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A Vague Theory of Choice over Time*

Paola Manzini and Marco Mariotti

Abstract

We propose a novel approach to modelling time preferences, based on a cognitive shortcoming of human decision makers: the perception of future events becomes increasingly ‘blurred’ as the events are pushed further in time. Our model explains behavioural ‘anomalies’ such as preference reversal and cyclical choice.

KEYWORDS: time preferences, hyperbolic discounting, preference reversal, cycles, intransitive preferences

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1 Introduction

The standard model of choice over time is the exponential discounting model (EDM), according to which an outcome $x$ available at time $t$ is evaluated as $\delta^t u(x)$, with $\delta$ a constant discount factor and $u$ the instantaneous utility of $x$. However, in the last decade evidence has accumulated that is incompatible with the EDM. In particular, individuals often exhibit the so-called preference reversal phenomenon, whereby an initial preference between an earlier outcome and a later one is reversed once both outcomes are delayed by the same amount of time.

An alternative model, which is consistent with this type of evidence, has gathered increasing prominence and success in the literature. This is the model of hyperbolic discounting (HDM), in which the discount factor is a hyperbolic function of time. This functional form captures the idea that the rate of time preference between alternatives is not constant but varies, and in particular decreases as delay increases. The simplest and most widely used form of hyperbolic discounting occurs in a two-parameter model. In this $(\beta, \delta)$ model, the rate of time preference between a present alternative and one available in the next period is $\beta \delta$, whereas the rate of time preference between two consecutive future alternatives is $\delta$. So we may have for example $u(x) > \beta \delta u(y)$ and $\delta^{t+1} u(y) > \delta^t u(x)$, ‘rationalising’ a preference reversal. The HDM approach is extremely fruitful in addressing some of the issues raised by the experimental evidence, so much so that it has almost become a new consensus.

Yet there is also evidence against HDM. For example Rubinstein’s experiments ([26] and [27]) show that precisely the same type of decision situations that creates a difficulty for the EDM may also be problematic for the HDM. And Read [22] shows that the existing experimental evidence is consistent with other explanations, most notably subadditivity of discounting.\(^3\)

But human decision-makers have been shown to depart in an even more fundamental way from standard models: they may make intransitive choices. Although most data in this direction come from choices under risk, what evidence is available for time preferences suggests that violations of transitivity are even more frequent in this domain. Tversky et al. [35] show that a substantial 15% of subjects exhibited cyclical patterns of choice that could not be explained by ‘framing effects’. Roelofsma and Read’s [23] experiment is even more striking: they found that the majority of intertemporal choices

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\(^1\)See Fishburn and Rubinstein [7].
\(^2\)See for instance Phelps and Pollack [19], Loewenstein and Prelec [14], Laibson [13], Frederick et al. [8], O’Donoghue and Rabin [16] and [17].
\(^3\)See Frederick et al. [8] for a review of these issues.
were intransitive. This type of evidence, which we consider crucial in guiding our intuition about time preferences, has been somewhat downplayed in the mainstream economic literature on time preferences.\footnote{4}{A notable exception is the very general paper by Ok and Masatlioglu [18], who have axiomatised a class of models of choice over time. These include as particular cases HDM, Rubinstein’s ([26], [27]) similarity theory and Read’s subadditive discounting among several others, and some of the representations admitted by their axioms are compatible with intransitivities.}

We view the body of evidence against the EDM and the HDM as indicating that human agents follow heuristic procedures to make choices over time that differ fundamentally from the simple maximisation of a complete preference ordering or of a discounted utility function. For this reason we believe that looking directly at the possible cognitive sources of the ‘anomalies’ can be useful. The hypothesis we study in this paper is that at the root of phenomena like preference reversals and intransitivities is a basic cognitive shortcoming, noted by several thinkers on the topic and in popular wisdom: the perception of events that will occur in the future is ‘blurred’, and possibly it becomes increasingly blurred as the events are pushed farther in time\footnote{5}{For example, according to Pigou [20] “our telescopic faculty is defective, and we, therefore, see future pleasures, as it were, on a diminished scale”. Also, Böhm-Bawerk [4] stated that “…we limn a more or less incomplete picture of our future wants and especially of the remotely distant ones.”}

The decision-maker may be in two ‘states’, depending on the pair of alternatives he is evaluating: ‘vague’ or ‘lucid’. When a decision maker is ‘lucid’ he makes ‘rational’ trade-offs between the time and outcome dimensions; when he is ‘vague’ and unable to evaluate a trade-off, he resorts to a heuristic-based criterion. In modelling this criterion we follow some evidence and theory in the psychology literature. As for example Tversky \textit{et al.} [36] note:

‘Because it is often unclear how to trade one attribute against another, a common procedure for resolving conflict in such situations is to select the option that is superior on the more important attribute. This procedure, which is essentially lexicographic, has two attractive features. First, \textit{it does not require the decision maker to...}’

\footnote{6}{These notions are well established in proverbs, from “we’ll cross that bridge when we come to it” to “hindsight is better than foresight”, “it will all be the same in a hundred years” for the anglosaxon tradition, to “En cien años todos calvos” and “Tra tre anni beato chi ha un occhio” from the Spanish and Italian tradition, respectively (which translate as “In a hundred years we’ll all be bald” and “Blessed he who still has got an eye left in three years”. Interestingly, for the latter proverb the user is free to change the length of time as she sees fit). See for instance Taylor [32], Simpson and Speake [29], Rovira [24], Pittano [21].}
assess the trade-off between the attributes, thereby reducing mental effort and cognitive strain. Second, it provides a compelling argument for choice that can be used to justify the decision to oneself as well as to others.’ (p. 505, our italics)

So our ‘vague’ decision maker, too, is assumed to go through a lexicographic consideration of the two dimensions (time and outcome in our context) involved in the comparison. That is to say, when unable to resolve a trade off, the decision maker chooses the alternative that is better according to the dimension that he regards as the most important.

