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PARTIALLY HYPERBOLIC SETS WITH POSITIVE MEASURE AND *ACIP* FOR PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. In [23] Xia introduced a simple dynamical density basis for partially hyperbolic sets of volume preserving diffeomorphisms. We apply the density basis to the study of the topological structure of partially hyperbolic sets. We show that if Λ is a strongly partially hyperbolic set with positive volume, then Λ contains the global stable manifolds over $\alpha(\Lambda^d)$ and the global unstable manifolds over $\omega(\Lambda^d)$.

We give several applications of the dynamical density to partially hyperbolic maps that preserve some *acip*. We show that if f is essentially accessible and μ is an *acip* of f, then $\mathrm{supp}(\mu) = M$, the map f is transitive, and μ-a.e. $x \in M$ has a dense orbit in M. Moreover if f is accessible and center bunched, then either f preserves a smooth measure or there is no *acip* at all.

1. **Introduction.** Let M be a closed, connected, n-dimensional manifold, $r > 1$ and $f \in \mathrm{Diff}^r(M)$ be a C^r diffeomorphism on M. A compact f-invariant subset $\Lambda \subset M$ is said to be *partially hyperbolic* if there are a continuous Tf-invariant splitting of $T_xM = E^s_x \oplus E^c_x \oplus E^u_x$ for every $x \in \Lambda$, a smooth Riemannian metric g on M for which we can choose continuous positive functions $\nu, \tilde{\nu}, \gamma$ and $\tilde{\gamma}$ on Λ with $\nu, \tilde{\nu} < 1$ and $\nu < \gamma \le \tilde{\gamma}^{-1} < \tilde{\nu}^{-1}$ such that, for all $x \in \Lambda$ and for all unit vectors $v \in E^s_x$, $w \in E^c_x$ and $v' \in E^u$,

$$\|Tf(v)\| \le \nu(x) < \gamma(x) \le \|Tf(w)\| \le \tilde{\gamma}(x)^{-1} < \tilde{\nu}^{-1}(x) \le \|Tf(v')\|. \qquad (1)$$

The notation here is taken from [10]. Such a metric is called *adapted* (see [16]). If both E^s and E^u are nontrivial, then we say Λ is strongly partially hyperbolic. In particular the map f is called a (strongly) partially hyperbolic diffeomorphism if M itself is a (strongly) partially hyperbolic set. It is well known that E^s and E^u are uniquely integrable and tangent to the stable lamination $\mathcal{W}^s$ and the unstable lamination $\mathcal{W}^u$ respectively.

In [23] Xia introduced a simple dynamical density basis for general partially hyperbolic sets. Namely let $\delta > 0$, $W^s(x, \delta)$ be the local stable manifold through $x \in \Lambda$ of radius δ. Let $B^s_n(p) = f^n W^s(f^{-n}p, \delta)$ for each $p \in \Lambda$ and $n \ge 0$. The collection of sets $\mathcal{S} = \{B^s_n(p) : n \ge 0, p \in \Lambda\}$ is called the *stable basis* on Λ (see

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[23]). Let $A \subset \Lambda$ be a measurable subset. A point $p \in A$ is said to be an $\mathcal{S}$-density point of A if

$$\lim_{n\to\infty} \frac{m_{W^s(p)}(B_n^s(p)\cap A)}{m_{W^s(p)}(B_n^s(p))} = 1,$$

where $m_{W^s(p)}$ is the leaf-volume induced by restricting the Riemannian metric on $W^s(p)$. Let A_s^d be the set of $\mathcal{S}$-density points of A. Following [23] we have:

Proposition. *Let $r > 1$, $f \in \mathrm{Diff}^r(M)$ and Λ be a partially hyperbolic set with positive volume. For each measurable subset $A \subset \Lambda$, m-a.e. point in A is an $\mathcal{S}$-density point of A, that is, $m(A\backslash A_s^d) = 0$. In words, $\mathcal{S}$ forms a density basis.*

This simply defined density basis turns out to be useful in the study of the topological structure of (partially) hyperbolic sets. There is an extensive literature discussing the topology of (partially) hyperbolic sets. We just name a few that are closely related to the results here. Bowen showed in [5], there exists C^1 horseshoe of positive volume (It is also showed in [6] that this *fat* horseshoe can not exist among the C^2 diffeomorphisms). In [2] Alves and Pinheiro showed that for a diffeomorphism $f \in \mathrm{Diff}^r(M)$, if Λ is a partially hyperbolic set that attracts a positive volume set, then Λ contains some local unstable disk (hence Λ cannot be a horseshoe-like set). Under a much stronger setting, we can get a useful characterization that serves well for later applications. More precisely let $\alpha(x)$ be the set of accumulation points of $\{f^n x : n \le 0\}$ of $x \in M$. For $E \subset M$, let $\alpha(E)$ be the closure of $\bigcup_{x\in E} \alpha(x)$. Similarly we can define $\omega(x)$ and $\omega(E)$. We let A_u^d denote the set of unstable density points if Λ is strongly partially hyperbolic and $A^d = A_s^d \cap A_u^d$. Similarly $m(A\backslash A^d) = 0$ for each measurable subset $A \subset \Lambda$. Then we have

Theorem A. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ be a partially hyperbolic set with positive volume. Then Λ contains the global stable manifolds over $\alpha(\Lambda_s^d)$, that is, $W^s(x) \subset \Lambda$ for each $x \in \alpha(\Lambda_s^d)$.*

In particular if Λ is a strongly partially hyperbolic set with positive volume, then Λ contains the global stable manifolds over $\alpha(\Lambda^d)$ and the global unstable manifolds over $\omega(\Lambda^d)$.

The argument here relies on the bounded distortion estimates and the absolute continuity of stable and unstable laminations, which fail for C^1 maps. See [5, 20]. Although $\alpha(\Lambda^d)$ is nonempty, the volume of $\alpha(\Lambda^d)$ could be zero (even in the hyperbolic case). In fact Fisher [13] constructed several hyperbolic sets Λ with nonempty interior such that $\alpha(\Lambda^d)$ are repellers and $\omega(\Lambda^d)$ are attractors (hence their volume must be zero).

