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Torsion In The Cohomology Of Congruence Subgroups Of SL (4. Z) And Galois Representations

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TORSION IN THE COHOMOLOGY OF CONGRUENCE SUBGROUPS OF $SL(4,\mathbb{Z})$ AND GALOIS REPRESENTATIONS

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Abstract. We report on the computation of torsion in certain homology theories of congruence subgroups of $\mathrm{SL}(4,\mathbb{Z})$. Among these are the usual group cohomology, the Tate-Farrell cohomology, and the homology of the sharbly complex. All of these theories yield Hecke modules. We conjecture that the Hecke eigenclasses in these theories have attached Galois representations. The interpretation of our computations at the torsion primes 2,3,5 is explained. We provide evidence for our conjecture in the 15 cases of odd torsion that we found in levels ≤ 31.

1. Introduction

In a series of papers [AGM02, AGM08, AGM] we have computed the cohomology H^5 of certain congruence subgroups of $\mathrm{SL}(4,\mathbb{Z})$ (with 5 being the degree of most interest for reasons explained in [AGM02]). The coefficients in the cohomology of these papers consists of the trivial module $\mathbb{C}$, or its stand-in, $\mathbb{F}_p$ for a large prime p.

We are now beginning a series of computations of the torsion in the cohomology with $\mathbb{Z}$-coefficients, from which we can also deduce the the cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficients for all primes p.

We are primarily interested in the cohomology as a module for the Hecke algebra and in the connections with Galois representations. In the future, we hope to deal with twisted mod p-coefficients and to be able to test examples of the Serre-type conjectures enunciated in [AS00, ADP02, Her09]. Looking at the details of the these conjectures convinces us that any non-Eisensteinian example likely to be within the range of feasible computation will involve non-trivial coefficients.

There are also conjectures that are a sort of converse of the conjectures above. The prototype of these is Conjecture B in [Ash92]. We have been led, through our computational work, to generalize these conjectures from the group cohomology itself to a variety of related theories.

Conjecture 1. *A Hecke eigenclass in any reasonable (co)homology theory of an arithmetic subgroup of* $\mathrm{GL}(m,\mathbb{Z})$ *should have a Galois representation attached.*

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We will give a precise version, including what "reasonable" means, in Conjecture 5 of Section 3 below.

As of now, our programs compute with trivial coefficients. As the level N of the congruence group gets large, the time required by the computations also becomes large. In this paper we will:

(1) report on our results for $H^5(\Gamma_0(N), \mathbb{Z})$ over $\mathbb{Z}$ with $N \leq 31$ and test Conjecture 5 where possible;

(2) clarify what it is we are actually computing when $p = 2, 3, 5$, since the group $\Gamma_0(N)$ contains torsion elements of orders $2, 3, 5$; and

(3) make some specific conjectures along the lines of Conjecture 1 and relate them to each other.

Among our preliminary results in (1), we find torsion classes of orders $2, 3, 5$. For these, we are actually testing one of these new conjectures, contained in Conjecture 5. (For reasons explained below we cannot handle the prime 2 at present.)

In fact, we will discuss these issues in the reverse order (3)–(2)–(1).

2. Homology theories and Hecke operators

In this section we rely heavily on Brown's book [Bro94]. Our "value added" is showing that the Hecke algebra acts on the various kinds of homology theories defined below, and that the main exact sequence below is Hecke equivariant.

In [Bro94] the theory we discuss is developed for any virtual duality group (see [Bro94, p. 229] for the definition). Any arithmetic group is a virtual duality group, as proved by Borel and Serre [BS73]. So let G be a reductive $\mathbb{Q}$-group, Γ an arithmetic subgroup of $G(\mathbb{Q})$, and S a subsemigroup of $G(\mathbb{Q})$ such that (Γ, S) is a Hecke pair. Let $\mathcal{H} = H(\Gamma, S)$ denote the Hecke algebra of double cosets with $\mathbb{Z}$-coefficients, which we assume to be commutative. The paper [Ash92] contains an introduction to these terms.

First recall that if M is any S-module, $\mathcal{H}$ acts naturally on $H_*(\Gamma, M)$ and $H^*(\Gamma, M)$. The action is given by composing a twisted restriction map with a corestriction map. See, for example, [Ash92, Def. 1.7].

We let St denote the dualizing module for Γ, also known as the *Steinberg module*. It is isomorphic to $H^n(\Gamma, \mathbb{Z}\Gamma)$ where n is the virtual cohomological dimension of Γ.

Choose a complete resolution (F, P, ϵ) for Γ. Recall [Bro94, p. 273] this means that F is an acyclic chain complex of projective $\mathbb{Z}\Gamma$-modules together with an ordinary projective resolution $\epsilon\colon P \to \mathbb{Z}$ over $\mathbb{Z}\Gamma$ such that F and P coincide in sufficiently high dimensions. By the results of Borel and Serre already mentioned, a complete resolution for Γ exists.

For any Γ-module M set [Bro94, p. 277]

$$\widehat{H}^*(\Gamma, M) = H^*(\mathrm{Hom}_\Gamma(F, M)).$$

(We use Brown's conventions for the definition of Hom [Bro94, p. 5]. Further M is to be thought of as a complex concentrated in dimension 0.)

