respect to $\mathscr{Y}$. An overall social choice correspondence $f^* : R \to 2^X \setminus \emptyset$ can be extended to the $\mathscr{Y}$ agenda domain by

$$f(Y, \succsim) = f^*(\succsim^Y) \quad \text{for all } Y \in \mathscr{Y}, \text{ all } \succsim \in R, \tag{3}$$

where $\succsim^Y$ is a strong Y-lifting (the ε does not matter) from $\succsim$.[11] If f^* is a function, so is f.

To be a valid social choice correspondence, f must be feasible, that is, $f(Y, \succsim) \subseteq Y$ for all Y in $\mathscr{Y}$. This is guaranteed by the following lemma.

Lemma 4.6. *Let R be a strongly admissible preference domain with respect to the minimal agenda domain. If an overall social choice correspondence $f^* : R \to 2^X \setminus \emptyset$ is Strongly Unanimous and Pareto Monotonic, then $x \notin f^*(\succsim)$ whenever x is weakly Pareto dominated at $\succsim$.*

Proof. Suppose by contradiction that y weakly Pareto dominates x at $\succsim$ but $x \in f^*(\succsim)$. Pick $\varepsilon < |x - y|$ and let $\succsim'$ be a strong $(\{x, y\}, \varepsilon)$-lifting from $\succsim$. Notice that $\succsim'$ is an $\{x\}$-advancement of $\succsim$. Pareto Monotonicity requires $x \in f^*(\succsim')$ but Strong Unanimity requires $f^*(\succsim') = \{y\}$. □

[10]By requiring the implication to hold only on Y-advancements, this condition is weaker than the standard Monotonicity as the relative ranking elements in y may fall against Pareto dominated alternatives (hence the term "Pareto Monotonic"). As in MT (see their footnote 10), two set inclusions are required. First, all alternatives in Y should remain chosen ($Y \subseteq f^*(\succsim')$); Second, no previously losing alternatives should become chosen ($f^*(\succsim') \subseteq f^*(\succsim)$). When f^* is a function (hence Y is a singleton), this definition simply says Y remains chosen.

[11]There are multiple strong Y-lifting for each $\succsim$ and Y. One can pick any one of them to define an extension. Due to the IIF part of Proposition 4.7, all our results are unaffected by the choice of the lifting, hence the choice of the extension.

Notice that the application of this lemma requires the $\succsim^Y$ in Equation (3) to be a *strong* Y-lifting (rather than just a lifting). Otherwise, alternatives outside Y may not be Pareto dominated at $\succsim^Y$.

Proposition 4.7. *Let R be a strongly admissible domain with respect to some agenda domain $\mathscr{Y}$. If an overall social choice correspondence $f^* : R \to 2^X \setminus \emptyset$ is Strongly Unanimous and Pareto Monotonic, then the social choice correspondence $f : \mathscr{Y} \times R \to 2^X \setminus \emptyset$ satisfies SU, IIF and ILA. Moreover, if f^* is upper hemi-continuous, so is f.*

Proof. SU: Follows by definition of Strong Unanimity.

IIF: Suppose $\succsim$ and $\succsim'$ agree on $Y \in \mathscr{Y}$. By Lemma 4.6, $f^*(\succsim^Y) \subseteq Y$ and $f^*(\succsim'^Y) \subseteq Y$. Notice that $\succsim'^Y$ is a $f^*(\succsim^Y)$-advancement of $\succsim^Y$; and $\succsim^Y$ a $f^*(\succsim'^Y)$-advancement of $\succsim'^Y$. Pareto Monotonicity requires $f^*(\succsim^Y) = f^*(\succsim'^Y)$. By Equation (3), $f(Y, \succsim) = f(Y, \succsim')$.