A point x is said to be backward recurrent if $x \in \alpha(x)$, and to be recurrent if $x \in \alpha(x) \cap \omega(x)$. An interesting case is when most points are recurrent. This will hold in particular if $\mu(\Lambda) > 0$ for some *absolutely continuous invariant probability* measure (*acip* for short) $\mu \ll m$. For simplicity we assume that $\Lambda = \mathrm{supp}\mu$.

Corollary B. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ a strongly partially hyperbolic set supporting some acip μ. Then Λ is bi-saturated, that is, for each point $p \in \Lambda$, the global stable manifolds and the global unstable manifolds over p lie in Λ.*

In particular we give a dichotomy for maps $f \in \mathrm{Diff}^r(M)$: either f is a transitive Anosov diffeomorphism, or each f-invariant hyperbolic set Λ is *acip*-null, that is, $\mu(\Lambda) = 0$ for every *acip* μ.

Theorem C. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$, μ be an acip and Λ be a hyperbolic set with positive μ-measure. Then $\Lambda = M$ and f is a transitive Anosov diffeomorphism on M.*

The similar result has been proved if the *acip* μ assumed to be equivalent to m (see [4, 23]). Moreover it is proved in [6] that for a C^r transitive Anosov diffeomorphism, the *acip* must have Hölder continuous density with respect to the volume and be an ergodic (indeed *Bernoulli*) measure, see Remark 2. Also note that the condition that Λ has positive μ-measure for some *acip* is nontrivial and see [13] for counter-examples.

Theorem C motivates the analogous generalizations from hyperbolic dynamics to accessible partially hyperbolic dynamics. Recall that an f-invariant measure μ is said to be *weakly ergodic* if for μ-a.e. x, $\mathcal{O}(x)$ is dense in $\mathrm{supp}(\mu)$. Following generalizes the well known result of Brin [7] to the presence of *acip*.

Theorem D. *Let $f \in \mathrm{SPH}^r(M)$ for some $r > 1$ be essentially accessible. If there exists some acip μ of f, then $\mathrm{supp}(\mu) = M$, the map f is transitive, and the acip μ is weakly ergodic. In particular $\mathcal{O}(x)$ is dense in M for μ-a.e. $x \in M$.*

In the following we assume $r = 2$ for simplicity. Burns and Wilkinson proved in [10] that if a map $f \in \mathrm{SPH}^2(M)$ is center bunched, then every measurable bi-essentially saturated set is essentially bi-saturated. Applying this to *acip* we have

Proposition. *Let $f \in \mathrm{SPH}^2(M)$ be essentially accessible and center bunched. If there exists some acip μ, then μ must be equivalent to the volume. In particular μ is ergodic.*

Note that the arguments in [10] still work if $f \in \mathrm{SPH}^r(M)$ for $r > 1$, as long as we assume *strong center bunching* (see [10, Theorem 0.3]). So our results also extend to this setting. Then applying the *cohomology theory* developed in [22], we show that the *acip* is a *smooth measure*, that is, the density $\frac{d\mu}{dm}$ of μ with respect to m is Hölder continuous on M, bounded and bounded away from zero.

Theorem E. *Let $f \in \mathrm{SPH}^2(M)$ be accessible and center bunched. If there exists some acip, then the acip must have Hölder continuous density with respect to the volume of M. In words, either f preserves some smooth measure or there is no acip for f.*

Combining the results in [12] we have the following direct corollary:

Corollary F. *The set of maps that admit no acip contains a C^1 open and dense subset of C^2 strongly partially hyperbolic and center bunched diffeomorphisms. In particular the set of maps that admit no acip contains a C^1 open and dense subset of C^2 strongly partially hyperbolic diffeomorphisms with $\dim(E^c) = 1$.*

Finally we remark that although the volume measure need not be f-invariant, there always exists some f-invariant measures. The density argument combines the dynamics of *acip* and the dynamics of volume on M. This is why most results of volume-preserving partially hyperbolic systems have parallel generalizations to the systems with *acip*.

2. **Dynamical density basis for partially hyperbolic sets.** In this section we will consider C^r diffeomorphisms for some $r > 1$ and partially hyperbolic invariant sets with positive volume. More precisely let M be a closed and connected smooth manifold. Each Riemannian metric g on M induces a (geodesic) distance d on M and a normalized volume measure m on M. Let $\mathcal{B}$ be the Borel σ-algebra of M. A Borel probability measure μ on M is said to be *absolutely continuous* with respect to m, denoted $\mu \ll m$, if $\mu(A) = 0$ for each set $A \in \mathcal{B}$ with $m(A) = 0$, and to be *equivalent to* m if $\mu \ll m$ and $m \ll \mu$. It is evident for any other Riemannian metric g' compatible with g, the induced volume of g' is equivalent to m.

Let $f \in \mathrm{Diff}^r(M)$ for $r > 1$ and Λ be a compact partially hyperbolic invariant set with positive volume. In the following we always assume that the stable subbundle E^s is nontrivial on Λ and m is the normalized volume measure on M induced by some Riemannian metric adapted to the invariant splitting (see [16]).

Since $r > 1$, it is well known that the stable bundle E^s is Hölder continuous over Λ (the Hölder exponent may be much smaller than $r-1$, see [9]) and is tangent to the stable lamination $\mathcal{W}^s$ over Λ. (A lamination over Λ is a partial foliation which may not foliate an open neighborhood of Λ, see [17].) In case that $\Lambda = M$, $\mathcal{W}^s$ turns out to be a foliation. As in the hyperbolic case, the family $\mathcal{W}^s$ is transversally absolutely continuous (see [8, 9, 1]). We use $W^s(x, \delta)$ to denote the local stable manifold through $x \in \Lambda$ centered at x and of radius δ in the stable leaf $W^s(x)$. It is worth to point out that the definition of density point given by (3) dose not depend on the choice of δ (see Remark 1). By slightly increasing ν and decreasing δ if necessary, we can assume that for each $x \in \Lambda$ the following holds:

$$\text{if } p, p' \in W^s(x, \delta), \text{ then } d(fp, fp') \leq \nu(p) d(p, p'). \tag{2}$$

In particular we have $fW^s(x, \delta) \subseteq W^s(fx, \delta \cdot \nu(x))$ for all $x \in \Lambda$.

