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Abstract

We analyze a costly voting model with more than three candidates. We show that there are two different types of equilibria. In the first one, all candidates receive votes and have an equal chance of winning, independent of their popular support levels. In the second type of equilibrium, only two candidates receive votes and have a positive winning probability. Furthermore, in both types of equilibria, all voters who cast votes do so for their most preferred candidate, i.e., there is no "strategic voting."
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1 Introduction

The literature analyzing the decision whether or not to vote when voting is costly generally focuses on settings in which there is an election between exactly two candidates (Palfrey and Rosenthal (1983, 1985); Ledyard (1984) and the literature discussed in Section 2 below). The advantage of assuming that only two candidates compete in the election is that there are only two weakly undominated strategies for each citizen: Abstain, or vote for one's favorite candidate.

However, many real-life elections involve competition between more than two parties or candidates. In this paper, we analyze a costly voting model with multiple candidates, in which citizens' preferences over candidates are drawn from a distribution that is common knowledge, but each citizen only knows his own realized type. Citizens decide strategically whether to vote at all (taking this action requires a citizen to pay a voting cost c), and if they choose to vote, also which candidate to vote for.

This is a potentially very complicated setting because, in addition to the participation decision, we allow for citizens to vote strategically for other candidates than their most preferred one, and it is well known that only voting for one's least-preferred candidate is a dominated strategy in a multi-candidate election. In spite of this, the model remains surprisingly tractable. Our analysis focuses on the case of three candidates, but it will be clear that our qualitative results generalize to the general case of m candidates.

We show that two structurally types of equilibria arise when the number of voters is large. In the first one, all three candidates receive the same positive expected number of votes, and each wins with probability 1/3. In the second type of equilibrium, only two candidates receive a positive expected number of votes, and each of these relevant candidates wins with probability 1/2; there are three different equilibria of this second type equilibrium, one for each subset of two candidates (i.e., one in which candidates A and B are the relevant candidates; another one in which A and C are the relevant candidates, and a last one in which B and C are the relevant candidates).

Interestingly, in all equilibria, all voters vote sincerely, that is, for their most preferred candidate. This result contrasts with the literature on strategic voting in settings where the set of participating voters is exogenous (see Myerson and Weber 1993; Messner and Polborn 2011) and where there are generally many equilibria in which strategic voting occurs.

Intuitively, the reason for our result that "strategic voting" (i.e., voting for a candidate who is not the voter's most preferred one) does not occur in equilibrium when voting is costly and society is large is as follows: A voter who votes strategically for his second-most preferred candidate has a lower marginal benefit from voting than a voter who votes for the same candidate, but ranks him highest among the candidates. Because the strategic voter has to be at least indifferent between voting and not voting, the second type of voter must strictly prefer participation over abstention. When the number of citizens is large, this cannot be the case, because then, the number of participating voters would be very high, and consequently, the probability that each voter is pivotal would be very close to zero.

The paper proceeds as follows: Section 2 places our paper in the literature. We present the model in Section 3, and our results in Section 4. Section 5 concludes.

2 Related literature

Our paper contributes to the literature on costly voting (and, more generally, endogenous participation models) pioneered by Ledyard (1984) and Palfrey and Rosenthal (1983, 1985), and developed further by a large number of papers.

In particular, the costly voting framework has been used and modified to understand stylized facts about participation in elections (Feddersen and Sandroni, 2006; Herrera and Martinelli, 2006; Levine and Palfrey, 2007), as well as the more normative question whether a social planner should encourage citizens to participate in elections (Börgers, 2004; Krasa and Polborn, 2009; Taylor and Yildirim, 2010a,b; Krishna and Morgan, 2012, among others). These models, and – to the best of our knowledge – all other costly voting models assume that citizens have to choose between only two candidates if they vote. Clearly, focusing on the two candidate case simplifies the analysis (because voters effectively only have to decide whether to participate, while their vote decision if they participate is trivial), and thus enables these authors to focus on other interesting questions in the framework. Our paper complements this literature by focusing on the basic costly voting model, and analyzing the case of more than two candidates.

Our paper also contributes to another literature, namely the one analyzing strategic voting in multi-candidate elections. It is well known that, in voting games with multiple candidates, the only weakly dominated strategy for voters is to vote for one's least preferred candidate. Even iterated elimination of weakly dominated strategies usually does not narrow down the set of possible equilibrium outcomes (Dhillon and Lockwood, 2004). Myerson and Weber (1993) and Messner and Polborn (2011) consider different trembling refinements, and Messner and Polborn (2007) consider refinements based on coordination between different voter groups. These models primarily aim to increase our understanding of the empirical regularity known as "Duverger's Law" which states that, under plurality rule, most voters vote for one of two "main" candidates, with all other candidates receiving very few votes because those voters who like them expect that a vote for their favorite candidate would be wasted because he has no chance of winning. Therefore, these voters are better off voting for the main candidate whom they like better than his main competitor.

A similar effect is present in those equilibria of our model in which only two candidates receive a positive expected vote share: However, in contrast to the models above, the supporters of the third candidate in such an equilibrium abstain completely, rather than vote for one of the two main candidates. Our model also admits an equilibrium in which all three candidates receive a positive vote share and even have the same probability of winning. In contrast, in Messner and Polborn (2007, 2011), this cannot happen in equilibrium. Myerson and Weber (1993) also obtain an equilibrium in which all three candidates receive votes and tie; however, this equilibrium is based on a reduced form modeling of the pivot event.[1]

3 The model

We consider an election game with three candidates whom we call A, B and C. The voting system is plurality rule, i.e., there is a single round of voting, and the candidate who receives the most votes is elected.

The players of our game are N *citizens* (to avoid confusion, we reserve the term "voter" to a citizen who chooses to actually vote). Each citizen decides simultaneously whether to abstain or to vote, and if he votes, which candidate to vote for. Voters receive different utilities from different candidates. A citizen of type ij receives a benefit of 1 if his most preferred candidate i wins the election, λ if his second preferred candidate j wins, and 0 if his least preferred candidate $k \neq i, j$ wins. Table 1 shows the six possible voter preference types and their payoffs.