Up to isomorphism $H^*(\mathrm{Hom}_\Gamma(F, M))$ is independent of the choice of F, and it is called the *Farrell cohomology* of Γ. Since restriction and corestriction maps exist in the theory of Farrell cohomology, the action of $\mathcal{H}$ on it is defined in the usual way.

Besides the ordinary cohomology and the Farrell cohomology, we need to consider a third homology theory, which we will refer to as *Steinberg homology*. This is

defined by [Bro94, p. 279]

$$\widetilde{H}_*(\Gamma, M) = H_*(\Gamma, St \otimes_{\mathbb{Z}} M)).$$

Again, $\mathcal{H}$ acts on this in the usual way using restriction and corestriction maps.

These three homology theories fit together in a long exact sequence. To see that this sequence is $\mathcal{H}$-equivariant, we must recall how it is obtained [Bro94, p. 280]. We choose a finite type projective resolution $\epsilon\colon P \to \mathbb{Z}$ and a finite type projective resolution $\eta\colon Q \to St$, both over $\mathbb{Z}\Gamma$. If A is any complex of $\mathbb{Z}\Gamma$-modules, denote by $\overline{A} = \mathrm{Hom}_{\mathbb{Z}\Gamma}(A, \mathbb{Z}\Gamma)$ the dual complex. We let $\Sigma^k A$ denote the k-th suspension of A.

Recall that n denotes the virtual cohomological dimension of Γ. Proposition (2.5) of [Bro94] states that we may choose the complete resolution F so that $\overline{F}$ is the mapping cone of a chain map $\phi\colon \Sigma^{-n}Q \to \overline{P}$.

It follows immediately as on [Bro94, p. 280], that for any coefficient module M, the exact homology sequence of this mapping cone yields the long exact sequence

$$\cdots \to \widetilde{H}_{n-i} \to H^i \to \widehat{H}^i \to \widetilde{H}_{n-1-i} \to H^{i+1} \to \cdots$$

Theorem 2. *Assume that St has a resolution by S-modules that are projective as Γ-modules. Then this exact sequence is equivariant for the action of the Hecke algebra $\mathcal{H}$.*

Proof. Up to isomorphism, this exact sequence doesn't depend on the choice of resolutions P and Q. Unfortunately, [Bro94] assumes that P and Q are of finite type, i.e. in each degree they are finitely generated as $\mathbb{Z}\Gamma$-modules. We will need to use projective resolutions $\epsilon'\colon P' \to \mathbb{Z}$ and $\eta'\colon Q' \to St$ over $\mathbb{Z}\Gamma$ that are not of finite type. So we first want to show that we can replace P by P' and Q by Q' in the construction of the mapping cone and yet derive the same exact sequence.

To see this, first keep P fixed. Given Q and Q' as above, choose a homotopy equivalence $f\colon Q' \to Q$. Define the chain map $\phi'\colon \Sigma^{-n}Q' \to \overline{P}$ by $\phi' = \phi \circ f$. Let the mapping cone of the chain map $\phi \otimes_\Gamma M\colon \Sigma^{-n}Q \otimes_\Gamma M \to \overline{P} \otimes_\Gamma M$ be denoted by C. Let the mapping cone of the chain map $\phi' \otimes_\Gamma M\colon \Sigma^{-n}Q' \otimes_\Gamma M \to \overline{P} \otimes_\Gamma M$ be denoted by C'. We have the following diagram of complexes, where the horizontal lines are exact and the vertical arrow g is induced by f:

$$\begin{tikzcd}
0 \arrow[r] & \overline{P} \otimes_\Gamma M \arrow[r] \arrow[d, equal] & C \arrow[r] & \Sigma\Sigma^{-n}Q \otimes_\Gamma M \arrow[r] & 0 \\
0 \arrow[r] & \overline{P} \otimes_\Gamma M \arrow[r] & C' \arrow[r] \arrow[u, "g"'] & \Sigma\Sigma^{-n}Q' \otimes_\Gamma M \arrow[r] \arrow[u, "\Sigma\Sigma^{-n} f \otimes_\Gamma 1"'] & 0
\end{tikzcd}$$

Taking homology we obtain two long exact sequences as follows: using the fact that f and hence $\Sigma\Sigma^{-n} f \otimes_\Gamma 1$ are homotopy equivalences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i \arrow[r] \arrow[d, equal] & \widehat{H}^i \arrow[r] & \widetilde{H}_{n-1-i} \arrow[r] & \cdots \\
\cdots \arrow[r] & H^i \arrow[r] & H_i(C') \arrow[r] \arrow[u, "g_*"'] & \widetilde{H}_{n-1-i} \arrow[r] \arrow[u, "h"'] & \cdots
\end{tikzcd}$$

Since f and hence $\Sigma\Sigma^{-n} f \otimes_\Gamma 1$ are homotopy equivalences, the maps marked h are isomorphisms. Hence so are the maps marked g_*.

Therefore $H_i(C') \approx \widehat{H}^i$ and we can compute Farrell cohomology in this way, using Q' in place of Q. We now keep Q' fixed and repeat the argument with P' in place of P, using the fact that if $x\colon P \to P'$ is a homotopy equivalence then $\overline{x}\colon \overline{P}' \to \overline{P}$ is a homotopy equivalence. We deduce that we can compute Farrell cohomology and the long exact sequence from the mapping cone, using P', Q' in place of P, Q respectively.