Before moving on, let's fix some notations as in [10]. Let $S \subset M$ be a submanifold of M, m_S be the volume measure on S induced by the restricted Riemannian metric $g|_S$ on S. In particular if $S = W^s(x)$, we abbreviate the induced measure as $m_{s,x}$. Denote $m_{s,x}(A)$ the restricted submanifold measure for a measurable subset $A \subseteq W^s(x)$. This should not be confused with conditional measures. Let $\eta = \min\{\|Tf(v)\| : v \in TM \text{ with } \|v\| = 1\}$ and $\overline{\nu} = \sup_{p \in \Lambda} \nu(p)$. Clearly $0 < \eta \leq \nu(p) \leq \overline{\nu} < 1$ by compactness. For each $p \in \Lambda$ we let $p_i = f^i p$ for $i \in \mathbb{Z}$, $\nu_0(p) = 1$ and $\nu_n(p) = \nu(p_{n-1}) \cdots \nu(p_0)$ for $n \geq 1$. Let $B^s_n(p) = f^n W^s(p_{-n}, \delta)$. Since Λ is f-invariant, we have $B^s_n(p) \subset W^s(p, \delta \cdot \nu_n(p_{-n}))$.

Since each stable manifold is a C^r submanifold of the Riemannian manifold M and f is C^r for $r > 1$, the *stable Jacobian* $J^s(f, x)$ of the restricted map $Tf : T_x W^s(p) \to T_{fx} W^s(fp)$ (for $x \in W^s(p, \delta)$ and $p \in \Lambda$) is well defined and Hölder continuous with uniform Hölder exponent and Hölder constant. That is, there exist $\alpha > 0$ and $C_0 > 0$ such that for any $p \in \Lambda$ and $x, y \in W^s(p, \delta)$ we have $|J^s(f, x) - J^s(f, y)| \leq C_0 d(x, y)^\alpha$. Also there exists $J^* \geq 1$ such that $1/J^* \leq J^s(f, x) \leq J^*$ for all $x \in W^s(p, \delta)$ and $p \in \Lambda$. Decreasing δ again if necessary we assume $C_1 = \prod_{k=0}^{\infty} \frac{(1 + J^* C_0 \delta^\alpha \overline{\nu}^{k\alpha})}{(1 - J^* C_0 \delta^\alpha \overline{\nu}^{k\alpha})} < \infty$.

Let $\mathcal{S} = \{B^s_n(p) : n \geq 0, p \in \Lambda\}$ be the stable basis of Λ. It is easy to see that $\{B^s_n(p) : n \geq 0\}$ forms a nesting sequence of neighborhoods of $p \in \Lambda$ relative to $W^s(p, \delta)$ and $B^s_n(p)$ shrinks to p as $n \to \infty$. Note that the basis here is in the leafwise sense and may have infinite eccentricity. The proposition below states that the stable basis $\mathcal{S}$ behaves well in the sense of [19]:

Proposition 1. *The following properties hold for the stable basis $\mathcal{S}$:*

1. *For any $p \in \Lambda$, $m_{s,p}(B^s_n(p)) \to 0$ if and only if $n \to \infty$.*
2. *For any $k \geq 0$, there exists $c_k \geq 1$ such that $\frac{m_{s,p}(B^s_n(p))}{m_{s,p}(B^s_{n+k}(p))} \leq c_k$ for all $p \in \Lambda$, $n \geq 0$.*
3. *There exists $L \in \mathbb{N}$ such that for any $p, q \in \Lambda$, $n \geq 0$, if $B^s_{n+L}(p) \cap B^s_{n+L}(q) \neq \emptyset$, then $B^s_{n+L}(p) \cup B^s_{n+L}(q) \subseteq B^s_n(p) \cap B^s_n(q)$.*

The properties listed above appeared in [19] (in a general setting) and is named to be *volumetrically engulfing* (also see [23] for example). The proof mainly uses the distortion estimates of C^r maps.

Let $A \in \mathcal{B}_\Lambda$ be a measurable subset of Λ. Recall that a point $x \in A$ is said to be an $\mathcal{S}$-density point of A if

$$\lim_{n\to\infty} \frac{m_{s,p}(B^s_n(p) \cap A)}{m_{s,p}(B^s_n(p))} = 1. \tag{3}$$

Let A^d_s be the set of $\mathcal{S}$-density points of A.

Remark 1. For different δ's, the induced stable bases are *internested* (see [10, Lemma 2.1] for details). So the definition of $\mathcal{S}$-density point given by (3) is independent of the choice of the radius δ of the local stable manifolds and the choice of the adapted Riemannian metric on M.

For each $A \in \mathcal{B}_\Lambda$ and each $p \in \Lambda$, $A \cap W^s(p, \delta)$, the intersection of two Borel measurable subsets, is a Borel measurable subset of the submanifold $W^s(p, \delta)$. (Note that if A is Lebesgue measurable, above relation will hold for m-a.e. $p \in \Lambda$ by Fubini's Theorem.) Let us denote A^d_p the set of $\mathcal{S}$-density points of $A \cap W^s(p, \delta)$. Clearly we have $A^d_s = \bigcup_{p\in\Lambda} A^d_p$.

Proposition 2. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ be a partially hyperbolic set with positive measure. For each subset $A \in \mathcal{B}_\Lambda$, we have*

1. *for each $p \in \Lambda$, $m_{s,p}$-a.e. point in $W^s(p, \delta) \cap A$ is an $\mathcal{S}$-density point of A: $m_{s,p}(W^s(p, \delta) \cap A \backslash A^d_p) = 0$.*
2. *m-a.e. point of A is an $\mathcal{S}$-density point of A: $m(A \backslash A^d_s) = 0$.*

Moreover if $A \in \mathcal{B}_\Lambda$ is f-invariant, so is A^d_s.