Table 1: Preference Types

Preferences	ABC	ACB	BAC	BCA	CAB	CBA
A	1	1	λ	0	λ	0
B	λ	0	1	1	0	λ
C	0	λ	0	λ	1	1

Following Myerson (2000), we assume that the number of citizens of type ij is drawn from a Poisson distribution with parameter N_{ij}, for all types ij. Note that N_{ij} is the expected number of citizens of type ij. The analytical advantage of the Poisson game assumption is

[1]The equilibrium concept of Myerson and Weber (1993) is based on a vector of pivot beliefs held by the voters (essentially summarizing their beliefs that any pair of candidates will be tied). Myerson and Weber allow for the pivot belief to take any value between 0 and 1 if two candidates are tied (and not just 1/2), and this feature is essential to support their 3 candidate equilibria.

that, from the point of view of a particular citizen of type ij, the number of other citizens of type ij is still Poisson distributed with parameter N_{ij}.

Our equilibrium concept is a quasi-symmetric (mixed strategy) equilibrium, that is, a strategy profile in which all players of preference type ij participate with the same probability p_{ij} and vote for the same candidate (not necessarily their most preferred candidate, although we will show that, in any equilibrium, they will vote for candidate i). Note that, while we look for an equilibrium in which voters of the same type play the same strategy, the deviations that we consider are, of course, unilateral deviations by one voter (both with respect to participation, and with respect to which candidate to vote for in case of participation).

4 Analysis

4.1 Equilibria with two relevant candidates

We start by analyzing the possibility of an equilibrium in which all citizens who vote do so for one of two candidates, while no player will ever vote for the third candidate. Without loss of generality, we focus on an "AB-equilibrium" in which the two "relevant" candidates are A and B.

It is intuitively clear that the expectation that candidate C is ignored is self-sustaining: If C is ignored by the other voters, and if the expected number of votes for the two other candidates is not too small (which translates into a very mild condition on the cost of voting in Proposition 1 below), then it follows that C's winning probability with only one vote would be very small.

The voter types AB, AC, and CA prefer A over B, so if they vote in an AB-equilibrium, they vote for candidate A. Similarly, if types BA, BC, and CB vote in an AB-equilibrium, they vote for candidate B. The equilibrium is characterized by a vector p_{ij} denoting, for each type ij, the probability that a voter of the respective type votes. However, it is notationally more convenient to focus rather on the expected number of *votes* from preference group ij, v_{ij}, given by $v_{ij} = p_{ij}N_{ij}$ (see Table 2). By Myerson (1998) (see also Johnson and Kotz (1969)), the number of votes from group ij is Poisson distributed with parameter v_{ij}.

Preferences	ABC	ACB	BAC	BCA	CAB	CBA
Exp. number	v_{AB}	v_{AC}	v_{BA}	v_{BC}	v_{CA}	v_{CB}

Table 2: Expected number of voters of each type

The following Proposition 1 shows that, provided that the cost of voting is not too large, equilibria with two relevant candidates exist, and then characterizes them. In particular, we show that the only citizen preference types who vote with positive probability are those types who are maximally motivated because they rank the one relevant candidate highest and the other relevant candidate lowest.

Intuitively, in a sufficiently large economy, no voter type can vote with probability 1 because then the probability of being pivotal for the election outcome would go to zero, making it not worthwhile for a citizen to spend the cost of voting. Thus, those voter types who participate with positive probability must be indifferent between voting and not voting.

In an AB-equilibrium, citizens of type AC receive a benefit of 1 if A wins and of 0 if B wins, and are therefore maximally motivated to change the outcome of the election. If they are indifferent, in equilibrium, between voting and not voting, it must be true that citizens of type AB (who also get a payoff of 1 if A wins, but of λ if B wins) have a smaller benefit from swinging the election, and thus strictly prefer to abstain.

Proposition 1. *Suppose that $c \leq 1/4$. Then, for each pair of candidates $i, j \in \{A, B, C\}$, there exist equilibria in which exactly two candidates, i and j, receive a positive expected number of votes, and they win with equal probability.*

Furthermore, in every such two-candidate equilibrium, only two of the six preference types vote, namely the ones who rank candidate i highest and candidate j lowest, and vice versa.

Proof. Without loss of generality, consider a mixed strategy equilibrium in which the relevant candidates i and j are A and B. Denote the expected number of A-voters by v_A, and that of B-voters by v_B. Note that v_A is Poisson distributed with parameter $v_{AB} + v_{AC} + v_{CA}$,[2] and analogously for v_B.

The expected benefit of voting for candidate A for an AC type is

$$\sum_{x=0}^{\infty} \left[\frac{1}{2} e^{-(v_A)} \frac{(v_A)^x}{x!} e^{-(v_B)} \frac{(v_B)^x}{x!} + \frac{1}{2} e^{-(v_A)} \frac{(v_A)^x}{x!} e^{-(v_B)} \frac{(v_B)^{(x+1)}}{(x+1)!} \right] = c. \tag{1}$$

Here, $e^{-(v_A)} \frac{(v_A)^x}{x!} e^{-(v_B)} \frac{(v_B)^x}{x!}$ is the probability that both A and B receive x votes from the other voters, so that, if the citizen we consider votes, he breaks the tie and thus increases A's winning probability by 1/2, from 1/2 to 1. The second term is the probability that, among the other voters, A is one vote behind B, and thus an additional vote for A gets A into a tie, and thus increases A's winning probability again by 1/2 (from 0 to 1/2). In a mixed strategy equilibrium, AC types are indifferent between voting or not, i.e., the left-hand side of (1) is equal to cost c.

[2]This follows because, if X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively, then $X + Y$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

We now consider the other voter types who might vote for A, namely types CA and AB. Note that both types of voters have the same probability of being pivotal as in (1) because in a Poisson game, a voter does not learn any information about the preference distribution of other voters from learning his own type. Thus, the expected benefit of voting for a CA type is

$$\sum_{x=0}^{\infty}\left[\frac{\lambda}{2}e^{-(v_A)}\frac{(v_A)^x}{x!}e^{-(v_B)}\frac{(v_B)^x}{x!}+\frac{\lambda}{2}e^{-(v_A)}\frac{(v_A)^x}{x!}e^{-(v_B)}\frac{(v_B)^{(x+1)}}{(x+1)!}\right]=\lambda c<c. \tag{2}$$

Thus, a CA type strictly prefers to abstain. Analogously, an AB-type's benefit of voting is $(1-\lambda)c<c$, so that they also strictly prefer to abstain.

