Frame-invariant proper-force

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Here we try to find "the best joints for carving up" the phenomenon of accelerated-motion so as to obtain (i) the least need for extended-networks of synchronized-clocks as well as (ii) the greatest frame-independence. The acceleration four-vector's invariant magnitude, and a number of other quantities from the traveler's point of view, show promise for broadening student understanding (and perhaps even practical application) of accelerated motion perspectives at both low and high speeds.

CONTENTS

I. INTRODUCTION

Relativists have long expressed unhappiness with coordinate-acceleration and coordinate-force (for good reason[1,2]), but have also pointed out that general-relativity makes a case for the local-validity of Newton's laws in all frames[3–5] provided that we consider geometric (frame-dependent or "connection-coefficient") forces as well as proper-forces whenever we find ourselves in a non-"free-float" trajectory[6]. In this paper we explore an approach to accelerated motion designed to be: (i) the most frame-independent, and (ii) the least in need of synchronized-clock arrays. These latter might be difficult to come by on accelerated platforms and in curved spacetime.

The first proper-time derivative of an accelerated traveler's 4-vector position has lightspeed c as its invariant magnitude. Here we simply define simultaneity using bookkeeper coordinates and then examine the second proper-time derivative of position, as seen from the proper reference-frame[3] of that accelerated traveler.

In the process we show: (a) that the distinction between proper and geometric forces is already quite useful for introductory physics, (b) that via the metric equation a lot can be done with only a single extended map-frame of yardsticks and synchronized clocks, and (c) that the traveler's view of anyspeed-acceleration is less frame-variant than the map perspective. We also exploit the frame-invariance of proper-force in an empirical observation exercise on the electrostatic origin of magnetism, which provides some visceral experience with length-contraction at the same time.

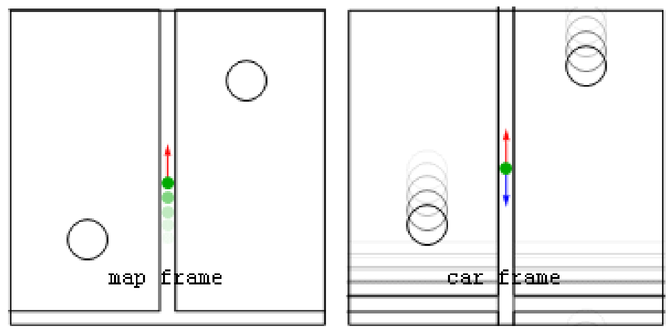

FIG. 1. Two views of proper (red) and geometric (dark blue) forces on leaving a stop sign.

II. FRAME DEPENDENCE & SYNCHRONY

The value of frame-independence in the modeling of relativistic-motion and curved-spacetime goes without saying. The frame-invariance of lightspeed c (the magnitude of the **velocity 4-vector** $U^\lambda \equiv dX^\lambda/d\tau$) has been central to our understanding of spacetime from the beginning[7]. Proper-time (the magnitude of the **displacement 4-vector** X^λ) is finding increasing use by introductory text authors as we speak.

The Lorentz-transform view of proper-time, of course, is that it is time-passing on the synchronized clocks of a tangent but co-moving free-float-frame in flat spacetime. The metric equation's view of proper-time is simpler but more general, i.e. as a quantity measured on a single clock under any conditions i.e. accelerated or not, in curved space-time or not.

Proper-time is frame-invariant in the sense that its

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TABLE I. Accelerated-motion definitions in flat (3+1)D spacetime. Note that acceleration/force magnitudes are spacelike, while the others are timelike along a traveler's worldline, and that we've defined x and y as spatial coordinates || and $\perp$ to the direction of proper-acceleration 3-vector $\vec{\alpha}$.