Proof. The first item follows by applying Theorem 3.1 in [19] for the stable basis $\mathcal{S}$ to each intersection $A \cap W^s(p, \delta)$. Proposition 1 ensures that $\mathcal{S}$ forms a density basis in this leafwise sense.

Using the absolute continuity of the stable foliation $\mathcal{W}^s$ and the relation $A^d_s = \bigcup_{p\in\Lambda} A^d_p$, we have $m(A \backslash A^d_s) = 0$. Hence $\mathcal{S}$ also forms a density basis in the ambient sense.

For the last item, we note that each local leaf $W^s(s, \delta)$ is a C^r submanifold of M and the restriction of f on local stable manifolds is diffeomorphic onto their images. So $p \in \Lambda$ is an $\mathcal{S}$-density point of $A \cap W^s(p, \delta)$ (or equally, of A) if and only if fp is an $\mathcal{S}$-density point of $A \cap W^s(fp, \delta)$. This completes the proof. $\square$

3. **The topological structure of partially hyperbolic sets.** In this section we give some descriptions of the topological structure of partially hyperbolic sets with positive volume. As in Section 2 we let M be a closed connected manifold, $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ a partially hyperbolic set with positive volume.

Given a Borel subset $A \subset \Lambda$, we consider the family of functions η_n on Λ as

$$\eta_n(x) = m_{s,x}(B_n^s(x)\backslash A)/m_{s,x}(B_n^s(x)).$$

The following result shows the increasing occupation of an invariant set A in the local stable manifolds along the backward iterates of an $\mathcal{S}$-density point of A.

Lemma 3.1. *There exists a constant $C \geq 1$ such that given an f-invariant subset $A \in \mathcal{B}_\Lambda$, $m_{s,x_{-n}}(W^s(x_{-n},\delta)\backslash A) \leq C \cdot \eta_n(x)$ for each $x \in \Lambda$ and $n \geq 0$.*

Proof. We only need to adapt the notations in [23, Lemma 3.2], since the proof is essentially the same. Let A be an invariant subset of Λ and $x \in \Lambda$ be fixed. Let $B_n^k = f^k W^s(x_{-n},\delta)$ and $D_n^k = B_n^k \backslash A$ for each $0 \leq k \leq n$. Note that $B_n^0 = W^s(x_{-n},\delta)$ is a local stable leaf and $B_n^n = B_n^s(x)$ is an element in the stable basis $\mathcal{S}$. Then using the constant C_5 given by [23, Page 816], we have $m_{s,x_{-n}}(D_n^0) \leq C_5 \cdot \eta_n(x) \cdot m_{s,x_{-n}}(B_n^0)$. Applying $B_n^0 = W^s(x_{-n},\delta)$ and $D_n^0 = W^s(x_{-n},\delta)\backslash A$, we finish the proof with a uniform constant $C = C_5 \cdot \max_{p\in\Lambda} m_{s,p}(W^s(p,\delta))$. $\square$

Recall that $\alpha(x)$, the α-set of x, is the set of accumulation points along the backward orbit $\{x, f^{-1}x, \cdots\}$. Let $\alpha(E)$ be the closure of $\bigcup_{x\in E}\alpha(x)$. Note that for each point $x \in \Lambda$ and each subset $E \subset \Lambda$, the sets $\alpha(x)$ and $\alpha(E)$ are compact f-invariant subsets of Λ.

Theorem 3.2. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ a partially hyperbolic set with positive volume. Then Λ contains the global stable manifolds over $\alpha(\Lambda_s^d)$.*

Proof. First let us consider $y \in \alpha(x)$ for some $x \in \Lambda_s^d$. Pick a sequence of times $n_i \to +\infty$ such that $x_{-n_i} \to y$ (clearly all these points are in Λ). By Lemma 3.1 we have $m_{s,x_{-n}}(W^s(x_{-n},\delta)\backslash\Lambda) \leq C \cdot \eta_n(x)$. (Note that $\eta_n(x) \to 0$ as $n \to \infty$.) Passing to a subsequence if necessary, we can assume that $W^s(x_{-n_i},\delta) \cap \Lambda$ contains a $\frac{1}{i}$-dense subset $E_{x_{-n_i},i}$ of $W^s(x_{-n_i},\delta)$. Let $E = \limsup_{i\to\infty} E_{x_{-n_i},i} := \bigcap_{k\geq 1}\overline{\bigcup_{i\geq k} E_{x_{-n_i},i}}$. It is clear that $E \subset \Lambda$ since Λ is compact. By continuity of the stable manifolds, E contains a dense subset of $W^s(y,\delta)$, and hence $W^s(y,\delta) \subset E$. So $W^s(y,\delta) \subset \Lambda$ for each $y \in \alpha(x)$ and each $x \in \Lambda_s^d$.

Still by the compactness of Λ, $W^s(y,\delta) \subset \Lambda$ for each $y \in \alpha(\Lambda_s^d)$. By the invariance of Λ and $\alpha(\Lambda_s^d)$, the global stable manifolds $W^s(y) \subset \Lambda$ for each $y \in \alpha(\Lambda_s^d)$. $\square$

If Λ is a strongly partially hyperbolic set, we can also consider the unstable density basis $\mathcal{U} = \{B_n^u(p) = f^{-n}W^u(f^n p,\delta) : n \geq 0, p \in \Lambda\}$. Let A_u^d be the set of $\mathcal{U}$-density point of A and $A^d = A_s^d \cap A_u^d$. Following exactly the same line as in Section 2 we get $m(A\backslash A^d) = 0$ for each measurable subset A of Λ. Similarly we consider the ω-sets $\omega(x)$ and $\omega(E)$. For strongly partially hyperbolic sets we have

Theorem 3.3. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ a strongly partially hyperbolic set with positive volume. Then Λ contains the global stable manifolds over $\alpha(\Lambda^d)$ and the global unstable manifolds over $\omega(\Lambda^d)$.*

So every partially hyperbolic set with positive volume is far from being a topological horseshoe-like set. Although the sets $\alpha(\Lambda^d)$ and $\omega(\Lambda^d)$ are always nonempty, we do not know how large they could be and when they could intersect with each other. This can be improved if we require that Λ admits some recurrence.