Similarly, the benefit of voting for a BC-type citizen is

$$\frac{P(\text{B-votes} - \text{A-votes} \in \{0,-1\})}{2}=\frac{1}{2}\sum_{x=0}^{\infty}\left[e^{-v_B}\frac{(v_B)^x}{x!}e^{-v_A}\frac{(v_A)^x}{x!}+e^{-v_B}\frac{(v_B)^x}{x!}e^{-v_A}\frac{(v_A)^{(x+1)}}{(x+1)!}\right]=c \tag{3}$$

and is larger than the benefit of voting for either BA-types or CB-types.

Lemma 1 in the Appendix proves that, for any $c\in[0,1/2]$, there exists a unique pair $v_A(c), v_B(c)$, with $v_A(c)=v_B(c)$ such that the left-hand sides of (1) and (3) are equal to c. Letting $v(c)\equiv v_A(c)=v_B(c)$, and notationally suppressing the dependence on c, this implies that, in equilibrium, we have that

$$\frac{1}{2}\sum_{x=0}^{\infty}\left[e^{-2v}\frac{v^{2x}}{x!x!}+e^{-2v}\frac{v^{2x+1}}{x!(x+1)!}\right]=c. \tag{4}$$

Lemma 1 also proves that the solution of (4) is strictly decreasing in c.

It remains to show that voter types who abstain in equilibrium do not want to deviate and vote for candidate C. Consider first types CA and CB. If an individual of this type votes for C, he is pivotal for the election outcome in the following cases:

- First, if there are no votes for A or B; in this case, the voter's expected utility increases from $\frac{1+\lambda+0}{3}$ to 1, so the benefit is $\frac{2-\lambda}{3}$.
- Second, if there is one vote each for A and B, respectively; in this case, the voter's expected utility increases from $\frac{\lambda+0}{2}$ to $\frac{1+\lambda+0}{3}$, so the benefit is $\frac{2-\lambda}{6}$.
- Third, if there is one vote for A and zero votes for B, or vice versa; in one of these cases, the voter's expected utility increases from 0 to $\frac{1}{2}$, and in the other case from λ to $\frac{1+\lambda}{2}$.

Multiplying with the respective probabilities, we have that the expected benefit of voting for C for a CA and CB-type citizen is equal to

$$\begin{aligned}\frac{2-\lambda}{3}\left(\frac{v^0}{0!}e^{-v}\right)^2+\frac{2-\lambda}{6}\left(\frac{v^1}{1!}e^{-v}\right)^2+\left[\frac{1}{2}+\frac{1-\lambda}{2}\right]\frac{v^0}{0!}e^{-v}\frac{v^1}{1!}e^{-v}&\\ =e^{-2v}\left[\frac{2-\lambda}{3}+\frac{1}{6}v^2+\frac{1}{6}(1-\lambda)v^2+(1-\frac{\lambda}{2})v\right]&.\end{aligned} \tag{5}$$

We want to prove that (5) is less than the left-hand side of (4) (and thus less than c). Because (5) is decreasing in λ, we can substitute $\lambda = 0$ to see that (5) is smaller than $e^{-2v}\left[\frac{2}{3}+\frac{1}{3}v^2+v\right]$. Furthermore, since all terms on the left-hand side of (4) are positive, it is larger than $\frac{1}{2}\sum_{x=0}^{2}\left[e^{-2v}\frac{v^{2x}}{x!x!}+e^{-2v}\frac{v^{2x+1}}{x!(x+1)!}\right]$. Consequently, a sufficient condition for (5) to be smaller than c is that

$$\frac{1}{2}\sum_{x=0}^{2}\left[e^{-2v}\frac{v^{2x}}{x!x!}+e^{-2v}\frac{v^{2x+1}}{x!(x+1)!}\right]-e^{-2v}[\frac{2}{3}+\frac{1}{3}v^2+v]\geq 0. \tag{6}$$

Simplifying and rearranging (6), we get

$$\frac{e^{-2v}}{2}\left[-\frac{1}{6}-\frac{1}{2}v+\frac{1}{6}v^2+\frac{1}{4}v^3+\frac{1}{8}v^4+\frac{1}{24}v^5\right]\equiv\frac{e^{-2v}}{2}k(v). \tag{7}$$

Thus, a sufficient condition for CA and CB-types to prefer not to vote is that $k(v) \geq 0$ for the equilibrium value of v as defined by (4).

The derivative of $k(v)$ is $k^{'}(v) = -\frac{1}{2}+\frac{1}{3}v+\frac{3}{4}v^2+\frac{1}{2}v^3+\frac{5}{24}v^4$, and the second derivative is $k^{''}(v) = \frac{1}{3}+\frac{3}{2}v+\frac{3}{2}v^2+\frac{5}{6}v^3$, which is positive for all $v > 0$. Since $k^{'}(1) = -\frac{1}{2}+\frac{1}{3}+\frac{3}{4}+\frac{1}{2}+\frac{5}{24} > 0$, and $k^{'}$ is increasing, it follows that $k^{'}(v) > 0$ for all $v \geq 1$.

Thus, if, for some value of $v_1 > 1$, $k(v_1) > 0$, then $k(v) > k(v_1) > 0$ for all $v \geq v_1$. When $c = \frac{1}{4}$, the value of v that solves (4) is approximately 1.56, and substituting this yields $k(1.56) \approx 1.53 > 0$. Since v is decreasing in c, this proves that for all $c < 1/4$, CA and CB-types do not benefit from voting for C.

Finally, by arguments analogous to the ones given above, it follows that citizens of other preference types have an even lower benefit from voting for C. □

4.2 Equilibria with three relevant candidates

We now turn to the second fundamental type of equilibrium, namely one in which all three candidates receive a positive expected number of votes. We call this type of equilibrium an "ABC equilibrium," and will show the following fundamental properties of such an equilibrium. First, the expected number of votes for each of the three candidates is equal, and so are their winning probabilities. Second, all citizen types will vote for their most preferred candidates (if they choose to vote at all, of course); in other words, like in the two candidate equilibria discussed above, there is no strategic voting in an ABC equilibrium, either.

The intuition for these results is the following. If the expected number of votes for different candidates was different, different voter types would face different participation incentives; but this cannot be the case because all types face the same cost of participation and must be indifferent between voting and not voting.

Given that all candidates receive the same expected number of votes, the probability that a voter is pivotal is the same for all pairs of candidates. If a citizen votes for his favorite candidate, he may thus replace his second-ranked or third-ranked candidate by his top choice, depending on which type of pivot event occurs. In contrast, if a citizen votes for his second-ranked candidate, he may replace either his third choice (which is good) or his first choice (which is bad). However, it is quite clear that in expectation, voting for one's second-ranked candidate is less attractive than voting for one's top choice in an ABC equilibrium.