4-vector	magnitude	time-components	|| to spatial 3-vector $\vec{\alpha}$	$\perp$ to spatial 3-vector $\vec{\alpha}$
power/force	$\Sigma F_o \equiv \frac{dp_o}{d\tau}$	$\frac{\mathrm{P}}{c} = \left(\frac{1}{c}\right)\frac{dE}{d\tau}$	$F_{\vert\vert} \equiv \frac{dp_{\vert\vert}}{d\tau}$	$F_\perp \equiv \frac{dp_\perp}{d\tau}$
acceleration	$\alpha = \frac{\Sigma F_o}{m}$	$c\frac{d\gamma}{d\tau} = \frac{\mathrm{P}}{mc} = \gamma\frac{P}{mc}$	$\frac{dw_{\vert\vert}}{d\tau} = \frac{F_{\vert\vert}}{m} = \gamma\frac{f_{\vert\vert}}{m}$	$\frac{dw_\perp}{d\tau} = \frac{F_\perp}{m} = \gamma\frac{f_\perp}{m}$
energy/momentum	mc	$\frac{E}{c} = \gamma mc$	$p_{\vert\vert} = mw_{\vert\vert}$	$p_\perp = mw_\perp$
velocity	c	$c\gamma \equiv c\frac{dt}{d\tau} = \frac{E}{mc}$	$w_{\vert\vert} \equiv \frac{dx}{d\tau} = \frac{p_{\vert\vert}}{d\tau} = \gamma v_{\vert\vert}$	$w_\perp \equiv \frac{dx}{d\tau} = \frac{p_\perp}{d\tau} = \gamma v_\perp$
coordinate	$c\tau$	ct	x	y

TABLE II. Relationship between variables: Here τ is traveler-time elapsed from "turnaround" (when $\gamma \equiv \gamma_o$) for as long as proper acceleration $\vec{\alpha}$ doesn't change, and $\gamma_\pm \equiv \sqrt{(\gamma_o \pm 1)/2}$. The right arrow $\rightarrow$ denotes the non-relativistic limit.

4-vector	invariant	time-components/c	|| to spatial 3-vector $\vec{\alpha}$	$\perp$ to spatial 3-vector $\vec{\alpha}$
accel.	α	$\frac{d\gamma}{d\tau} = \frac{\alpha}{c}\gamma_+ \sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \left(\frac{\alpha}{c}\right)^2\tau$	$\frac{dw_{\vert\vert}}{d\tau} = \alpha\cosh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \alpha$	$\frac{dw_\perp}{d\tau} = \alpha\gamma_- \sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow 0$
velocity	c	$\gamma = \gamma_-^2 + \gamma_+^2\cosh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow 1$	$w_{\vert\vert} = c\gamma_+ \sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \alpha\tau$	$w_\perp = c\sqrt{2}\gamma_- \cosh\left[\frac{\alpha\tau}{2c\gamma_+}\right]^2 \rightarrow v_\perp$
coord.	τ	$t = \gamma_-^2\tau + \gamma_+^{3/2}\frac{c}{\alpha}\sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \tau$	$x = \gamma_+^2\frac{2c^2}{\alpha}\sinh\left[\frac{\alpha\tau}{2c\gamma_+}\right]^2 \rightarrow \frac{1}{2}\alpha\tau^2$	$y = \gamma_+\gamma_- c\tau + \gamma_+\frac{c^2}{2\alpha}\sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow v_\perp\tau$

value may be agreed upon using *any* general-relativistic book-keeper coordinates that we choose. These book-keeper coordinates are alone used to define extended simultaneity (i.e. the global place and time of events), while the frame-invariance of proper-time drastically improves the transformation-properties of quantities differentiated with respect to it. The proper-velocity 3-vector $\vec{w} \equiv d\vec{x}/d\tau$ (which unlike coordinate velocity $\vec{v} \equiv d\vec{x}/dt$ adds vectorially with appropriate rescaling of the "out-of-frame" component) and the proper-acceleration 3-vector (discussed here) are cases in point.