Now we choose P' and Q' so they are complexes of S-modules, where the S-module structure extends that of Γ. For example, we can let P' be the standard resolution of $\mathbb{Z}$ for the group $G(\mathbb{Q})$. The existence of Q' is given by hypothesis.

We now have the exact sequence of S-module complexes:

$$0 \longrightarrow \overline{P}' \otimes_\Gamma M \longrightarrow C' \longrightarrow \Sigma\Sigma^{-n}Q' \otimes_\Gamma M \longrightarrow 0$$

If X is an S-module, or a complex of S-modules, then $X \otimes_\Gamma M = H_0(\Gamma, X \otimes_{\mathbb{Z}} M)$ has a natural $\mathcal{H}$-action, since it is a homology group of Γ with S-module coefficients. Since $\mathcal{H}$ thus acts on $\overline{P}' \otimes_\Gamma M$ and $\Sigma^{-n}Q' \otimes_\Gamma M$, it also acts on $\Sigma\Sigma^{-n}Q' \otimes_\Gamma M$ and on the mapping cone C'. Taking homology of the complexes in the last exact sequence, one obtains in this way a $\mathcal{H}$-action on the homology groups which coincides with the action on these groups as already defined. Following out the proof of the snake lemma one easily sees that all the maps in the long exact sequence of homology are $\mathcal{H}$-equivariant. □

We have not attempted to prove in general that St possesses a resolution Q' by projective Γ-modules that are also S-modules. However, in the case when $G = \mathrm{GL}(m)/\mathbb{Q}$, we can construct such a Q' as follows: let $Sh_\bullet$ be the sharbly complex, whose definition is recalled in Section 5 below. It is a resolution of St by S-modules, but it is not projective if Γ has nontrivial torsion. However, the tensor product of any Γ-module M with $\mathbb{Z}\Gamma$ becomes a free $\mathbb{Z}\Gamma$-module, as long as M is $\mathbb{Z}$-free [Bro94, Cor. 5.7]. So we can take the tensor product of $Sh_\bullet$ with a free resolution P' of $\mathbb{Z}$ by S-modules: $Q' = P' \otimes_{\mathbb{Z}} Sh_\bullet$. This proves the following:

Corollary 3. *Let (Γ, S) be a Hecke pair contained in $\mathrm{GL}(m, \mathbb{Q})$. Then for any S-module M, the exact sequence*

$$\cdots \to H_{n-i}(\Gamma, St \otimes_{\mathbb{Z}} M) \to H^i(\Gamma, M) \to \widehat{H}^i(\Gamma, M) \to H_{n-1-i}(\Gamma, St \otimes_{\mathbb{Z}} M) \to \cdots$$

is equivariant for the action of the Hecke algebra $\mathcal{H}$.

The last homology theory we consider is *sharbly homology* $H_*(\Gamma, Sh_\bullet \otimes_{\mathbb{Z}} M)$. Its relationship to $H_*(\Gamma, St \otimes_{\mathbb{Z}} M)$ will be considered in Section 5. It is also naturally a $\mathcal{H}$ module, since $Sh_\bullet$ and M are S-modules.

As we have already noted, if Γ possesses nontrivial torsion, then $Sh_\bullet$ will not be a projective resolution., So when Γ is a subgroup of $\mathrm{GL}(4, \mathbb{Z})$, we end up in actuality computing $H_*(\Gamma, Sh_\bullet \otimes_{\mathbb{Z}} M)$, rather than $H_*(\Gamma, St \otimes_{\mathbb{Z}} M)$, when Γ possesses nontrivial torsion. We have seen that we could compute the Steinberg homology by replacing $Sh_\bullet$ by $Q' = P' \otimes_{\mathbb{Z}} Sh_\bullet$, or by rigidifying the sharbly complex in some other way. However, any method of doing this that we have considered increases the number of cells in the relevant dimensions so much as to make actual computation infeasible.

3. Some conjectures and their interrelationships

In this section we state conjectures that state that Galois representations are attached to Hecke eigenclasses in any of our four homology theories. We make a few remarks on the relationships between these conjectures. We state these conjectures for the congruence subgroups $\Gamma_0(N)$ so we can be definite in our notation. They are easily modified for any Hecke pair (Γ, S) in $\mathrm{GL}(m, \mathbb{Q})$.

Let $\Gamma_0(N)$ be the subgroup of matrices in $\mathrm{SL}(m, \mathbb{Z})$ whose first row is congruent to $(*, 0, \ldots, 0)$ modulo N. Define S_N to be the subsemigroup of integral matrices in $\mathrm{GL}(m, \mathbb{Q})$ satisfying the same congruence condition and having positive determinant relatively prime to N.