The following Proposition 2 formally states these results.

Proposition 2. *Assume that $c < \frac{2}{3}(1-\frac{\lambda}{2})$. There exist equilibria in which all three candidates receive a positive expected number of votes.*

1. *In every such equilibrium, the expected number of voters for each candidate is the same: $v_A = v_B = v_C \equiv v(c)$. Moreover, $v(\cdot)$ is a decreasing function of c.*

2. *In every such equilibrium, every citizen who votes does so for his most preferred candidate.*

It is interesting to note that an ABC equilibrium may exist for values of participation costs that are too high for a 2-candidate equilibrium to exist For example, if λ is close to zero, then an ABC equilibrium exists for all cost values lower or equal to 2/3; in such an equilibrium, the expected number of votes for each candidate is very small, and so the main pivot event is actually that, among other voters, all three candidates receive no votes. In this case, the benefit of voting is about 2/3 because there is a chance of 2/3 that the voter's favorite candidate replaces one of the two candidates that the voter does not like. In contrast, in an equilibrium in which there are only two relevant candidates, the increase of a voter's utility in case of a pivot event is limited to 1/2.

Proof of Proposition 2. We start by showing the existence of an ABC equilibrium with the stipulated characteristics. Since the votes of AB and AC type voters are independent Poisson distributed random variables, the probability that there are n votes for candidate A is given by

$$P(A\ receives\ n\ votes) = e^{-(v_{AB}+v_{AC})}\frac{(v_{AB}+v_{AC})^n}{n!}$$

Similarly, the corresponding probabilities for candidates B and C are given by

$$P(B\ receives\ n\ votes) = e^{-(v_{BA}+v_{BC})}\frac{(v_{BA}+v_{BC})^n}{n!}$$

$$P(C\ receives\ n\ votes) = e^{-(v_{CA}+v_{CB})}\frac{(v_{CA}+v_{CB})^n}{n!}$$

Note that these probabilities only depend on the sums $v_{AB}+v_{AC} \equiv v_A$, $v_{BA}+v_{BC} \equiv v_B$ and $v_{CA}+v_{CB} \equiv v_C$.

Consider the expected benefit of voting for A for a voter of type AB. There are the following different possibilities of a pivot event:

1. Among the other voters, A receives the same number of votes as both B and C. With an additional vote, A wins outright, replacing either B or C with probability 1/3 each. The expected benefit of voting is therefore $\frac{1}{3}(1-\lambda)+\frac{1}{3}=\frac{2-\lambda}{3}$.

2. Among the other voters, A is one vote behind both B and C. With an additional vote for A, there is a chance that A replaces either B (probability $\frac{1}{3}\frac{1}{2}=\frac{1}{6}$; with a benefit of $1-\lambda$) or C (again probability $\frac{1}{6}$; but now benefit 1). Thus, the benefit is $\frac{1}{6}(1-\lambda)+\frac{1}{6}=\frac{2-\lambda}{6}$

3. A is one vote behind B, and ahead of or tied with C; the benefit of voting is $\frac{1-\lambda}{2}$.

4. A is one vote behind C, and ahead of or tied with B; the benefit of voting is $\frac{1}{2}$.

5. A and B are tied for the lead, with C strictly behind them; in this case, the benefit of voting is $\frac{1-\lambda}{2}$.

6. A and C are tied for the lead, with B strictly behind them; in this case, the benefit of voting is $\frac{1}{2}$.

Multiplying the probabilities of these events with the corresponding benefits, and adding up yields that the expected benefit of voting for our AB-type voter is

$$EB_{AB}(v_A, v_B, v_C) = e^{-(v_A+v_B+v_C)} \Bigg\{ \sum_{x=0}^{\infty} \left[\frac{v_A^x v_B^x v_C^x}{(x!)^3} \left(\frac{2-\lambda}{3}\right) + \frac{v_A^x v_B^{x+1} v_C^{x+1}}{x!((x+1)!)^2} \left(\frac{2-\lambda}{6}\right) + \frac{v_A^x v_B^{x+1} e^{v_C} \Gamma(x+1, v_C)}{(x!)^2 (x+1)!} \frac{1-\lambda}{2} + \frac{v_A^x v_C^{x+1} e^{v_B} \Gamma(x+1, v_B)}{(x!)^2 (x+1)!} \frac{1}{2} \right] + \sum_{x=1}^{\infty} \left[\frac{v_A^x v_B^x e^{v_C} \Gamma(x, v_C)}{(x-1)!(x!)^2} \frac{1-\lambda}{2} + \frac{v_A^x v_C^x e^{v_B} \Gamma(x, v_B)}{(x-1)!(x!)^2} \frac{1}{2} \right] \Bigg\}, \tag{8}$$

where $\Gamma(x, v) = (x-1)! \sum_{n=0}^{x-1} \frac{v^n}{n!} e^{-v}$ is the upper incomplete gamma function.

The probability that an A-voter is pivotal between A and B is equal to

$$Piv_{AB,A} = \left(\sum_{x=0}^{\infty} [\frac{v_A^x v_B^{x+1} e^{v_C} \Gamma(x+1, v_C)}{(x!)^2 (x+1)!}] + \sum_{x=1}^{\infty} [\frac{v_A^x v_B^x e^{v_C} \Gamma(x, v_C)}{(x-1)!(x!)^2}] \right) e^{-(v_A+v_B+v_C)} \tag{9}$$

and the probability that an A-voter is pivotal between A and C is equal to

$$Piv_{AC,A} = \left(\sum_{x=0}^{\infty} [\frac{v_A^x v_C^{x+1} e^{v_B} \Gamma(x+1, v_B)}{(x!)^2 (x+1)!}] + \sum_{x=1}^{\infty} [\frac{v_A^x v_C^x e^{v_B} \Gamma(x, v_B)}{(x-1)!(x!)^2}] \right) e^{-(v_A+v_B+v_C)} \tag{10}$$

It is useful to define

$$\sum_{x=0}^{\infty} \frac{v_A^x v_B^x v_C^x}{(x!)^3} =_0 F_2(;1,1; v_A v_B v_C)$$

and

$$\sum_{x=0}^{\infty} \frac{v_A^x v_B^x v_C^x}{x!((x+1)!)^2} =_0 F_2(;2,2; v_A v_B v_C),$$

where ${}_0F_2(;1,1;v_A v_B v_C)$ and ${}_0F_2(;2,2;v_A v_B v_C)$ are confluent hypergeometric functions.