The topic of this paper is in particular the frame-invariant magnitude of the **acceleration 4-vector**, in standard notation[3]:

$$A^\lambda := \frac{DU^\lambda}{d\tau} = \frac{dU^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu}U^\mu U^\nu \tag{1}$$

and uses for this vector's components (as power/force) when they are multiplied by frame-invariant rest-mass m. Here free-float or geodesic trajectories have $A^\lambda = 0$, so that we can think of coordinate acceleration $dU^\lambda/d\tau$ as a sum of proper and geometric terms, the latter depending on local space-time curvature through the 64-component affine-connection $\Gamma^\lambda_{\mu\nu}$ which gives rise to "apparent" forces in accelerated coordinate-systems and curved space-time. As usual greek indices run from 0 (time-component) to 3 (space-components) and obey the Einstein summation convention when repeated in a product. Because this proper-acceleration four-vector becomes purely space-like in a frame instantaneously-comoving with our traveler, its physical interpretation is simply the proper-force/mass felt to be "pressing on" our traveler, as well as the 3-vector proper-acceleration[8–10] $\vec{\alpha}$ seen by free-float observers in the co-moving frame.

In addition to a preference here for frame-invariance, the concept of simultaneity is a messy one in accelerated frames (e.g. using radar-time methods[11]) as well as in curved spacetime. Hence we take a "metric-first" approach to kinematics here by choosing a single "book-keeper" coordinate-system in terms of which both "map-time" t and "map-position" $\vec{x}$ are measured. Siimultaneity will be defined in terms of synchronized (but not always local e.g. in the case of Schwarzschild "far-time") clocks in this book-keeper frame.

In addition purely space-like vectors, along with frame-invariants, may be described as "synchrony-free" to use a word employed by William Shurcliff when discussing proper-velocity[12,13] $\vec{w} \equiv d\vec{x}/d\tau = \vec{p}/m$. These are quantities whose operational-definition does not require an extended network of synchronized-clocks, something of limited availability around gravitational-objects (like earth), and impossible to find on platforms (like spaceships) undergoing accelerated motion. The time-like energy of a moving object via its dependence on the Lorentz-factor $\gamma \equiv dt/d\tau$ is (like "mixed objects" such as coordinate-velocity $\vec{v} \equiv d\vec{x}/dt$) not synchrony-free, because it requires map-time t data from clocks at multiple locations.

The "traveler's point of view" that we argue offers the most direct way to communicate about an accelerated traveler is the frame that Misner, Thorne and Wheeler[3] refer to as "the proper reference frame of an accelerated traveler". One can always convert these to expressions written in terms of bookkeeper variables like map-time t and coordinate-velocity $\vec{v}$, but we show here that the algorithmically-simplest way to describe the effects of the local space-time metric on motion (following the critera above) involves the parameterization described here.

III. LOW SPEED APPLICATIONS

For applications at low speed, telling students about proper-forces as distinct from geometric-forces (that act on every ounce of a object's being) is a good start in preparing them for the value of Newton's laws in both free-float and accelerated frames. The simple example of a car leaving a stop-sign is illustrated by the animation[14] screen capture in Fig. 1, which shows the red proper-force seen by observers in both frames as pressing on the driver's back while the car accelerates. This of course is canceled *only in the car frame* by a geometric force which (like gravity) acts on every ounce of the driver's being.

We also recommend telling them that time itself is dependent on a given clock's location and state of motion, with the "speed of map-time" relative to a traveler's clock (i.e. $dt/d\tau$) an important clue to the traveling-clock's energy (potential and/or kinetic). These things may be done at the outset, followed by the assertion that introductory physics texts by default refer to map-time (t) since traveler-time (τ) differences at low speed are negligible, and they traditionally treat gravity as another proper-force even though we now know that it too is a geometric-force, caused not by a traveler's motion but by gravity's curvature of space-time around massive objects. Traditional treatments often further focus only on application of Newton's laws from "inertial-frame" perspectives, in which case geometric-forces (other than gravity) can be ignored. With these minor "metric-first" changes to the introduction, traditional introductory physics treatments remain perfectly self-consistent and intact.

IV. BRINGING IN THE METRIC

In order for teachers to feel grounded when addressing introductory issues in context of an intimidating Riemann-geometry framework, it is crucial that the consequences of their assumptions be easy to *for them* to verify. Thankfully the metric-equation, unlike Lorentz transforms, requires only one bookkeeper frame whose time-variable may (or may not) be possible to associate with time's passage on clocks synchronized across a meaningful region of spacetime.