ILA: Suppose $Z = f^*(\succsim) \cap Y = f(X, \succsim) \cap Y$ is non-empty. Let $\succsim^Y$ be a strong Y-lifting from $\succsim$. Notice that $\succsim^Y$ is a Z-advancement from $\succsim$. Pareto Monotonicity implies $Z \subseteq f^*(\succsim^Y) \subseteq f^*(\succsim)$. Meanwhile, Lemma 4.6 requires $f^*(\succsim^Y) \subseteq Y$. Hence $f^*(\succsim^Y) \subseteq f^*(\succsim) \cap Y = Z$, which in turns implies $f^*(\succsim^Y) = Z$. Using Equation (3), $f(Y, \succsim) = Z = f(X, \succsim) \cap Y$.

Upper Hemi-continuity: Let $\succsim^0$ be the indifference profile. For any pairwise subset Y, write $\succsim^Y$ as the preference profile obtained from a strong Y-lifting from $\succsim^0$. Let $\{x_n\} \to x$ and suppose $x_n \in f(\{x_n, y\}, \succsim^0) = f^*(\succsim^{\{x_n, y\}})$. Upper Hemi-continuity of f^* implies $x \in f^*(\succsim^{\{x,y\}}) = f(\{x, y\}, \succsim^0)$. □

The following corollaries of Theorems 3.1 and 3.4 can be viewed as variations of the Muller-Satterthwaite Theorem (Muller and Satterthwaite, 1977):

Corollary 4.8.

(1) *Let R be a strongly comprehensive preference domain relative to the pairs and triplets agenda domain. Any overall social choice correspondence $f^* : R \to 2^X \setminus \emptyset$ that is Strongly Unanimous, Pareto Monotonic and Upper Hemi-continuous is Serially Dictatorial.*

(2) *Let R be a strongly admissible preference domain relative to the minimal agenda domain. Any overall social choice correspondence $f^* : R \to 2^X \setminus \emptyset$ that is Strongly Unanimous and Pareto Monotonic is Dictatorial.*

The requirement of strong admissibility may seems restrictive. Nonetheless, the strong closure under lifting requirement is relative only to the pairs and triplets agenda domain (or even the minimal agenda domain). By Fact 2.6, any preference domain that is comprehensive relative to an agenda domain containing the pairs and triplets domain is strongly comprehensive relative to the pairs and triplets domain. Similarly, any admissible preference domain is strongly admissible relative to the minimal agenda domain. Thus the preference domain restrictions in the above corollary are no stronger than those in other theorems in this note. In particular, the canonical preference domain of all complete, transitive and continuous preference relations over X is strongly comprehensive relative to the pairs and triplets agenda domain. So is the preference domain comprised of preferences representable by upper semi-continuous piecewise linear utility functions (Example 2.5).

While we call the above statements "variations of the Muller-Satterthwaite Theorem", they are substantially different from the original Muller-Satterthwaite Theorem. The agenda domain can be infinite. Weak preferences are allowed. The overall social choice can be multi-valued. Monotonicity is required only among Pareto undominated alternatives. To our knowledge, this is the first paper to document this very general statement.

4.3. **Candidate Stability.** Candidate stability concerns the effect of the unilateral withdrawal of a single alternative. Intuitively, it seems awkward to extend this concept to an infinite-alternative setting. Technically, if X is uncountably infinite, taking away a finite number of alternatives from X leaves a subset that is not closed, on which the social choice correspondence is undefined. For these reasons we will not attempt an infinite-alternative version of the results in Section 6 of MT.

Appendix A. Collection of Omitted Proofs

A.1. **Proof of Example 2.4.** Let R_i be the set of all complete, transitive and continuous individual preferences on X. Since X is a compact metric space, it is second-countable. Hence a preference belongs to R_i if and only if it can be represented by a continuous utility function (Debreu, 1954, see also Mehta, 1977, Theorem 1). It suffices therefore to consider continuous utility functions over X rather than preferences.

Choice Normality: The set of maximizers of a continuous function on a non-empty, compact set is non-empty and closed.