Using this notation, the expected benefit formulas can be expressed as follows

$$EB_{AB}(v_A, v_B, v_C) = \left\{ {}_0F_2(;1,1;v_A v_B v_C) \frac{2-\lambda}{3} +_0 F_2(;2,2;v_A v_B v_C) v_B v_C \frac{2-\lambda}{6} \right\} e^{-(v_A+v_B+v_C)} + Piv_{AB,A} \frac{1-\lambda}{2} + Piv_{AC,A} \frac{1}{2} \tag{11}$$

$$EB_{AC}(v_A, v_B, v_C) = \left\{ {}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Bv_C\frac{2-\lambda}{6} \right\} e^{-(v_A+v_B+v_C)} + Piv_{AB,A}\frac{1}{2} + Piv_{AC,A}\frac{1-\lambda}{2} \tag{12}$$

$$EB_{BA}(v_A, v_B, v_C) = \left\{ {}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Av_C\frac{2-\lambda}{6} \right\} e^{-(v_A+v_B+v_C)} + Piv_{BA,B}\frac{1-\lambda}{2} + Piv_{BC,B}\frac{1}{2} \tag{13}$$

$$EB_{BA}(v_A, v_B, v_C) = \left\{ {}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Av_C\frac{2-\lambda}{6} \right\} e^{-(v_A+v_B+v_C)} + Piv_{BA,B}\frac{1}{2} + Piv_{BC,B}\frac{1-\lambda}{2} \tag{14}$$

$$EB_{CA}(v_A, v_B, v_C) = \left\{ {}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Av_B\frac{2-\lambda}{6} \right\} e^{-(v_A+v_B+v_C)} + Piv_{CA,C}\frac{1-\lambda}{2} + Piv_{CB,C}\frac{1}{2} \tag{15}$$

$$EB_{CB}(v_A, v_B, v_C) = \left\{ {}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Av_B\frac{2-\lambda}{6} \right\} e^{-(v_A+v_B+v_C)} + Piv_{CA,C}\frac{1}{2} + Piv_{CB,C}\frac{1-\lambda}{2} \tag{16}$$

In a mixed strategy equilibrium, we must have that

$$EB_{ij}(v_A, v_B, v_C) = c \tag{17}$$

for all $i, j \in \{A, B, C\}$.

We now show that, in every ABC equilibrium, the expected number of voters for each candidate must be the same: $v_A = v_B = v_C$.

Since the benefit of voting is the same for each type, (17) implies that $EB_{AB}(v_A, v_B, v_C) = EB_{AC}(v_A, v_B, v_C) = c$. As the first two terms in both expressions coincide, we have

$$Piv_{AB,A}(\frac{1}{2} - \frac{\lambda}{2}) + Piv_{AC,A}\frac{1}{2} = Piv_{AB,A}\frac{1}{2} + Piv_{AC,A}(\frac{1}{2} - \frac{\lambda}{2})$$

which implies $Piv_{AB,A} = Piv_{AC,A}$. Analogously, we have that $Piv_{BA,B} = Piv_{BC,B}$ for B voters, and $Piv_{CA,C} = Piv_{CB,C}$ for C voters.

Using $Piv_{AB,A} = Piv_{AC,A}$ in (11), and the corresponding equality in EB_{BA}, we have that

$$\begin{aligned}
&\underbrace{\left[{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Bv_C\frac{2-\lambda}{6}\right]e^{-(v_A+v_B+v_C)} + Piv_{AB,A}(1-\frac{\lambda}{2})}_{EB_{AB}(v_A,v_B,v_C)} = \\
&\underbrace{\left[{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} +_0 F_2(;2,2;v_Av_Bv_C)v_Av_C\frac{2-\lambda}{6}\right]e^{-(v_A+v_B+v_C)} + Piv_{BA,B}(1-\frac{\lambda}{2})}_{EB_{BA}(v_A,v_B,v_C),}
\end{aligned} \tag{18}$$

where $EB_{AB} = EB_{BA}$ follows from (17).

Suppose, to the contrary of the claim, that $v_A > v_B$. Canceling the common first term in (18) and observing that $v_A > v_B$ implies ${}_0F_2(;2,2;v_Av_Bv_C)v_Bv_C <_0 F_2(;2,2;v_Av_Bv_C)v_Av_C$, so it follows that $Piv_{AB,A} > Piv_{BA,B}$.

However, if $v_A > v_B$, then, using (9) and the corresponding definition of $Piv_{BA,B}$ shows that

$$\begin{aligned}
Piv_{AB,A} &= \left(\sum_{x=0}^{\infty}[\frac{v_A^x v_B^{x+1} e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} = \\
&= \left(v_B\sum_{x=0}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} < \\
&< \left(v_A\sum_{x=0}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} = \\
&\left(\sum_{x=0}^{\infty}[\frac{v_A^{x+1} v_B^x e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} = Piv_{BA,B}
\end{aligned} \tag{19}$$

This contradiction proves that $v_A \leq v_B$. Analogously, one can show that $v_A \geq v_B$, which then implies $v_A = v_B$. Likewise, we can show $v_B = v_C$.

Using $v_A = v_B = v_C = v$, the expected benefit of voting for each type is

$$\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{x!^3}\frac{2-\lambda}{3}+\frac{v^{3x+2}}{x!(x+1)!^2}\frac{2-\lambda}{6}+\frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x}\frac{v^n}{n!}\frac{1}{2}+\frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x}\frac{v^n}{n!}\frac{1-\lambda}{2}\right]e^{-3v}$$
$$+\sum_{x=1}^{\infty}\left[\frac{v^{2x}}{x!x!}\sum_{n=0}^{x-1}\frac{v^n}{n!}\frac{1}{2}+\frac{v^{2x}}{x!x!}\sum_{n=0}^{x-1}\frac{v^n}{n!}\frac{1-\lambda}{2}\right]e^{-3v}=$$
$$e^{-3v}\left(1-\frac{\lambda}{2}\right)\left\{\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{x!^3}\frac{2}{3}+\frac{v^{3x+2}}{x!(x+1)!^2}\frac{1}{3}+\frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x}\frac{v^n}{n!}\right]+\sum_{x=1}^{\infty}\left[\frac{v^{2x}}{x!x!}\sum_{n=0}^{x-1}\frac{v^n}{n!}\right]\right\}. \tag{20}$$

Lemma 2 in the Appendix shows that (20) is decreasing in v, which then implies that the value of v for which the expected benefit of voting is equal to c is a decreasing function of c.