Definition 3.4. A point x is said to be *backward recurrent* if $x \in \alpha(x)$. The definition of *forward recurrent* is analogous. A point is said to be *recurrent* if it is both backward and forward recurrent.

Definition 3.5. Let E be a measurable subset of Λ. Then E is said to be *s-saturated* if for each $x \in E$, $W^s(x) \subset E$. Similarly we can define *u-saturated* sets. Then the set E is *bi-saturated* if it is s-saturated and u-saturated.

Corollary 1. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ be a strongly partially hyperbolic set supporting some acip μ. Then Λ is bi-saturated.*

Proof. By Poincaré recurrence theorem [15, Theorem 3.3], we have that μ-a.e. $x \in \Lambda$ is recurrent, that is, $\mu(\mathrm{Rec}_\Lambda) = 1$ where Rec_Λ is the set of recurrent points in Λ. Also we have $\mu(\Lambda \backslash \Lambda^d) = 0$ since $\mu \ll m$ and $m(\Lambda \backslash \Lambda^d) = 0$. So $\mu(\Lambda^d \cap \mathrm{Rec}_\Lambda) = 1$ and the closed set $\alpha(\Lambda^d)$ contains $\Lambda^d \cap \mathrm{Rec}_\Lambda$, which is a subset of full μ-measure and hence dense in $\mathrm{supp}\mu = \Lambda$. So $\alpha(\Lambda^d) = \Lambda$ and the set Λ is s-saturated by Theorem 3.3. Similarly we can show Λ is u-saturated. This completes the proof. □

4. **Regularity of *acip*: Hyperbolic case.**

In this section we consider the hyperbolic sets. We show that if a hyperbolic set is of positive *acip*-measure, then the map is a transitive Anosov diffeomorphism. It is well known that for a transitive Anosov diffeomorphism, the *acip* is not only equivalent to the volume, but also has a smooth density with respect to the volume. This motivates the generalization to partial hyperbolic systems in next section.

Theorem 4.1. *Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$, μ be an acip and Λ be a hyperbolic set with positive μ-measure. Then $\Lambda = M$ and f is a transitive Anosov diffeomorphism on M.*

Proof. By considering $\Lambda_\mu = \Lambda \cap \mathrm{supp}(\mu)$ and $\mu|_{\Lambda_\mu}$ if necessary, we can assume that $\Lambda = \mathrm{supp}(\mu)$. By Corollary 1, we have that Λ is bi-saturated. By the uniform hyperbolicity of Λ, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset \bigcup_{y \in W^u(x,\delta)} W^s(y, \delta) \subset \Lambda$ for each $x \in \Lambda$. So the set Λ is both closed and open, whence coincides with the whole manifold M. Since M is a hyperbolic set, the map f is an Anosov diffeomorphism. Moreover f is transitive since $M = \Lambda = \mathrm{supp}(\mu) \subset \Omega(f) \subset M$ (by spectral decomposition theorem, see [6]). □

Remark 2. Spectral decomposition theorem actually implies that f is mixing. Moreover by Corollary 4.13 and Theorem 4.14 in [6], μ coincides with the equilibrium state μ_{ϕ^u} of the potential $\phi^u(x) = -\log(J^u(f, x))$, and has Hölder continuous density with respect to m. Furthermore the smooth measure μ is ergodic and *Bernoulli*.

Remark 3. The regularity of $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ is an essential assumption in a two-fold sense. In [20] Robinson and Young constructed a C^1 Anosov diffeomorphism with non-absolutely continuous stable and unstable foliations, which does have some closed invariant set with positive volume. In [5] Bowen constructed a C^1 horseshoe Ω_B for some $f_B \in \mathrm{Diff}^1(M)$ with positive volume and absolutely continuous local stable and unstable laminations, such that f_B dose preserve the restriction m_{Ω_B}[1] (although f_B is not volume–preserving).

[1]This follows from his construction that the Jacobian $J(f_B, x) = 1$ for every point $x \in \Omega_B$.

5. **Regularity of *acip*: Partially hyperbolic case.** In this section we show analogous results in Section 4 hold for accessible strongly partially hyperbolic systems. Namely, let $f \in \mathrm{SPH}^r(M)$ for some $r > 1$ be a C^r strongly partially hyperbolic diffeomorphism and m be the volume measure associated to some Riemannian metric adapted to the partially hyperbolic splitting. Let $\mathcal{W}^s$ be the stable foliation tangent to the stable bundle and $\mathcal{W}^u$ the unstable foliation tangent to the unstable bundle.

Definition 5.1. Let E be a measurable subset of M. Then E is said to be *essentially s-saturated* if there exists an s-saturated set $\widehat{E}^s$ with $m(E\triangle\widehat{E}^s) = 0$. Similarly we can define *essentially u-saturated* sets. The set E is *essentially bi-saturated* if there exists a bi-saturated set $\widehat{E}^{su}$ with $m(E\triangle\widehat{E}^{su}) = 0$, and *bi-essentially saturated* if E is essentially s-saturated and essentially u-saturated.

Definition 5.2. A strongly partially hyperbolic diffeomorphism $f : M \to M$ is said to be *accessible* if each nonempty bi-saturated set is the whole manifold M. The map f is *essentially accessible* if every measurable bi-saturated set has either full or zero volume.

Theorem 5.3. *Let $f \in \mathrm{SPH}^r(M)$ be essentially accessible. If there exists some acip for f, then the support of the acip is the whole manifold and the map f is transitive.*

Before the proof, we mention that there exists a C^1 open set $\mathcal{U} \subset \mathrm{SPH}^1(M)$ such that each $f \in \mathcal{U}$ is accessible but non-transitive (see [18]).

Proof. Let μ be an *acip* of f. Then the support $\mathrm{supp}(\mu)$ of μ is a strongly partially hyperbolic set supporting μ, whence is a bi-saturated set by Corollary 1. Essential accessibility of f implies that $m(\mathrm{supp}(\mu)) = 1$. Hence $\mathrm{supp}(\mu) = M$ since $\mathrm{supp}(\mu)$ is closed.