Closure under Lifting: Let $u : X \to \mathbb{R}$ be a continuous utility function on X. Let $Y \in \mathscr{Y}$ be a non-empty closed subset of X. Adopt the shorthand $Y^c_\varepsilon = X \setminus B_\varepsilon(Y)$. Since both Y and Y^c_ε are closed subsets of X and are compact, the following maxima and minima are well-defined:

$$\overline{a} = \max_{y \in Y} u(y); \qquad \overline{b} = \max_{y \in Y^c_\varepsilon} u(y);$$
$$\underline{a} = \min_{y \in Y} u(y); \qquad \underline{b} = \min_{y \in Y^c_\varepsilon} u(y).$$

Denote also $y^* = \arg\max_{y \in Y} u(y)$ as the maximal element in Y according to the preference represented by u.

Construct a continuous function $\tilde{u} : Y \cup Y_\varepsilon^c \to [-1, 1]$ by

$$\tilde{u}(y) = \begin{cases} \frac{1}{3} & \text{if } y \in Y \text{ and } \overline{a} = \underline{a} \\ \frac{2u(y)+\overline{a}-3\underline{a}}{3(\overline{a}-\underline{a})} & \text{if } y \in Y \text{ and } \overline{a} > \underline{a} \\ -\frac{1}{3} & \text{if } y \in Y_\varepsilon^c \text{ and } \overline{b} = \underline{b} \\ \frac{2u(y)+\underline{b}-3\overline{b}}{3(\overline{b}-\underline{b})} & \text{if } y \in Y_\varepsilon^c \text{ and } \overline{b} > \underline{b}. \end{cases}$$

The restriction of $\tilde{u}$ to Y is a monotonic transformation of u restricted to Y. Similarly, its restriction to Y_ε^c is a monotonic transformation of u restricted to Y_ε^c. Moreover, if $y \in Y$ and $y' \in Y_\varepsilon^c$, then $\tilde{u}(y) > \tilde{u}(y')$. It remains to show that $\tilde{u}$ can be extended to a continuous function v over X such that for each $x \notin Y$, there is a $y \in Y$ such that $v(y) > v(x)$, and that this y depends only on $\tilde{u}$. To this end, first note that

$$\tilde{u}^{-1}\left(\left[\frac{1}{3}, 1\right]\right) = Y; \qquad \tilde{u}^{-1}\left(\left[-1, -\frac{1}{3}\right]\right) = Y_\varepsilon^c.$$

Construct $g : X \to [-\frac{1}{3}, \frac{1}{3}]$ by

$$g(X) = \begin{cases} \frac{1}{3} & \text{if } x \in Y \\ -\frac{1}{3} & \text{if } x \in Y_\varepsilon^c \\ \frac{|x-Y|}{|x-Y|+|x-Y_\varepsilon^c|}\left(-\frac{1}{3}\right) + \frac{|x-Y_\varepsilon^c|}{|x-Y|+|x-Y_\varepsilon^c|}\left(\frac{1}{3}\right) & \text{otherwise.} \end{cases}$$

This g is continuous and $|\tilde{u}(y) - g(y)| \leq 2/3$ for all $y \in Y \cup Y_\varepsilon^c$. For any $x \notin Y$, $y \in Y$, $g(x) < 1/3 = g(y)$. Moreover, since $\tilde{u}(y^*) = 1$, $\tilde{u}(y^*) - g(y^*) = 2/3$.

Now $\tilde{u} - g$ is a continuous function from $Y \cup Y_\varepsilon^c$ to $[-\frac{2}{3}, \frac{2}{3}]$. As X is a metric space, it is normal. By the Tietze Extension Theorem, $\tilde{u} - g$ can be extended to a continuous function $h : X \to \left[-\frac{2}{3}, \frac{2}{3}\right]$. Define now, for each $x \in X$,

$$v(x) = h(x) + g(x).$$

This v function is clearly continuous. For any $y \in Y \cup Y_\varepsilon^c$, $v(y) = \tilde{u}(y)$ so v is an extension of $\tilde{u}$[12]. For any $x \notin Y$,

$$\begin{aligned} v(x) &= h(x) + g(x) \\ &< \frac{2}{3} + \frac{1}{3} \\ &= v(y^*). \end{aligned}$$