To prove that a unique solution exists for all $c < \frac{2}{3}(1-\frac{\lambda}{2})$, it is then sufficient to prove that limit of benefit at 0 is $\frac{2}{3}(1-\frac{\lambda}{2})$ and at infinity is 0. Since the expected benefit in (20) is equal to $(1-\frac{\lambda}{2})f(v)$, it is enough to prove that $\lim_{v\to 0} f(v) = \frac{2}{3}$ and $\lim_{v\to\infty} f(v) = 0$.

Since $f(v)$ is continuous function, the limit at zero is just the value at 0, which is $\frac{2}{3}$. For the case of infinity, we use the squeeze theorem.

$$0 \le f(v) \le \left(\sum_{x=0}^{\infty}\left[\frac{v^{2x}}{x!x!}\sum_{n=0}^{x}\frac{v^n}{n!}\right]+\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x+1}\frac{v^n}{n!}\right]\right)e^{-3v} \le \left(\sum_{x=0}^{\infty}\frac{v^{2x}}{x!x!}+\sum_{x=0}^{\infty}\frac{v^{2x+1}}{x!(x+1)!}\right)e^{-2v}, \tag{21}$$

where certainly $\sum_{n=0}^{x}\frac{v^n}{n!} \le e^v$, for each x, because $e^v = \sum_{n=0}^{\infty}\frac{v^n}{n!}$. The right hand side of (21) can be expressed in terms of modified Bessell functions of order zero and one.

$$\left(\sum_{x=0}^{\infty}\frac{v^{2x}}{x!x!}+\sum_{x=0}^{\infty}\frac{v^{2x+1}}{x!(x+1)!}\right)e^{-2v} = \frac{I_0(2v)+I_1(2v)}{e^{2v}}$$

By formula 9.7.1 of Abramowitz and Stegun (1964), we have that

$$\lim_{v\to\infty}\frac{I_0(2v)+I_1(2v)}{e^{2v}} = 0.$$

Therefore,

$$0 \le \lim_{v\to\infty} f(v) \le \lim_{v\to\infty}\frac{I_0(2v)+I_1(2v)}{e^{2v}} = 0.$$

Finally, to prove that no strategic voting (i.e., voting for second preferred candidate) can occur in any ABC equilibrium, we show that a voter's benefit from voting is larger when voting for his most preferred candidate than when voting for his second choice. It is useful to remember that, because $v_A = v_B = v_C$, all combinations of candidates are equally probable in the following exhaustive list of pivot events.

1. All three candidates are tied. Voting for one's preferred candidate increases utility by

$$(1 - \frac{1+\lambda}{3}) = \frac{2-\lambda}{3},$$

while voting for one's second choice gives

$$\lambda - \frac{1+\lambda}{3} = \frac{2\lambda - 1}{3} < \frac{2-\lambda}{3}.$$

2. Two candidates are tied, while the third one is two or more votes behind. If voting for the preferred candidate, there is a 2/3 chance that he is among the two frontrunners, and conditional on that being the case, the chance that the other frontrunner is the second-ranked or the third-ranked candidate is 1/2 each; thus, the benefit of voting is

$$\frac{2}{3}\left(\frac{1}{2}\left(1 - \frac{1+\lambda}{2}\right) + \frac{1}{2}\left(1 - \frac{1}{2}\right)\right) = \frac{2-\lambda}{6}.$$

If voting for the second-ranked candidate, the benefit of voting is

$$\frac{2}{3}\left(\frac{1}{2}\left(\lambda - \frac{1+\lambda}{2}\right) + \frac{1}{2}\left(\lambda - \frac{\lambda}{2}\right)\right) = \frac{2\lambda - 1}{6} < \frac{2-\lambda}{6}.$$

3. Two candidates are tied for the lead, and the third one is one vote behind. The expected benefit of voting for one's favorite candidate is

$$\frac{1}{3}\left[\frac{1+\lambda}{3} - \frac{\lambda}{2}\right] + \frac{1}{3}\left[1 - \frac{1+\lambda}{2}\right] + \frac{1}{3}\left[1 - \frac{1+0}{2}\right] = \frac{4}{9} - \frac{2}{9}\lambda.$$

The expected benefit of voting for one's second-ranked candidate is

$$\frac{1}{3}\left[\frac{1+\lambda}{3} - \frac{1}{2}\right] + \frac{1}{3}\left[\lambda - \frac{1+\lambda}{2}\right] + \frac{1}{3}\left[\lambda - \frac{\lambda+0}{2}\right] = \frac{4}{9}\lambda - \frac{2}{9} < \frac{4}{9} - \frac{2}{9}\lambda.$$

4. One candidate is one vote ahead of the second candidate, and the third one is farther behind. The expected benefit of voting for one's favorite candidate is

$$\frac{1}{3}[1-1] + \frac{1}{3}\left[\frac{1+\lambda}{2} - \lambda\right] + \frac{1}{3}\left[\frac{1+0}{2} - 0\right] = \frac{2-\lambda}{6}.$$

The expected benefit of voting for one's second-ranked candidate is

$$\frac{1}{3}\left[\lambda - \lambda\right] + \frac{1}{3}\left[\frac{1+\lambda}{2} - 1\right] + \frac{1}{3}\left[\frac{\lambda+0}{2} - 0\right] = \frac{2\lambda - 1}{6} < \frac{2-\lambda}{6}.$$

5. One candidate is one vote ahead, and the two other ones are tied one vote behind. The expected benefit of voting for one's favorite candidate is

$$\frac{1}{3}\left[1 - 1\right] + \frac{1}{3}\left[\frac{1+\lambda}{2} - \lambda\right] + \frac{1}{3}\left[\frac{1+0}{2} - 0\right] = \frac{2-\lambda}{6}.$$

   The expected benefit of voting for one's second-ranked candidate is

$$\frac{1}{3}\left[\lambda - \lambda\right] + \frac{1}{3}\left[\frac{1+\lambda}{2} - 1\right] + \frac{1}{3}\left[\frac{\lambda+0}{2} - 0\right] = \frac{2\lambda - 1}{6} < \frac{2-\lambda}{6}.$$

Thus, in all of these cases, voting for one's most preferred candidate leads to a larger utility gain than voting for one's second-ranked candidate, and therefore the probabilities of the pivot events do not matter: In expectation, a vote for one's second-ranked candidate is dominated by a vote for one's top candidate, and therefore cannot occur in any *ABC* equilibrium. □

Observe that the symmetry of the equilibrium (in terms of the expected number of votes received by the candidates) is essential in the last part of the proof because there are, of course, instances in which voting for one's second-ranked candidate is better; for example, if one's second- and third-ranked candidates are tied, while one's top-candidate is more than a vote behind. The fact that the equilibrium is symmetric ensures that these cases are exactly as probable as that one's first- and third-ranked candidates are tied, and that the second-ranked candidate is behind, so that, in expectation, it is better to vote for one's top candidate.