Let $\mathcal{H}(N)$ denote the $\mathbb{Z}$-algebra of double cosets $\Gamma_0(N)S_N\Gamma_0(N)$. Then $\mathcal{H}(N)$ is a commutative algebra that acts on the cohomology and homology of $\Gamma_0(N)$ with coefficients in any $\mathbb{Z}[S_N]$ module. When a double coset is acting on cohomology or homology, we call it a Hecke operator. Clearly, $\mathcal{H}(N)$ contains all double cosets of the form $\Gamma_0(N)D(\ell, k)\Gamma_0(N)$, where ℓ is a prime not dividing N, $0 \leq k \leq m$, and

$$D(\ell, k) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ell & & \\ & & & & \ddots & \\ & & & & & \ell \end{pmatrix}$$

is the diagonal matrix with the first $m-k$ diagonal entries equal to 1 and the last k diagonal entries equal to ℓ. It is known that these double cosets generate $\mathcal{H}(N)$. When we consider the double coset generated by $D(\ell, k)$ as a Hecke operator, we call it $T(\ell, k)$.

Definition 4. Let A be a ring and V an $\mathcal{H}(N) \otimes_{\mathbb{Z}} A$-module. Suppose that $v \in V$ is a simultaneous eigenvector for all $T(\ell, k)$ and that $T(\ell, k)v = a(\ell, k)v$ with $a(\ell, k) \in A$ for all prime $\ell \not| N$ and all $0 \leq k \leq m$. If

$$\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}(m, A)$$

is a continuous representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside $L \in \mathbb{N}$, and

$$\sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k = \det(I - \rho(\mathrm{Frob}_\ell)X) \tag{1}$$

for all $\ell \not| LN$, then we say that ρ is attached to v (or that v corresponds to ρ).

Let p be a prime, and $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$. Let M be a S_N-module that is a finite-dimensional vector space over $\mathbb{F}_p$ on which S_N acts via its reduction modulo p. We call such a module an admissible S_N-module. Then we make the following conjectures:

Conjecture 5. *Fix p, m, N, M as above. Let v be a Hecke eigenclass in*
(a) $H_*(\Gamma_0(N), St \otimes_{\mathbb{Z}} M)$,
(b) $H^*(\Gamma_0(N), M)$,
(c) $\widehat{H}^*(\Gamma_0(N), M)$, or
(d) $H_*(\Gamma_0(N), Sh_\bullet \otimes_{\mathbb{Z}} M)$.

Then there exists a continuous representation unramified outside pN

$$\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}(m, \overline{\mathbb{F}}_p)$$

attached to v.

It follows immediately from the exact sequence of Corollary 3 that if any two of Conjectures 5 (a),(b),(c) hold, then the third one holds. We can express this also in the following way.

Let $\mathcal{M}(N)$ denote a set of representatives of all admissible $\mathbb{F}_p[S_N]$-modules modulo isomorphism. If V is an $\mathcal{H}(N) \otimes_{\mathbb{Z}} \mathbb{F}_p$-module, let $\mathcal{G}(V)$ denote the set of all Galois representations (modulo isomorphism) attached to Hecke eigenvectors in V.

Theorem 6. *We have the following inclusions of sets of Galois representations:*

$$\mathcal{G}(\oplus_{M\in\mathcal{M}(N)} \oplus_k \widehat{H}^k(\Gamma_0(N), M)) \subset \mathcal{G}(\oplus_{M\in\mathcal{M}(N)} \oplus_k H^k(\Gamma_0(N), M)). \tag{2}$$

$$\mathcal{G}(\oplus_{M\in\mathcal{M}(N)} \oplus_k H_k(\Gamma_0(N), St \otimes_{\mathbb{Z}} M)) \subset \mathcal{G}(\oplus_{M\in\mathcal{M}(N)} \oplus_k H^k(\Gamma_0(N), M)). \tag{3}$$

Proof. As explained on p. 278 of [Bro94], there is dimension shifting in both directions on Farrell cohomology. We wish to dimension shift upwards. To do this, we use induced modules. Fix a torsionfree subgroup Γ' of finite index in $\Gamma_0(N)$. Let $I(M)$ denote the induced module $\mathbb{F}_p\Gamma_0(N) \otimes_{\mathbb{F}_p\Gamma'} M$. Let K denote the kernel of the natural map $I(M) \to M$. Then

$$\widehat{H}^k(\Gamma_0(N), M)) \approx \widehat{H}^{k+1}(\Gamma_0(N), K)).$$

Now $I(M)$ is isomorphic to the module denoted $I(\Gamma', \Gamma_0(N); M)$ in Definition 1.5 p. 240 of [Ash92] and in particular has the structure of $\mathbb{F}_pS_N$-module. Therefore $\widehat{H}^k(\Gamma_0(N), M))$ and $\widehat{H}^{k+1}(\Gamma_0(N), K))$ are isomorphic as $\mathcal{H}$-modules.

Lemma 1.6 of [Ash92] implies that $I(M)$ and K are admissible. Repeating this construction, we can dimension shift as high as we like. So if n is the virtual cohomological dimension of $\Gamma_0(N)$, we obtain

$$\mathcal{G}(\oplus_{M\in\mathcal{M}(N)} \oplus_k \widehat{H}^k(\Gamma_0(N), M)) \subset \mathcal{G}(\oplus_{M\in\mathcal{M}(N)} \oplus_{k>n} \widehat{H}^k(\Gamma_0(N), M)).$$

But above dimension n, the Farrell and ordinary cohomology are isomorphic as $\mathcal{H}$-modules. This proves the inclusion in (2).