Our first step, namely choosing the metric parameterization to describe a specific problem, is especially important because it defines both the meaning of measurments and our (perhaps implicit) definition of simultaneity. This is good news for introductory teachers, since *its bad enough to be talking about different times on different clocks, without having to also be juggling multiple definitions of simultaneity.*

For general relativity applications in a world where time is measured on watches, and distances are measured with yardsticks, whenever possible we will seek metric parameterizations whose time-variable corresponds to clocks that can be synchronized. We therefore follow Newton in flat-space settings by choosing a set of free-float (e.g. inertial or un-accelerated) frame variables like coordinate-time t and coordinate-position $\vec{x}$ to describe accelerated motion.

As teachers, once we have a metric and a corresponding definition of what simultaneity means, we are back on familiar territory. The caveat is that frame-independence may be attributed only to four-vector magnitudes, and no longer to time-intervals, distances, or rates of momentum-change. For the flat-space (1+1)D case, for instance, the proper time-interval $\delta\tau$ and derivatives with respect to τ yield the following frame-invariant magnitudes:

$$(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2, \tag{2}$$

with the lightspeed constant c

$$c^2 = \left(c\frac{\delta t}{\delta\tau}\right)^2 - \left(\frac{\delta x}{\delta\tau}\right)^2, \tag{3}$$

and proper-acceleration α:

$$-\alpha^2 = \left(c\frac{\delta^2 t}{\delta\tau^2}\right)^2 - \left(\frac{\delta^2 x}{\delta\tau^2}\right)^2. \tag{4}$$

Given this, the challenge of finding the integrals of the motion e.g. for constant acceleration is much like that challenge of showing that $x = \frac{1}{2}at^2$ via the same derivative relations, but using Newton's assumptions that coordinate-intervals and coordinate-acceleration $a \equiv \frac{\delta^2 x}{\delta t^2}$ are frame-invariant. Simple-form versions of the metric-based integrals are tabulated in context of the discussions to follow.

V. ANY SPEED APPLICATIONS

Table I defines notation for describing accelerated motion in (3+1)D flat spacetime. Table II shows the instantaneous relationship between these varables (also at low speed), as parameterized by the "traveler-time τ and Lorentz-factor γ_o from turnaround" were the instantaneous proper acceleration to remain constant (cf. Appendix A). In both tables, only values in the "time-components" column rely on synchrony between map-frame clocks at more that one location. Values in the spatial-coordinate columns to the right are synchrony-free, while values in the column to the left are frame-invariant as well.

Of course a map-frame observer's measurements of map-position as a function of map-time (along with deliverables like inferred coordinate-forces) will be parameterized in terms of synchrony-dependent map-time instead of frame-invariant traveler-time. Although map-frame observers can calculate synchrony-free quantities like momentum and proper-velocity in terms of synchrony-dependent parameters, it will take extra steps going to there from what they measure, and perhaps also going from there to what they want to infer.

If on the other hand the traveler measures their "felt proper-acceleration", as well as the rates at which they pass "map-landmarks" on their route, the equations to