It is interesting to relate our result that there is no strategic voting in multi-candidate elections when voting is costly to Krishna and Morgan (2012). They analyze a model in which voters are differentially informed and may either be forced to vote, or may decide whether to vote. If voting is mandatory, voters may have to randomize their vote and will sometimes vote strategically for a candidate that their private information does not indicate as the best candidate, because they have to condition on the event that their vote is pivotal for the outcome of the election. This is the famous swing voter's curse (Feddersen and Pesendorfer, 1996). However, if voters have a cost of voting and can decide not to vote, then, rather than voting strategically against one's signal, it is actually better to abstain.

Consequently, strategic voting (in the sense of voting against the candidate indicated by one's private information) disappears in Krishna and Morgan (2012).

Similarly, in our model, strategic voting may occur in equilibrium if participation were exogenously imposed, or if the cost of voting was so low that all citizens would have a positive expected net benefit from participation, even if all other citizens participate. However, if there are sufficiently many voters, not all voters will participate in equilibrium, and those who do must be indifferent between voting for their top candidate and abstaining. But if this is true, voting for one's second-ranked candidate must lead to a lower gross utility gain from voting, and is thus less attractive, implying that citizens would rather abstain than vote for their second-ranked candidate.

5 Discussion and conclusion

In this article, we have developed a theory of voting in elections with three candidates when voting is costly. We have shown that there are two fundamentally different types of equilibria.

In the first one, only two candidates receive a positive expected number of votes; these relevant candidates can be any subset of two from the candidate set. Moreover, the only citizens who vote in this equilibrium with positive probability are the ones who are maximally motivated because they rank one of the relevant candidates highest and the other one lowest.

In the second type of equilibrium, all three candidates receive the same expected number of votes and win with probability 1/3 each. Here, it is possible that all citizen preference types participate in the election with positive probability. While there are multiple equilibria that differ with respect to the participation rate of the different subgroups, the overall participation rate is the same in every equilibrium. Furthermore, all equilibria share the feature that all voters, in equilibrium, vote sincerely.

How would our results change if there were more than three candidates? It is quite clear that the two candidate equilibrium type would extend to any setting with more than three candidates: Any subset of two candidates can be the set of the relevant candidates, and with a sufficiently large electorate, the only citizens to vote would be those who are maximally motivated because they rank one of the relevant candidates highest and his opponent lowest.

How does the three-candidate equilibrium generalize to a scenario with more than three candidates? Intuitively, there is likely to be an "all-candidate equilibrium" in which all candidates receive the same expected number of votes (and thus win with equal probability), and where symmetry ensures that there is no strategic voting (i.e., because all ties are equally probable, voting for one's favorite candidate yields a higher expected utility than voting for any lower-ranked candidate).

What is somewhat more speculative is whether there are any equilibria in which the set of relevant candidates consists of 3 or more candidates, but is a strict subset of the set of all candidates. If such an equilibrium exists, the citizens who vote with positive probability are likely those who are maximally motivated to participate because they rank one of the relevant candidates highest and all the competitors (among the relevant candidates) lowest.

6 Appendix

Lemma 1. *For each $c \in (0, 0.5)$, there is exactly one pair (v_A, v_B) that satisfies equations* (1)*and* (3)*. Moreover $v_A = v_B$ for every value of c, and v_A and v_B are decreasing in c.*

Proof. Recall that the formulas for expected benefits are given by (1)and (3). Using formula 9.6.10 from Abramowitz and Stegun (1964), these equations are equivalent to

$$\frac{I_0(2\sqrt{v_A v_B})}{e^{v_A+v_B}} + \frac{\sqrt{v_B}}{\sqrt{v_A}} \frac{I_1(2\sqrt{v_A v_B})}{e^{v_A+v_B}} =: \Sigma_{11} + \Sigma_{21} \tag{22}$$

and

$$\frac{I_0(2\sqrt{v_A v_B})}{e^{v_A+v_B}} + \frac{\sqrt{v_A}}{\sqrt{v_B}} \frac{I_1(2\sqrt{v_A v_B})}{e^{v_A+v_B}} =: \Sigma_{12} + \Sigma_{22}, \tag{23}$$

where I_0 and I_1 are modified Bessel functions of order 0 and 1. Since the (22) and (23) are equal to $2c$, it follows that

$$\frac{I_0(2\sqrt{v_A v_B})}{e^{v_A+v_B}} + \frac{\sqrt{v_B}}{\sqrt{v_A}} \frac{I_1(2\sqrt{v_A v_B})}{e^{v_A+v_B}} = \frac{I_0(2\sqrt{v_A v_B})}{e^{v_A+v_B}} + \frac{\sqrt{v_A}}{\sqrt{v_B}} \frac{I_1(2\sqrt{v_A v_B})}{e^{v_A+v_B}}.$$

Since, for positive values, modified Bessel functions are strictly positive, and since $\Sigma_{11} = \Sigma_{12}$, it follows that $\Sigma_{21} = \Sigma_{22}$, which implies that

$$\frac{\sqrt{v_B}}{\sqrt{v_A}} = \frac{\sqrt{v_A}}{\sqrt{v_B}},$$

and this immediately implies that $v_A = v_B$; denote this value by v. The expected benefit of voting for AC and BC types is the same and is equal to

$$\sum_{x=0}^{\infty} [\frac{1}{2} e^{-2v} \frac{v^{2x}}{x!x!} + \frac{1}{2} e^{-2v} \frac{v^{2x+1}}{x!(x+1)!}] = c. \tag{24}$$

Equation (24) can be simplified to this expression using formula 9.6.10 from Abramowitz and Stegun (1964) to yield

$$\sum_{x=0}^{\infty} [e^{-2v} \frac{v^{2x}}{x!x!} + e^{-2v} \frac{v^{2x+1}}{x!(x+1)!}] = \frac{I_0(2v) + I_1(2v)}{e^{2v}} = 2c,$$

Denote

$$f(v) = \frac{I_0(2v) + I_1(2v)}{e^{2v}}$$

Using formula 9.7.1 in Abramowitz and Stegun (1964), we have

$$\lim_{v\to 0} f(v) = 1, \qquad \text{and} \qquad \lim_{v\to\infty} f(v) = 0. \tag{25}$$

Since modified Bessel functions are strictly positive and continuous in $(0, \infty)$, this proves the existence of v for each cost value $c < 0.5$.