The inclusion in (3) now follows from the exact sequence of $\mathcal{H}$-modules of Corollary 3, as in the proof of Theorem 3.1 of [Ash92]. □

Remark: the inclusion (2) is strict, in general. For example, if $m = p - 1$, Theorem 0.2 of [Ash92] plus Theorem 7.3 of [Bro94] implies that all Galois representations attached to the Farrell cohomology of $\Gamma_0(N)$ with any admissible coefficient module are of a very simple type, namely, induced from a character. On the other hand, if now $m = 4, p = 5$ we can reduce modulo 5 any of the Hecke eigenclasses found in [AGM02, AGM08, AGM] that have non-induced Galois representations attached to them.

We do not know if the inclusion (3) is strict. We guess that it is.

As explained in Section 5, the conjecture we actually test with our computations on GL(4) is Conjecture 5 (d) for $m = 4$. In Section 5, we will explain the relationship between Conjecture 5 (d) and Conjecture 5 (a).

Theorem 7. *In all cases, Conjecture 5 (b) implies Conjecture 5 (a) and Conjecture 5 (c).*

Now suppose that p, m, N are such that there is no p-torsion in $\Gamma_0(N)$.

(1) Conjecture 5 (c) is trivially true since the Farrell cohomology vanishes.

(2) Conjectures 5 (a) and (b) and (d) are equivalent.

Next suppose that $p = 5, m = 4$. Then

(3) Conjecture 5 (c) is true.

(4) Conjectures 5 (a) and (b) are equivalent.

Proof. The first assertion follows immediately from Theorem 6.

Now suppose there is no p-torsion in $\Gamma_0(N)$. Then the Farrell cohomology vanishes identically [Bro94, Exercise 2, p. 280]. Also, by Lemma 9 below, the Steinberg cohomology is isomorphic to the ordinary group cohomology, and to the sharbly cohomology, because $\Gamma_0(N)$ is p-torsionfree. Because of the long exact sequence of Corollary 3, (1) implies (2).

Next suppose $p = 5, m = 4$. Then Theorem 6.4.3 of [Ash92] and Theorem 7.3 of [Bro94] imply that there is a Galois representation attached to any Hecke eigenclass in the Farrell cohomology. This implies (3), and (4) then follows from the long exact sequence of Corollary 3. □

If there is p-torsion in $\Gamma_0(N)$, then Lemma 9 will give us a relationship between Conjectures 5 (a) and (d).

4. The primes $2, 3, 5, p > 5$

The torsion primes for $\mathrm{GL}(4, \mathbb{Z})$ are $2, 3, 5$. Setting $n = 4$ in our notation, we have:

Lemma 8. *For any $N \geq 1$, the subgroup $\Gamma_0(N)$ of $\mathrm{SL}(4, \mathbb{Z})$ has 2- and 3-torsion. It has 5-torsion if and only if N is not divisible by 25 and every prime divisor p of N not equal to 5 satisfies the condition $5 \mid p - 1$.*

Proof. Since $\Gamma_0(N)$ contains subgroups isomorphic to $\mathrm{GL}(3, \mathbb{Z})$, it contains 2- and 3-torsion. Now consider 5-torsion. Because the class number of $\mathbb{Q}(\zeta_5)$ is 1, any subgroup of $\mathrm{SL}(4, \mathbb{Z})$ of order 5 is conjugate to the one generated by the element

$$Z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

So $\Gamma_0(N)$ contains 5-torsion if and only if there exists $A \in \mathrm{SL}(4, \mathbb{Z})$ such that $A^{-1}ZA \in \Gamma_0(N)$ iff there exists a primitive vector $v \in \mathbb{Z}^4$ and an integer λ prime to N such that $vZ \equiv \lambda v$ modulo N. Keeping in mind that v must be primitive, one sees that this is so if and only if there exists an integer a (namely any a congruent to λ modulo N) such that $1 + a + a^2 + a^3 + a^4 = 0$. The conclusion of the lemma follows easily. □

We now describe our computations of p-torsion in terms of the various p.

The prime 2: Unfortunately, at present, the algorithm in [Gun00] for reducing sharblies, which is essential for computing the Hecke operators on the sharbly homology, involves division by 2. So we cannot compute the Hecke action on 2-torsion classes at this time. Since there is generally a lot of 2-torsion in the homology, fixing

the algorithm to remove this division by 2 is a pressing desideratum that we plan to address in future work.

The prime 3: We have examples of 3-torsion that verify Conjecture 5 (d).

The prime 5: We have examples of 5-torsion that verify Conjecture 5 (d). Whether or not $\Gamma_0(N)$ contains 5-torsion, Lemma 7 shows that when $p = 5$ Conjectures 5 (a) and 5 (c) are equivalent. When $\Gamma_0(N)$ does not contain 5-torsion, Theorem 7 tells us that Conjectures 5 (a) and 5 (d) are equivalent. If $\Gamma_0(N)$ does not contain 5-torsion, the situation is the same as in the next paragraph.

Primes $p > 5$: We have examples of p-torsion that verify Conjecture 5 (d). Since $\Gamma_0(N)$ does not contain p-torsion, Conjectures 5 (a), (b) and (d) are all equivalent by Lemma 7. So in these cases we are truly verifying the original conjecture B of [Ash92].