Suppose on the contrary that f is not transitive. That is, there exists an f-invariant nonempty open set U such that $M\backslash\overline{U} \neq \emptyset$. So the set $\Lambda = M\backslash U$ is f-invariant, closed with nonempty interior. Hence $\mu(\Lambda) > 0$ and $\mu|_\Lambda$ is again an *acip*. Corollary 1 implies that Λ is bi-saturated. Since f is essentially accessible, we have $m(\Lambda) = 1$ and $m(U) = 0$. This contradicts the openness of U. □

Generally for a transitive map f, the set Tran_f of points with dense orbit could be measure-theoretically meagre (although topologically residual). In [21, Section 5.7] they extracted the following property which can be viewed as a stronger form of transitivity (or a weaker form of ergodicity).

Definition 5.4. An f-invariant measure μ is said to be *weakly ergodic* if the set of points with dense orbit in $\mathrm{supp}(\mu)$ has full μ-measure.

Clearly ergodicity implies weak ergodicity, and weak ergodicity implies the transitivity of the subsystem $(f, \mathrm{supp}(\mu))$. In the following we show some analogous results in [7, 11, 21] hold for *acip*. To this end let us introduce some necessary notations. Let μ be an *acip* of $f \in \mathrm{SPH}^r(M)$ for some $r > 1$ and $\phi = \frac{d\mu}{dm}$ be the *Radon-Nikodym derivative* of μ relative to m. Note that the *Jacobian* $J_f : M \to \mathbb{R}, x \mapsto \mathrm{Jac}(Df : T_xM \to T_{fx}M)$ is a Hölder continuous function on M, bounded and bounded away from 0. For each measurable subset $A \subset M$ we have:

$$\int_A \phi(x)dm(x) = \mu(A) = \mu(fA) = \int_{fA} \phi(y)dm(y) = \int_A \phi(fx)J_f(x)dm(x).$$

So the following holds:

$$\phi(fx)J_f(x) = \phi(x) \text{ for } m-\text{a.e. } x \in M. \tag{4}$$

Let us consider the set $E = \{x \in M : \phi(x) > 0\}$. Clearly E is measurable and $m(E) > 0$. By (4) we see that E is also f-invariant. Restricted to the set E, the measure $m|_E$ is equivalent to μ. So '(P) for m-a.e. $x \in E$' is the same as '(P) for μ-a.e. $x \in E$'. In this case we will say '(P) a.e. $x \in E$' for short.

Proposition 3. *Let $f \in \mathrm{SPH}^r(M)$, μ be an acip, ϕ be the Radon-Nikodym derivative of μ and $E = \{x \in M : \phi(x) > 0\}$. Then E is bi-essentially saturated.*

Note that all essential saturations are defined with respect the volume. If f is volume preserving, then every invariant set is always bi-essentially saturated by the Hopf argument. See [9, Lemma 6.3.2] and [21, Theorem 5.5].

Proof. It suffices to prove that E is essentially s-saturated. Let $B_n^s(x) = f^n W^s(x_{-n}, \delta)$ and E_s^d be the set of $\mathcal{S}$-density points of E. By Proposition 2 we have $m(E \backslash E_s^d) = 0$.

Consider the functions $\eta_n(x) = m_{W^s(x)}(B_n^s(x)\backslash E)/m_{W^s(x)}(B_n^s(x))$ for $n \geq 1$. So $\eta_n(x) \to 0$ as $n \to +\infty$ for a.e. $x \in E$. For each $\epsilon > 0$, there exists a subset $E_\epsilon \subset E$ with $m(E\backslash E_\epsilon) < \epsilon$ on which η_n converges uniformly to zero. Note that a.e. $x \in E_\epsilon$ is recurrent. For a recurrent point $x \in E_\epsilon$, let $\{n_i : i \geq 1\}$ be an increasing sequence of the forward recurrent times of x with respect to E_ϵ, that is, $f^{n_i}x \in E_\epsilon$ for each i.

By Lemma 3.1, there exists a uniform constant $C \geq 1$ such that for the point $y = f^{n_i}x$ and $n = n_i$ the following holds:

$$m_{W^s(x)}(W^s(x,\delta)\backslash E) \leq C \cdot \eta_{n_i}(f^{n_i}x).$$

Passing n_i to ∞ we have $m_{W^s(x)}(W^s(x,\delta)\backslash E) = 0$ for a.e. $x \in E_\epsilon$.

Since ϵ can be arbitrary small, we have $m_{W^s(x)}(W^s(x,\delta)\backslash E) = 0$ for a.e. $x \in E$. Since E is f-invariant and f is smooth between leaves of $\mathcal{W}^s$, $m_{W^s(x)}(f^{-n}W^s(f^n x, \delta)\backslash E) = 0$ for each $n \geq 1$ and a.e. $x \in E$. Hence $m_{W^s(x)}(W^s(x)\backslash E) = 0$ for a.e. $x \in E$. It follows from the absolute continuity of $\mathcal{W}^s$ that E is essentially s-saturated. Similarly we can show E is essentially u-saturated. This completes the proof. $\square$

Theorem 5.5. *Let $f \in \mathrm{SPH}^r(M)$ be essentially accessible. Then every acip is weakly ergodic. In particular if μ is an acip, then the orbit $\mathcal{O}(x)$ is dense in M for μ-a.e. $x \in M$.*

This result is well known if the system is volume preserving (see [7, 11, 21]). The idea of the proof is similar to Lemma 5 in [11]. Also see Proposition 5.17 in [21].

Proof. Let ϕ be the Radon-Nikodym derivative of μ with respect to m and $E = \{x \in M : \phi(x) > 0\}$. Then $\overline{E} = \mathrm{supp}(\mu) = M$ by Theorem 5.3 since f is essentially accessible. By Proposition 3, we have E is bi-essentially saturated.

Step 1. We will show that for each open ball B, $\mathcal{O}(x) \cap B \neq \emptyset$ for m-a.e. point $x \in E$. To the end we first consider $G(B)$, the subset of points x which has a neighborhood U of x such that $\mathcal{O}(y) \cap B \neq \emptyset$ for m-a.e. $y \in U \cap E$. Evidently $G(B)$ is a nonempty open subset (and f-invariant).