Thus this v is the desired continuous utility function. Notice that y^* depends only on the preference over Y. Therefore, if we apply the above construction on each of the individuals,

[12]We do not apply the Tietze Extension Theorem directly on $\tilde{u}$ since we need to guarantee the existence of a $y \in Y$ such that $v(y) > v(x)$ for all $x \notin Y$.

two individuals with the same preference over Y would have the same y^*, satisfying condition (ii) of a lifting.

Indifference Profile: A constant function is continuous.

A.2. **Proof of Example 2.5.** Let R_i be the set of individual preferences over X with an upper semi-continuous piecewise linear utility function representation and let $R = \prod_{i \in N} R_i$.

Choice Normality: The set of maximizers of an upper semi-continuous function on a non-empty compact set is non-empty and closed.

Closure under Strong Lifting: Let $\succsim \in R$ and let $u_i : X \to \mathbb{R}$ be the upper semi-continuous piecewise linear function representing $\succsim_i$. Let $b = \max_x u_i(x) + 1$. Given a closed set $Y \subseteq X$, the function $v_i : X \to \mathbb{R}$ defined by

$$v_i(x) = \begin{cases} u_i(x) + b & \text{if } x \in Y \\ u_i(x) & \text{if } x \notin Y \end{cases}$$

is also upper semi-continuous and piecewise linear. Let $\succsim'$ be the preference profile in which each $\succsim_i'$ is the preference represented by v_i. Then $\succsim' \in R$. Notice that $\succsim'$ is a strong Y-lifting of $\succsim$ — for each i, $v_i(y) > v_i(x)$ whenever $y \in Y$ and $x \notin Y$.

Inclusion of the Indifference Profile: Any constant function over X is upper semi-continuous and piecewise linear.

A.3. **Proof of Fact 2.6.** For this and the next proof, let Y be a finite subset of X. Since X is metric, there is an ε_0 sufficiently small such that the ε_0-balls around each element of Y are pairwise disjoint. Take an $\varepsilon < \varepsilon_0$.

For each individual i let $Y_i^1, \ldots, Y_i^n$ be the indifference classes in Y according to $\succsim_i$. Order these sets such that for any $y_j \in Y_i^j$ and $y_k \in Y_i^k$, $y_j \succ_i y_k$ if and only if $j > k$. Construct a sequence of preferences $\{\succsim_i^j\}$ as follows: $\succsim_i^0 = \succsim_i$; and for each $j > 0$, $\succsim_i^j$ is a (Y_j^i, ε)-lifting from $\succsim_i^{j-1}$. Let $\succsim_i' = \succsim_i^n$. Let $\succsim' = (\succsim_1', \ldots, \succsim_N')$. This construction is feasible since R is a product set and is closed under lifting.

We would like to show that $\succsim'$ is a strong (Y, ε)-lifting from $\succsim$. To this end we show that it satisfies conditions (i) and (ii') in Definition 2.2.

(i): For each i, we first show that $\succsim_i'$ and $\succsim_i$ agree on Y. Suppose $y_j \in Y_i^j$ and $y_k' \in Y_i^k$. If $j = k$, since Y_i^j has an empty intersection with $\bigcup_{l<j} B_\varepsilon(Y_i^l)$, $\succsim_i^{j-1}$ agrees with $\succsim_i^0 = \succsim_i$ on Y_i^j. Lifting requires $\succsim_i^j$ to agree with $\succsim_i^{j-1}$ on Y_i^j. Meanwhile, Y_i^j has empty intersection with $\bigcup_{l>j} B_\varepsilon(Y_i^l)$ so $\succsim_i^j$ agrees with $\succsim_i^n = \succsim_i'$ on Y_i^j. Therefore $y_j \sim_i' y_k$. If instead $j > k$, then since $j \notin B_\varepsilon(Y_i^k)$, $y_k \succ_i^k y_j$. Again, $Y_i^j \cup Y_i^k$ has empty intersection with $\bigcup_{l>k} B_\varepsilon(Y_i^l)$. Thus $\succsim_i'$ agrees with $\succsim_i^k$ on $Y_i^j \cup Y_i^k$. Hence $y_k \succ_i' y_j$.