5. What we are computing

Let (Γ, S) be a Hecke pair contained in $\mathrm{GL}(m, \mathbb{Q})$.

In [AGM02] we explained in detail how to compute with the spectral sequence

$$E_1^{p,q} = \sum_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, M_\sigma) \Rightarrow H_\Gamma^{p+q}(W, M).$$

This spectral sequence was derived as in [Bro94] from the double complex

$$\mathrm{Hom}_\Gamma(P_\bullet, C^\bullet(W, M)).$$

Here, we have chosen a resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ by free $\mathbb{Z}S$-modules, and W is a contractible Γ-cell complex. (In practice it is the well-rounded retract [Ash80].)

When the torsion in Γ is invertible on M, then $E_1^{p,q} = 0$ for $q > 0$. Then $E_1^{p,0} = H_\Gamma^p(W, M)$ is isomorphic to $H^p(\Gamma, M)$. The Hecke algebra $\mathcal{H}(\Gamma, S)$ acts on $E_1^{p,0}$ naturally via this isomorphism. This is what we computed in [AGM02, AGM08, AGM].

However, when the torsion in Γ is not invertible on M, then the higher rows of the spectral sequence do not vanish and the Hecke algebra does not act on the individual terms of the spectral sequence, because S does not preserve the cellulation. In other words, $E_1^{p,0}$ is no longer computing the object in Conjecture 5(b). We will now show that $E_1^{p,0}$ is actually computing the object in Conjecture 5(d), the sharbly homology. The papers [AGM02, AGM08, AGM] computed both 5(b) and 5(d)—they were the same because we were using $\mathbb{C}$ coefficients. The present paper extends those papers, working in the setting 5(d).

Recall that the sharbly complex $Sh_\bullet \to St$ gives a resolution of the Steinberg module by $\mathbb{Z}S$-modules. But it is not Γ-projective if Γ is not torsionfree.

Facts about the sharbly complex may be found in [Ash94]. The term Sh_k is the $\mathbb{Z}\mathrm{GL}(m, \mathbb{Q})$-module generated by symbols $[v_1, \ldots, v_{m+k}]$ where the v_i are primitive vectors in $\mathbb{Z}^m$, subject to the relations

(i) $[v_{\sigma 1}, \ldots, v_{\sigma(m+k)}] = (-1)^\sigma [v_1, \ldots, v_{m+k}]$ for all permutations σ;

(ii) $[v_1, \ldots, v_{m+k}] = 0$ if $v_1, \ldots, v_{m+k}$ do not span all of $\mathbb{Q}^m$.

We can form the double complex $P_\bullet \otimes_\Gamma Sh_\bullet$. From this we get in the usual way a first quadrant homology spectral sequence:

$$E_{p,q}^1 = H_q(\Gamma, Sh_p \otimes_\mathbb{Z} M) \Rightarrow H_{p+q}(\Gamma, St \otimes_\mathbb{Z} M).$$

Let $\mathcal{H}$ denote the Hecke algebra $\mathcal{H}(\Gamma, S)$.

Lemma 9. *(a) For any $1 \le r \le \infty$, $\mathcal{H}$ acts on every term $E^r_{p,q}$ in the E^r page of the spectral sequence and commutes with all differentials. The abutment morphism is equivariant for $\mathcal{H}$.*

(b) Let d be the product of all the torsion primes of Γ. Suppose that d acts invertibly on M. Then $E_1^{p,q} = 0$ for $q > 0$. Then the sharbly homology $H_p(Sh_p \otimes_{\mathbb{Z}\Gamma} M)$ is isomorphic to the Steinberg homology $H_p(St \otimes_{\mathbb{Z}\Gamma} M)$. This isomorphism is equivariant for $\mathcal{H}$.

Proof. Since $P_\bullet$ is a resolution of S-modules, as is $Sh_\bullet$, S acts on every term in the E^0 page of the spectral sequence and the differentials are S-module maps. It follows that $\mathcal{H}$ acts on every term in the E^1 page of the spectral sequence and the differentials are $\mathcal{H}$-module maps. Then (a) follows immediately.

It follows directly from the definition of the sharbly complex that for each i, Sh_i as a Γ-module is isomorphic to a direct sum of induced Γ-modules. Indeed, let R be a set of representatives of the Γ-orbits on "pure" p-sharblies, i.e. on the set of symbols $[v_1, \dots, v_{m+p}]$ modulo the action by permutation of the primitive vectors $v_j \in \mathbb{Z}^m$. For $r \in R$, let Γ_r denote the stabilizer, which is a finite group. Then Sh_p is isomorphic to $\oplus_{r\in R}\mathbb{Z}[\Gamma/\Gamma_r]$.

Then

$$H_q(\Gamma, Sh_p \otimes_{\mathbb{Z}} M) \approx \oplus_{r\in R} H_q(\Gamma, \mathbb{Z}[\Gamma/\Gamma_r] \otimes_{\mathbb{Z}} M) \approx \oplus_{r\in R} H_q(\Gamma_r, M)$$

the last isomorphism following by Shapiro's lemma. Now assume d acts invertibly on M. Since d divides the order of Γ_r for every r, each term in the last direct sum is 0 if $q > 0$, and so $H_q(\Gamma, Sh_p \otimes_{\mathbb{Z}} M) = 0$ if $q > 0$.