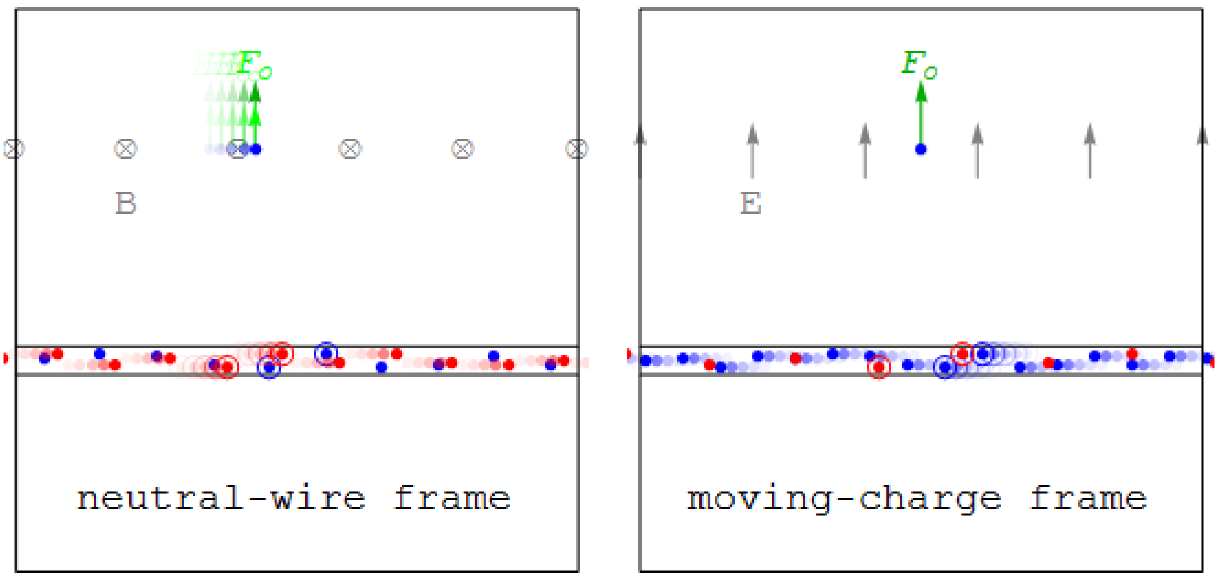

FIG. 2. Two views of proper force on a moving charge from a neutral current-carrying wire, with 40 millisecond time-steps between after-images. The shorter light-arrow in the wire-frame is the coordinate-force $f \equiv dp/dt = F_o/\gamma_\perp$. Effects of the depicted forces on the charge-motion are ignored, as is the B-field in the moving-charge frame which has no effect.

everything else are simpler and organically related as shown in Table II. Plus, everything that the traveler measures and reports on (except for elements in the time-component column of the table) will either be synchrony-free or frame-invariant.

The connection between the traveler control-parameters and Table II is reinforced if we imagine long-distance travel in a spacecraft with traveler control over thust (i.e. proper-force) magnitude and direction. The table connects proper-acceleration's magnitude and direction to instantaneous values of "proper-time from turnaround" and $v_\perp$, which in turn are related via the same table to navigational objectives (like the x and y values for the turnaround-point itself).

Although variable-rearrangement is complicated relative to the low-speed case via "gamma-factor" coupling between directions, a wide range of puzzles involving high-speed navigation in free-space may be addressed with this table. Of most interest perhaps to beginning students are of course the possibilities that relativity opens up for constant proper-acceleration (e.g. 1 "gee") round-trips between distant locations. Not only are these equations even simpler than the (3+1)D case, but the real limiting factor (namely the payload to launch-mass ratio) is quite simple to calculate as well.

A practical classroom application of the frame-independence of proper-force in this context involves an empirical observation exercise for students interested in the electrostatic origins of the magnetic force between moving charges. In essence, students are asked to take data in real time from animations (cf. Fig. 2) showing neutral-wire and moving-charge perspectives on the proper-force felt by the moving charge[15].

Simple ratios (in either space or time) allow students to quantify the length-contraction, the currents and charge densities from these two perspectives, and a variety of other physical quantities. In order to see significant differences in these quantities from the two perspectives, of course, charge velocities have to be relativistic. Since velocities are also perpendicular to observed forces, a significant difference between the coordinate-force observed in the neutral wire frame, and the proper-force felt by the moving charge, also shows up.

VI. DISCUSSION

As mentioned above, extended arrays of synchronized clocks are difficult to come by in curved spacetime (cf. the relativistic corrections needed to make global positioning estimates accurate). They are perhaps even more difficult to come by on accelerated platforms (cf. discussions of accelerated-frame "Rindler coordinates").

"Lorentz-transform first" analyses of any-speed motion of course require at least two relativistically co-moving frames of synchronized clocks. No wonder accelerated motion is of little interest in that context.