Claim. $G(B)$ is bi-saturated. So $m(G(B)) = 1$ since f is essentially accessible.

Proof of Claim. Let us prove $G(B)$ is s-saturated. It suffices to show that $q \in G(B)$ for each $q \in W^s(z, \delta)$ and each $p \in G(B)$, where the size δ is fixed. So the justification lies in a local foliation box X of $\mathcal{W}^s$ around p. Note that we can (and we do) replace E by its saturate $\widehat{E}^s$ in the definition of $G(B)$ since E is essentially s-saturated. For a point $x \in X$, denote $W^s_X(x)$ the component of $W^s(x) \cap X$ that contains x. Let U be a small neighborhood of p with $\mathcal{O}(y) \cap B \neq \emptyset$ for m-a.e. $y \in U \cap \widehat{E}^s$. Let R be the set of recurrent points $z \in U \cap \widehat{E}^s$ whose orbits enter B. Note that $m(U \cap \widehat{E}^s \backslash R) = 0$ since $m|_E$ is equivalent to the invariant measure μ and $m(E \triangle \widehat{E}^s) = 0$. So we can pick a smooth transverse T of $\mathcal{W}^s_X$ in U such that $T \cap W^s_U(p) \neq \emptyset$ and $m_T(\widehat{E}^s \backslash R) = 0$, where m_T is the induced volume on T (It is helpful to keep in mind that $\widehat{E}^s$ is not only essentially s-saturated, but s-saturated). Now we have

(I) For each $x \in R$ and $y \in W^s_X(x)$, we have $\mathcal{O}(y) \cap B \neq \emptyset$. This follows from that $d(f^n x, f^n y) \to 0$ and the point x is recurrent: the orbit of x will enter B infinite many times.

(II) The set $\bigcup_{x \in T \cap R} W^s_X(x)$ has full m-measure in the set $\bigcup_{x \in T \cap \widehat{E}^s} W^s_X(x)$. This follows from that both sets are measurable and $\mathcal{W}^s_X$-saturated, $\mathcal{W}^s_X$ is a transversally absolutely continuous foliation of X and $m_T(\widehat{E}^s \backslash R) = 0$.

(III) The set $\bigcup_{x \in T} W^s_X(x)$ contains an open neighborhood V of q. This follows from that the holonomy maps along $\mathcal{W}^s_X$ are homeomorphisms.

Also note that $\bigcup_{x \in T \cap \widehat{E}^s} W^s_X(x) = \big(\bigcup_{x \in T} W^s_X(x)\big) \cap \widehat{E}^s$. So $\mathcal{O}(y) \cap B \neq \emptyset$ for m-a.e. $y \in V \cap \widehat{E}^s$. This implies $q \in G(B)$ and hence $G(B)$ is s-saturated. Similarly we have $G(B)$ is also u-saturated and hence $m(G(B)) = 1$ by the essential accessibility of f. This completes the proof of Claim.

Now let $F(B) = \{x \in E : \mathcal{O}(x) \cap B \neq \emptyset\}$. We need to show that $m(E \backslash F(B)) = 0$. To derive a contradiction we assume $m(E \backslash F(B)) > 0$ and let $p \in G(B)$ be a Lebesgue density point of $E \backslash F(B)$ (here we use $m(G(B)) = 1$). So there exists an open neighborhood U of p such that $\mathcal{O}(x) \cap B \neq \emptyset$ for a.e. $x \in U \cap E$. Then we have $m(U \cap E \backslash F(B)) = 0$. But this is impossible since we choose p as a Lebesgue density point of $E \backslash F(B)$. So we have $m(E \backslash F(B)) = 0$ for each open ball B.

Step 2. Since M is compact, there exists a countable collection of open balls $\{B_n : n \geq 1\}$ which forms a subbasis of the topology on M. Let $F(B_n)$ be given by Step 1. We have $m(E \backslash F) = 0$ where $F = \bigcap_{n \geq 1} F(B_n)$. Now for each $x \in F$, $\mathcal{O}(x) \cap B_n \neq \emptyset$ for each $n \geq 1$. So the orbit $\mathcal{O}(x)$ is dense in M for each point $x \in F$. Equivalently we see μ-a.e. $x \in M$ has a dense orbit. So the *acip* μ is weakly ergodic. This completes the proof. □

Next we recall a famous conjecture due to Pugh–Shub [19]:

Conjecture. *If $f \in \mathrm{SPH}^2_m(M)$ is volume preserving with essential accessibility property, then (f, m) is ergodic.*

Similarly we can ask: if $f \in \mathrm{SPH}^r(M)$ is essentially accessible and preserves some *acip* μ, is μ an ergodic measure? This is closely related to the uniqueness of *acip*. Clearly the ergodicity follows if there is a unique *acip*. On the other hand, let us assume that exist two *acip*'s: $\mu = \phi m$ and $\nu = \psi m$. Let $E = \{x : \phi(x) > 0\}$ and $F = \{x : \psi(x) > 0\}$. If $m(E \triangle F) > 0$ we can further assume E and F are disjoint. Proposition 3 implies that both E and F are bi-essentially saturated (and nontrivial). In particular none of them can be essentially bi-saturated.

We do not know whether such example can exist, or generally a bi-essentially saturated set is automatically essentially bi-saturated. However Burns and Wilkinson showed that a sufficient condition for the later property is center bunching. From now on we assume $r = 2$ for simplicity.

Definition 5.6. A strongly partially hyperbolic diffeomorphism f is *center bunched* if the functions ν, $\tilde{\nu}$ and γ, $\tilde{\gamma}$ given in (1) can be chosen so that: $\nu < \gamma\tilde{\gamma}$ and $\tilde{\nu} < \gamma\tilde{\gamma}$.