Therefore, each term in the complex computing the sharbly homology, namely $E_1^{p,0} = H_0(\Gamma, Sh_p \otimes_{\mathbb{Z}} M)$, is isomorphic to the corresponding term of the complex that computes the Steinberg homology, namely $H_0(\Gamma, St \otimes_{\mathbb{Z}} M)$. The rest of (b) follows from (a). □

Remark 10. When d is not invertible on M, all we can assert is the following: (1) $E^\infty_{p,0}$ is a sub-$\mathcal{H}$-module of the sharbly homology $H_p(Sh_\bullet \otimes_\Gamma M) = E^2_{p,0}$. (2) $E^\infty_{p,0}$ is a quotient-$\mathcal{H}$-module of the Steinberg homology $H_p(St \otimes_\Gamma M)$. These assertions follow from standard facts about spectral sequences.

Let us now consider what we actually computed in [AGM02, AGM08, AGM]. Let W be the well-rounded retract for GL(4). Consider a cell σ of dimension $d > 0$ in W with minimal vectors $v_1, \dots, v_k$. Dual to this cell is the Voronoi cell with the same minimal vectors. Then $k + d = 10$ and σ corresponds to the sharbly $[v_1, \dots, v_k]$. Our computations involve only $d = 4, 5, 6$. (If $d = 0$, there is a cell σ with $k + d > 10$. For this cell, we would have to use a simplicial subdivision of the dual Voronoi cell and then convert to sharblies. Although we have no need to compute in this dimension, similar phenomena will appear widely for GL(m) with $m > 4$.)

Let us call a sharbly $[v_1, \dots, v_k]$ such that $v_1, \dots, v_k$ are the minimal vectors of a cell in W a *V-sharbly*. Then in the range $k = 4, 5, 6$, the $\mathbb{Z}$-spans of these V-sharblies form a subcomplex of the sharbly complex. We will also call any element of this span a V-sharbly. Write $[\sigma]$ for the V-sharbly corresponding to the cell σ. When we view this sharbly inside the Γ-coinvariants, as when computing the homology of Γ in $Sh_\bullet \otimes M$, we write it as $[\sigma]_\Gamma$.

We compute the bottom row of Brown's spectral sequence, i.e. $\mathrm{Ker}(d_1)/\mathrm{Im}(d_1)$, at the $E_1^{5,0}$ node. A typical cohomology class is thus represented by a linear combination $\sum a_\sigma \sigma$ where $a_\sigma \in M$ and σ runs over a set of representatives of 5-chains modulo Γ. We convert this to the V-sharbly $\sum a_\sigma [\sigma]_\Gamma$. We then compute the Hecke operator $T = \Gamma s \Gamma = \sum s_\alpha \Gamma$ on this sharbly, obtaining $\sum \sum a_\sigma [\sigma]_\Gamma | s_\alpha$. The algorithm of [Gun00] is then used to rewrite this last expression as homologous to some V-sharbly, $\sum b_\sigma [\sigma]_\Gamma$. This is reconverted into a homology class in the homology of the first spectral sequence at the same node, $\sum b_\sigma \sigma$ and we thus get a matrix for T in terms of the chosen basis of cycles.

When working away from the torsion primes of Γ, namely 2, 3 and possibly 5 (cf. Lemma 8), each of the two spectral sequences are zero above the first row, and they compute the same thing, Hecke-equivariantly. But at a torsion prime, there is no such vanishing, and the Hecke operators don't act on Brown's spectral sequence.

In effect, at primes that divide the torsion in Γ, we are computing as Hecke-module, not the cohomology of Γ with coefficients in M, but rather the homology of Γ with coefficients in $Sh_\bullet \otimes M$ at the $(1,0)$ node. The connection between this and the group cohomology, such as it is, may be seen by Corollary 2 and the remark following Lemma 9.

Since we have generalized (in Conjecture 5 (d)) the conjecture that expects Galois representations to be attached to Hecke-eigenclasses in the group cohomology, we can expect Galois representations to be attached to the Hecke-eigenclasses we are computing. This indeed happens in the examples we have computed so far—see Section 6.

There are two possible problems with these computations, both of which we conjecture do not arise in practice. First, it may not be possible to reduce $\sum \sum a_\sigma [\sigma]_\Gamma | s_\alpha$ to a homologous cycle of V-sharblies. However, if the algorithm of [Gun00] always terminates (as it always has in practice) such a reduction is always possible.

The second problem is that the sharbly cycle handed to us to perform Hecke operations on, namely $\sum a_\sigma [\sigma]_\Gamma$, could be a boundary in the sharbly complex even though it is not a boundary in the V-part of the sharbly complex. In this case, any Hecke eigenvalues we compute would be spurious, since they would be the "eigenvalues" of an operator on the 0-vector. If we could show that all elements in Sh_2 are homologous to V-sharblies (either by an algorithm or some other way) then this problem is irrelevant. We conjecture that this problem doesn't happen, as suggested by our data. The Hecke eigenvalues we compute never appear to be nonsensical, and we are always able to attach Galois representations to our putative Hecke eigenclasses. If this problem were happening, then we should be obtaining random numbers for supposed Hecke eigenvalues.