"Metric-first" approaches require only one such map-frame, since proper-time on traveler clocks is a frame-invariant. The integrals of constant proper-acceleration, especially in (1+1)D e.g. as $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$ where $\eta \equiv \sinh[\frac{w}{c}]$, are also quite manageable. As shown Table III, which is a (1+1)D version of Tables I and II combined, the general *magnitude-inequality* between coordinate-force $\vec{f} \equiv d\vec{p}/dt$ (where we are using

TABLE III. Relationships between variables for acceleration in (1+1)D flat-spacetime: Here τ is traveler-time elapsed from "turnaround" for as long as proper acceleration α doesn't change. The right arrow $\rightarrow$ shows simplification when $\alpha\tau \ll c$.

4-vector	invariants	time-components/c	space-components
acceleration	$\alpha \equiv \frac{\Sigma F_o}{m}$	$\frac{d\gamma}{d\tau} = \frac{\mathrm{P}}{mc^2} = \frac{\gamma P}{mc^2} = \frac{\alpha}{c}\sinh\left[\frac{\alpha\tau}{c}\right] \rightarrow \left(\frac{\alpha}{c}\right)^2\tau$	$\frac{dw}{d\tau} = \frac{\Sigma F}{m} = \frac{\gamma\Sigma f}{m} = \alpha\cosh\left[\frac{\alpha\tau}{c}\right] \rightarrow \alpha$
velocity	c	$\gamma \equiv \frac{dt}{d\tau} = \frac{E}{mc^2} = \sqrt{1+(\frac{w}{c})^2} = \cosh\left[\frac{\alpha\tau}{c}\right] \rightarrow 1$	$w \equiv \frac{dx}{d\tau} = \frac{p}{m} = \gamma v = c\sinh\left[\frac{\alpha\tau}{c}\right] \rightarrow \alpha\tau$
coordinate	τ	$t = \frac{c}{\alpha}\sinh\left[\frac{\alpha\tau}{c}\right] \rightarrow \tau$	$x = \frac{c^2}{\alpha}\left(\cosh\left[\frac{\alpha\tau}{c}\right] - 1\right) \rightarrow \frac{1}{2}\alpha\tau^2$

TABLE IV. Relationship between variables for acceleration in (1+1)D gravity: Here τ is traveler-time from "turnaround" for fixed proper acceleration, while as usual $g \equiv \frac{GM}{r^2}$ and $r_s \equiv \frac{2GM}{c^2}$. Here $\simeq$ neglects changes in g and $\rightarrow$ assumes that $\alpha\tau \ll c$.

4-vector	invariants	time-components/c	space-components
acceleration	$\alpha \equiv \frac{\Sigma F_o}{m}$	$\frac{d\gamma}{d\tau} = \frac{\mathrm{P}}{mc^2} = \frac{\gamma P}{mc^2} \simeq \frac{\alpha-g}{c}\sinh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow \left(\frac{\alpha}{c}\right)^2\tau$	$\frac{dw}{d\tau} = \frac{\Sigma F}{m} = \frac{\gamma\Sigma f}{m} \simeq (\alpha-g)\cosh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow (\alpha-g)$
velocity	c	$\gamma \equiv \frac{dt}{d\tau} = \frac{E}{mc^2} == \sqrt{\frac{1+(\frac{w}{c})^2}{1-\frac{r_s}{r}}} \rightarrow \sqrt{\frac{1}{1-\frac{r_s}{r}}}$	$w \equiv \frac{dr}{d\tau} = \frac{p}{m} = \gamma v \simeq c\sinh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow (\alpha-g)\tau$
coordinate	τ	$t \simeq \frac{c}{\alpha-g}\sinh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow \tau$	$r \simeq \frac{c^2}{\alpha-g}\left(\cosh\left[\frac{(\alpha-g)\tau}{c}\right] - 1\right) \rightarrow \frac{1}{2}(\alpha-g)\tau^2$

the relativistic momentum $\vec{p}$) and proper-acceleration $\vec{\alpha}$, namely $|\Sigma\vec{f}| \leq |m\vec{\alpha}|$, also becomes the more familiar-looking *signed-equality* $\Sigma f = m\alpha$.