To see that $\succsim_i'$ and $\succsim_i$ agree on $X \setminus B_\varepsilon(Y)$, notice that $(X \setminus B_\varepsilon(Y)) \subseteq \left(X \setminus \bigcup_{j=1}^n B_\varepsilon(Y_i^j)\right)$. The preference order over this set is never altered during the construction.

Finally we show that $y \succ_i' x$ whenever $y \in Y$ and $x \notin B_\varepsilon(Y)$. If $x \notin B_\varepsilon(Y)$, then $y_1 \succ_i^1 x$ for any $y_1 \in Y_i^1$. Yet $\{x\} \cup Y_i^1$ has an empty intersection with $\bigcup_{j>1} B_\varepsilon(Y_i^j)$. Thus $\succsim_i^1$ agrees with $\succsim_i'$ on $\{x\} \cup Y_i^1$. By (i), $y \succsim_i' y_1 \succ_i' x$ for all $y \in Y$.

(ii'): Consider $x \in B_\varepsilon(Y) \setminus Y$ and let $y = \arg\min_{y' \in Y} |x - y'|$ (this y is unique since the ε-balls are disjoint). For individual i, let Y_i^j be the indifference class to which y belongs. At the j-th step of the lifting, there exists a $y^* \in Y_i^j$ such that $y_j \succ_i^j x$. Since $y \sim_i^j y^*$, $y \succ_i^j x$. Since $\{x\} \cup Y_i^j$ has empty intersection with $\bigcup_{l>j} B_\varepsilon(Y_i^l)$, $\succsim_i^j$ agrees with $\succsim'$ on $\{x\} \cup Y_i^j$, thus $y \succ_i' x$. Since the choice of y is independent of i, for each $x \notin Y$, there is a $y \in Y$ such that $y \succ_i' x$ for all $i \in N$.

A.4. **Proof of Fact 2.7.** Let $\succ$ be a strict preference profile on Y. For each i, order elements in Y such that $y_i^1 \prec_i \cdots \prec_i y_i^n$. Construct the sequence of preferences $\left\{\succsim_i^j\right\}$ as in the proof of Fact 2.6, using an arbitrary $\succsim_i^0 \in R_i$ and taking $Y_i^j = \left\{y_i^j\right\}$. The proof of the agreement between $\succsim'$ and $\succ$ on Y is the same as the proof of (i) for Fact 2.6. The proof that $y \succ_i' x$ for all i whenever $x \notin Y$ and $y \in \arg\min_{y' \in Y} |x - y'|$ is the same as the the last part of (i) and (ii') for Fact 2.6.

References

Arrow, K. J. (1959): "Rational Choice Functions and Orderings," *Economica*, 26(102), 121–127.

——— (1963): *Social Choice and Individual Values*. Yale Univeristy Press, New Haven, 2nd edn.

Debreu, G. (1954): "Representation of a Preference Ordering by a Numerical Function," in *Decision Processes*, ed. by R. M. Thrall, C. H. Coombs, and R. L. Davis, chap. XI, pp. 159 – 165. Wiley, New York.

Man, P., and S. Takayama (2013): "A Unifying Impossibility Theorem," *Economic Theory*, forthcoming.

Mehta, G. (1977): "Topological Ordered Spaces and Utility Functions," *International Economic Review*, 18(3), 779 – 782.

Muller, E., and M. A. Satterthwaite (1977): "The Equivalence of Strong Positive Association and Strategy-proofness," *Journal of Economic Theory*, 14(2), 412–418.