6. Computational data

In this section we present the results of our current torsion computations. This data comes in fact from computations of the $(1,0)$ node of the spectral sequence computing the homology of $Sh_\bullet \otimes M$, where the module M is $\mathbb{Z}$ with Γ acting trivially. Recall that the conjectures above concern modules that are $\mathbb{F}_p$-vector spaces. By the universal coefficient theorem, the same packages of Hecke eigenvalues we present also occur in the homology with coefficients in the trivial module $\mathbb{F}_p$.

We computed homology and Hecke operators using the same techniques in [AGM02, AGM08, AGM]. In particular, as in [AGM02] we use a slight modification of the

Level N	prime p	dimension	Level N	prime p	dimension
11	5	1	27	3	2
19	3	1	29	5	1
19	5	1	29	7	1
22	5	3	30	5	1
23	11	1	31	5	1
25	5	2			

TABLE 1. Odd torsion classes

sharbly complex $Sh_\bullet$ from Section 5 that includes the extra relation

$$[v_1, v_2, \ldots, v_{m+k}] - [-v_1, v_2, \ldots, v_{m+k}] = 0.$$

The resulting complex is homotopy equivalent to $Sh_\bullet$ as long as 2 is invertible in the coefficients. This causes no trouble for the current work, since as mentioned before we only report on p-torsion for p odd.

Table 1 shows the levels ≤ 31 that have p-torsion for p odd, and gives the dimension of the relevant homology group as an $\mathbb{F}_p$-vector space. We note that we have examples for $p = 3, 5, 7$, and that for $p = 5$ we have examples both where Γ has 5-torsion and where Γ doesn't (Lemma 8). If a level $N \leq 31$ doesn't appear in Table 1, it means that there was no odd torsion.

Table 2 gives the Hecke eigenvalue data. In all cases we computed $T(\ell, k)$ for $\ell = 2, 3, 5, 7$ and $k = 1, 2, 3$ (note that if $\ell \mid N$ then we actually compute an analogue of the U-operator by choosing different coset representatives). For level 30, we also computed operators at $\ell = 11, 13$. We give the eigenvalue data by presenting the Hecke polynomials of each eigenclass; that is, we give the polynomial on the left of (1).

In all cases we were able to match our eigenclasses to Galois representations. In particular, let ε denote the p-adic cyclotomic character of $G_{\mathbb{Q}}$. Thus $\varepsilon(\mathrm{Frob}_\ell) = \ell$. Then in all cases our eigenclasses matched the Galois representations $1 \oplus \varepsilon \oplus \varepsilon^2 \oplus \varepsilon^3$, except when $N = 30$. For this level, our data matches that produced by the representation $\beta \oplus \varepsilon \oplus \varepsilon^2 \oplus \beta\varepsilon^3$, where β is the even character $(\mathbb{Z}/30\mathbb{Z})^\times \to (\mathbb{Z}/5\mathbb{Z})^\times$ that takes 2 to -1. As usual, we do not know how to prove that these Galois representations are actually attached to their respective Hecke eigenclasses.

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$N = 11$, $p = 5$			
T_2	$1 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 19$, $p = 3$			
T_2	$1 + X^2 + X^4$	T_5	$1 + X^2 + X^4$
T_3	$1 - X$	T_7	$1 - X - X^3 + X^4$
$N = 19$, $p = 5$			
T_2	$1 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 22$, $p = 5$, eigenclass 1			
U_2	$1 - X - X^2 - X^3 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 22$, $p = 5$, eigenclass 2			
U_2	$1 + 2X + X^2 - 2X^3 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 22$, $p = 5$, eigenclass 3			
U_2	$1 + X - X^2 + X^3 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 23$, $p = 11$			
T_2	$1 - 4X + 4X^2 + X^3 - 2X^4$	T_5	$1 - 2X + 4X^2 + 3X^3 + 5X^4$
T_3	$1 + 4X + 5X^2 - 2X^3 + 3X^4$	T_7	$1 - 4X - 4X^2 + 3X^3 + 4X^4$
$N = 25$, $p = 5$, both eigenclasses			
T_2	$1 - X^4$	U_5	1
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 27$, $p = 3$, both eigenclasses			
T_2	$1 + X^4$	T_5	$1 + X^2 + X^4$
U_3	1	T_7	$1 - X - X^3 + X^4$
$N = 29$, $p = 5$			
T_2	$1 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$
$N = 29$, $p = 7$			
T_2	$1 - X - X^3 + X^4$	T_5	$1 - 2X - 2X^2 + 2X^3 + X^4$
T_3	$1 + 2X - 2X^2 - 2X^3 + X^4$	T_7	$1 - X$
$N = 30$, $p = 5$			
U_2	$1 - 2X + X^3 - X^4$	T_7	$1 - 2X + 2X^2 - X^3 - X^4$
U_3	$1 + 2X - 2X^2 - X^4$	T_{11}	$1 + X + X^2 + X^3 + X^4$
U_5	1	T_{13}	$1 + X - 2X^2 + 2X^3 - X^4$
$N = 31$, $p = 5$			
T_2	$1 - X^4$	T_5	$1 - X$
T_3	$1 - X^4$	T_7	$1 - X^4$

TABLE 2. Hecke polynomials for odd torsion classes

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