Proposition 4 (Corollary 5.2 in [10])**.** *Let $f \in \mathrm{SPH}^2(M)$ be center bunched. Then every measurable bi-essentially saturated subset is essentially bi-saturated.*

Corollary 2. *Let $f \in \mathrm{SPH}^2(M)$ be essentially accessible and center bunched. If there exists some acip, then the acip must be equivalent to the volume. Moreover the acip μ is ergodic.*

Proof. Let μ be an *acip* and ϕ be the Radon-Nikodym derivative of μ with respect to m. We showed that $E = \{x \in M : \phi(x) > 0\}$ is bi-essentially saturated. Center bunching implies that E is also essentially bi-saturated. Since f is essentially accessible and $m(E) > 0$, $m(E) = 1$ and hence μ is equivalent to the volume m. So the *acip* is unique and hence ergodic. □

Remark 4. In [10], a map f is said to be *volume preserving* if f preserves some invariant measure μ that is equivalent to the volume. They proved that if $f \in \mathrm{SPH}^2(M)$ is essentially accessible, center bunched and preserves some μ equivalent to the volume, then the measure μ is ergodic (and *Kolmogorov*). By Corollary 2, we also see that if $f \in \mathrm{SPH}^2(M)$ is essentially accessible and center bunched, then either f is volume preserving in the broad sense, or there exists no *acip* at all.

Followed by Corollary 2 we get that the density $\phi = \frac{d\mu}{dm}$ of an *acip* is positive a.e. on M. Now we use *Cohomology Theory* developed in [22] to improve the regularity of the Radon-Nikodym derivative of μ. Namely let $\psi : M \to \mathbb{R}$ be a potential on M and consider the cohomological equation on M:

$$\psi = \Psi \circ f - \Psi. \tag{5}$$

Proposition 5 (Theorem A, part *II* and *III*, in [22])**.** *Let $f \in \mathrm{SPH}^2(M)$ be accessible, center bunched, and volume-preserving in the broad sense. Let $\psi : M \to \mathbb{R}$ be a Hölder continuous potential. If there exists a measurable solution Ψ such that (5) holds for a.e. $x \in M$, then there is a Hölder continuous solution Φ of (5) with $\Phi = \Psi$ a.e. $x \in M$.*

Now we let $f \in \mathrm{SPH}^2(M)$ be essentially accessible and center bunched, μ be an *acip*. Let $\psi = -\log J_f$, $\phi = \frac{d\mu}{dm}$ and $\Psi = \log \phi$. Now ψ is a C^1 function and Ψ is a well defined measurable function on M. Corollary 2 implies that Ψ is a measurable solution of the cohomological equation (5). Then applying Proposition 5 we get a Hölder continuous solution Φ of (5) which coincides with Ψ a.e.. It is evident that $\mu = e^{\Phi} m$ and the derivative e^{Φ} is bounded and bounded away from zero on M. Such a measure μ is called a *smooth measure*. So we have

Theorem 5.7. *Let $f \in \mathrm{SPH}^2(M)$ be accessible and center bunched. If there exists some acip, then the acip must have a Hölder continuous derivative with respect to the volume of M which is also bounded and bounded away from 0. In words, either f preserves a smooth measure or there is no acip of f.*

In particular center bunching holds whenever E^c is one-dimensional. As a corollary, we obtain:

Corollary 3. *Let $f \in \mathrm{SPH}^2(M)$ be accessible and $\dim(E^c) = 1$. Then either f preserves a smooth measure or there is no acip of f.*

Let $\mathrm{CB}^2(M) \subset \mathrm{SPH}^2(M)$ be the collection of C^2 strongly partially hyperbolic diffeomorphisms that are center bunched. Clearly $\mathrm{CB}^2(M)$ forms an open subset of $\mathrm{SPH}^2(M)$. Applying Theorem 5.7 and the result in [12] we have

Theorem 5.8. *The set of maps that admit no acip contains a C^1 open and dense subset of $\mathrm{CB}^2(M)$. In particular the set of maps that admit no acip contains a C^1 open and dense subset of C^2 strongly partially hyperbolic diffeomorphisms with $\dim(E^c) = 1$.*

The main obstruction for C^2 denseness in Theorem 5.8 is that we do not know whether stable accessibility is C^2 dense in $\mathrm{SPH}^2(M)$.

Proof. Dolgopyat and Wilkinson proved in [12] that there is a C^1 dense subset of stably accessible diffeomorphisms in $\mathrm{SPH}^2(M)$ (also C^1 dense in $\mathrm{CB}^2(M)$). Starting with arbitrary $f \in \mathrm{CB}^2(M)$, we first perturb it to a stably accessible one, say f_1. By C^1 closing lemma, there exists $f_2 \in \mathrm{CB}^2(M)$ close to f_1 that has some periodic point. We can assume that f_2 is also stably accessible since we can make it arbitrary close to f_1. By Franks' Lemma [14] we can assume that the periodic point p is hyperbolic with period k and the Jacobian of $Tf_2^k : T_xM \to T_xM$ has absolute value different from 1. These properties hold robustly for all maps in a small neighborhood $\mathcal{U} \subset \mathrm{CB}^2(M)$ of f_2.

Let $g \in \mathcal{U}$ and p_g be the continuation of p. By the choice of $\mathcal{U}$, we know that g is accessible and center bunched. If g admits some *acip* μ, then by Theorem 5.7 $\mu = \phi m$ for some Hölder continuous function ϕ which is bounded and bounded away from zero. By Equation (4) we have $\phi(p_g) = J_{g^k}(p_g)\phi(g^k p_g) = J_{g^k}(p_g)\phi(p_g)$. This is impossible sice $|J_{g^k}(p_g)| \neq 1$ and $\phi(p_g) \neq 0$. So each $g \in \mathcal{U}$ admits no *acip*. Hence there exists an open set $\mathcal{U}$ of maps C^1 close to f in which each map admits no *acip*. This finishes the proof. □

Remark 5. It is well known that among C^2 Anosov diffeomorphisms the ones that admits no *acip* are open and dense, see [6, Corollary 4.15]. This is due to the fact that there are many periodic points for every Anosov diffeomorphisms. Recently Avila and Bochi [3] proved that a C^1-generic map in $C^1(M, M)$ has no *acip*. In particular a C^1-generic map in $\mathrm{Diff}^1(M)$ has no *acip*.

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