The approach also works in curved-spacetime. Table IV illustrates for the "radial-only" Schwarzschild case using the exact Lorentz-factor from Hartle[4], even though the integration (even in the Newtonian case) is simplest if we can ignore variations of g with r. The competition between velocity-related, and gravitational, time-dilation e.g. for GPS-system orbits is nonetheless quite clear.

Just as in flat-spacetime, the metric equation in general associates a set of $\{t, x, y, z\}$ bookkeeper-coordinates with each event. In the Schwarzschild case, however, clocks can only be synchronized at fixed-r. Hence a radar-time model[11] (or some such) of extended-simultaneity might be needed to answer the question "What time is it now at radius r?"

The good news for the case of Schwarzschild (and other steady-state metrics) is that $\gamma \equiv \frac{dt}{d\tau} = \frac{E}{mc^2}$ can be defined regardless of one's model for extended-simultaneity. Although in general momentum $\vec{p} \equiv \frac{d\vec{x}}{d\tau}$ remains synchrony-free, definitions of synchrony-dependent energy may encounter significant complication when the bookkeeper time-derivative $\frac{dt}{d\tau}$ becomes dependent on extended-simultaneity.

We further show that frame-invariance (where all frames agree) is quite valuable for illustrations. The synchrony-free nature of proper-velocity and momentum, as well as of force-components described as derivatives using proper-time τ instead of map-time t, also lead to a simpler and more robust picture of accelerated motion when examined from the point of view of the accelerated traveler.

ACKNOWLEDGMENTS

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Appendix A: derivations

The entries in Table II suggest a robust (3+1)D generalization of hyperbolic velocity-angle (or rapidity) for high-speed motion with respect to a "free-float-frame", namely that $\eta \equiv \sqrt{2/(\gamma_o - 1)}\alpha\tau/c$ where γ_o is the "turnaround" Lorentz-factor at $\tau = 0$, so we should say a few words here about their origin. One might for example begin with the equations for coordinate acceleration in terms of proper acceleration from Levy[16], written in the form:

$$\vec{a} \equiv \begin{bmatrix} a_{||} \\ a_{\perp} \end{bmatrix} = \begin{bmatrix} (\frac{v_{||}}{v})^2 + \gamma(\frac{v_{\perp}}{v})^2 \\ -(\gamma-1)\frac{v_{||}v_{\perp}}{v^2} \end{bmatrix} \frac{\alpha}{\gamma^3} \tag{A1}$$

.

Note that in this form the coordinate-velocity ratios can be replaced by proper-velocity ratios, making the equation one which simply relates coordinate-acceleration $\vec{a}$ components to the proper-acceleration 3-vector $\vec{\alpha}$ through the *fractional-velocity* components parallel and perpendicular to $\vec{\alpha}$.

Using this in the expression for 4-vector acceleration in terms of coordinate velocity and acceleration 3-vectors, namely:

$$A \equiv \frac{dU}{d\tau} = \begin{bmatrix} \frac{cd\gamma}{d\tau} \\ \frac{d\vec{w}}{d\tau} \end{bmatrix} = \begin{bmatrix} \gamma^4 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right) \\ \gamma^2\vec{a} + \gamma^4 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right) \frac{\vec{v}}{c} \end{bmatrix} \tag{A2}$$

one can obtain the energy-integral differential equation:

$$\frac{c^2}{\alpha}\ddot{\gamma} = \left(\frac{1+\gamma+(\frac{c}{\alpha}\dot{\gamma})^2}{1+\gamma}\right)\alpha \tag{A3}$$

where the dot refers to differentiation with respect to proper-time τ, and $w_{||} = (c^2/\alpha)d\gamma/d\tau$. This integrates pretty quickly to the contents of Table II. Table III entries then follow directly for the (1+1)D case by letting $\gamma_o \rightarrow 1$.