To show uniqueness, it is enough to prove that $f(v)$ is strictly monotone. Using formula 9.6.26 in Abramowitz and Stegun (1964), it follows that

$$\frac{d}{dv} I_0(2v) = 2I_1(2v) \tag{26}$$

$$\frac{d}{dv} I_1(2v) = 2I_0(2v) - \frac{I_1(2v)}{v} \tag{27}$$

The derivative of the $f(v)$ function, using this two formulas above, is

$$\begin{aligned} f^{'}(v) = \frac{(2I_1(2v) + 2I_0(2v) - \frac{I_1(2v))}{v})e^{2v} - 2e^{2v}(I_0(2v) + I_1(2v))}{e^{4v}} &= \\ = -\frac{I_1(2v)}{ve^{2v}} < 0, \end{aligned} \tag{28}$$

which shows that f is strictly decreasing. Thus, for each cost value $c < 0.5$, there exists a unique solution to (24). □

Lemma 2. *Equation* (20) *is a decreasing function of* v.

Proof. Let $f(v)$ denote e^{-3v} times the term in curly brackets in (20). Since $\sum_{n=0}^{x} \frac{v^n}{n!} = \frac{e^v \Gamma(x+1,v)}{x!}$, $f(v)$ can be rewritten

$$\begin{aligned} f(v) &= \left\{ \sum_{x=0}^{\infty} \left[\frac{v^{3x}}{(x!)^3}\frac{2}{3} + \frac{v^{3x+2}}{x!((x+1)!)^2}\frac{1}{3} + \frac{v^{2x+1}e^v\Gamma(x+1,v)}{x!x!(x+1)!} \right] + \sum_{x=1}^{\infty} \left[\frac{v^{2x}e^v\Gamma(x,v)}{x!x!(x-1)!} \right] \right\} e^{-3v} \\ &\equiv \left[f_1(v)\frac{2}{3} + f_2(v)\frac{1}{3} + f_3(v) + f_4(v) \right] e^{-3v}. \end{aligned} \tag{29}$$

Differentiating (29) yields

$$
\begin{aligned}
f'(v) = &\sum_{x=0}^{\infty}\left[2\frac{v^{3x+2}}{x!((x+1)!)^2}e^{-3v} + \frac{v^{3x+1}}{x!x!(x+1)!} - \frac{1}{3}\frac{v^{3x+1}}{x!((x+1)!)^2}\right] + \\
&+ \sum_{x=1}^{\infty}\left[\frac{2xv^{2x-1}e^{v}\Gamma(x,v) + v^{2x}e^{v}\Gamma(x,v) - v^{x-1}v^{2x}}{x!x!(x-1)!}\right]e^{-3v} + \\
&+ \sum_{x=0}^{\infty}\left[\frac{(2x+1)v^{2x}e^{v}\Gamma(x+1,v) + v^{2x+1}e^{v}\Gamma(x+1,v) - v^{x}v^{2x+1}}{x!x!(x+1)!}\right]e^{-3v} - \\
&- 3\left[\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{(x!)^3}\frac{2}{3} + \frac{v^{3x+2}}{x!((x+1)!)^2}\frac{1}{3} + \frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] + \sum_{x=1}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x,v)}{x!x!(x-1)!}\right]\right]e^{-3v},
\end{aligned}
\tag{30}
$$

where we use $\frac{\partial\Gamma(x,v)}{\partial v} = -v^{x-1}e^{-v}$. Simplifying the second and third lines of this expression, we get

$$
\begin{aligned}
&\sum_{x=1}^{\infty}\left[\frac{2xv^{2x-1}e^{v}\Gamma(x,v) + v^{2x}e^{v}\Gamma(x,v) - v^{3x-1}}{x!x!(x-1)!}\right] = \\
&= 2\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] + \sum_{x=1}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x,v)}{x!x!(x-1)!}\right] - \sum_{x=0}^{\infty}\left[\frac{v^{3x+2}}{x!((x+1)!)^2}\right] = \\
&= 2f_3(v) + f_4(v) - f_2(v)
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
&\sum_{x=0}^{\infty}\left[\frac{(2x+1)v^{2x}e^{v}\Gamma(x+1,v) + v^{2x+1}e^{v}\Gamma(x+1,v) - v^{x}v^{2x+1}}{x!x!(x+1)!}\right] = \\
&= \sum_{x=0}^{\infty}\left[\frac{(2(x+1)-1)v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] + \sum_{x=0}^{\infty}\left[\frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] - \sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right] = \\
&= 2\sum_{x=1}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x,v)}{x!x!(x-1)!}\right] + 2\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{x!x!x!}\right] - \sum_{x=0}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] + \\
&\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] - \sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right] = \\
&= 2f_4(v) + 2f_1(v) - \sum_{x=0}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right] + f_3(v) - \sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right],
\end{aligned}
\tag{32}
$$

where we use $\Gamma(x+1,v) = \frac{x!\Gamma(x,v)}{(x-1)!} + \frac{v^x}{e^v}$, and $\Gamma(1,v) = e^{-v}$. Therefore,

$$\begin{aligned}
f'(v) = e^{-3v}[&\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!} - \frac{1}{3}\frac{v^{3x+1}}{x!((x+1)!)^2}\right] + 2f_2(v) + 2f_3(v) + f_4(v) - \\
&- f_2(v) + 2f_4(v) + 2f_1(v) - \sum_{x=0}^{\infty}\left[\frac{v^{2x}e^v\Gamma(x+1,v)}{x!x!(x+1)!}\right] + f_3(v) - \sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right] - \\
&- 2f_1(v) - f_2(v) - 3f_3(v) - 3f_4(v)] = \\
&= -\frac{1}{3}\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!((x+1)!)^2}\right]e^{-3v} - \sum_{x=0}^{\infty}\left[\frac{v^{2x}e^v\Gamma(x+1,v)}{x!x!(x+1)!}\right]e^{-3v} < 0.
\end{aligned} \tag{33}$$

□

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