Our contribution is two-fold. First, we provide a class of parsimonious representations of an individual’s basic time-outcome trade-off, in which a standard utility function (of time and outcomes) is combined, in an additive way, with a ‘vagueness function’. At the formal level, these representations take the form of interval orders or semiorders. Second, we focus on a very simple such representation - which we dub the \((\sigma, \delta)\) model - and combine it with the ‘vagueness-breaking’ criterion described above in order to examine choice behaviour. Needless to say, heuristics by definition cannot be axiomatised; we show how a simple ‘rule of thumb’ which has been widely studied in the psychological literature can be used to complete the basic transitive partial order when the latter leaves the decision maker ‘vague’.

In this way we can explain for instance the phenomenon of preference reversal with an interpretation that seems to us more adherent to the psychological basis for it. More strikingly, the model can also account for cyclical patterns of choice, which are obviously incompatible with theories such as HDM and EDM, based as they are upon the maximisation of a preference ordering.

The structure of the paper is as follows. We start by postulating a partial order (the ‘primary criterion’) on the set of outcome-date pairs, together with a ‘secondary criterion’ which embodies the vagueness-breaking heuristics (section 2). Next, we narrow down further this partial order, and impose axioms which specialise it to an interval order or a semiorder, which can be represented in a standard, additively separable, way (section 3). In section 4 we present our \((\sigma, \delta)\) model and consider preference reversal, cycles and a possible basis for the experimental elicitation of the parameters of the model. In the concluding section 5 we study some relation with literature, notably with Rubinstein’s [25] similarity model, and we offer some remarks on possible extensions of our model.
2 The Model

Let $X$ indicate a set of outcomes (e.g. money amounts) on which there exists a reflexive, complete and transitive preference relation $R$. We denote the symmetric and asymmetric components of $R$ by $I$ and $P$, respectively. Let $T$ be the set of dates. The set of alternatives is a set of outcome-date pairs $A \subseteq X \times T$.

We imagine that the preference relation $\succeq^*$ of the individual on $A$ is constructed as follows. There exists a primary criterion on the basis of which the individual makes comparisons between alternatives. We interpret the primary criterion as resolving comparisons for which the trade-off between outcome and delay yields, in the perception of the individual, a decisive advantage to one of the alternatives: that is, for example, the loss in the time dimension of one alternative clearly outweighs its advantage in the outcome dimension. Because of the cognitive limitations discussed in the introduction, however, there may be pairs of alternatives that cannot be ranked by the primary criterion alone: the outcome-delay trade-off may not offer a basis for decision, and the individual has cognitive ‘holes’. In this case, and only in this case, a secondary criterion is used. The relation $\succeq^*$ is derived by a combination of primary and secondary criteria.

More formally, let the primary criterion be a strict partial order $\succ$ on $A$ (that is, an irreflexive and transitive binary relation on $A$). When a pair of alternatives $a, b \in A$ is not ordered by $\succ$ we denote this fact by $a \sim b$, i.e.

$$(\text{not } a \succ b, \text{not } b \succ a) \iff a \sim b$$

and say that $a$ and $b$ are vague. Under our assumptions the relation $\sim$ may not be transitive. As mentioned above, $\sim$ does not mean indifference between two alternatives, but simply the inability to express a strict preference. We call this situation vagueness.

The secondary criterion is used to rank alternatives which are vague. As explained in the introduction, there is psychological evidence that when individuals are unable to compare alternatives on the basis of a trade-off between the various dimensions of those alternatives, they rely on lexicographic-type comparisons, by focussing on the ‘dominant’ dimension\(^7\). In our model there are two dimensions, time and outcome. Thus, a natural secondary criterion is to allow the individual to rank two alternatives which are vague under $\succ$ first according to the preference ordering $R$ over the (pure) outcomes, and if this

\(^7\text{See e.g. Slovic [30], Tversky et al. [36] and Shafir et al. [28].}\)
still does not result in a strict preference, according to time precedence. Formally, let $\succeq^*$ denote a complete preference relation (not necessarily transitive) on $A$. So we consider the following model:

Outcome Prominence Model (OPM): Let $i = (x_i, t_i) \in A$. Then:

1. $a \succ^* b \iff$
   
   (a) $a \succ b$ (Primary Criterion), or
   
   (b) $(a \sim b, x_a P x_b) \lor (a \sim b, x_a I x_b, t_a < t_b)$ (Secondary Criterion)

2. $a \sim^* b \iff (a \sim b, x_a I x_b, t_a = t_b)$

The other natural model in place of OPM is:

Time Prominence Model (TPM): Let $i = (x_i, t_i) \in A$. Then:

1. $a \succ^* b \iff$
   
   (a) $a \succ b$ or
   
   (b) $(a \sim b, t_a < t_b) \lor (a \sim b, t_a = t_b, x_a P x_b)$

2. $a \sim^* b \iff (a \sim b, x_a I x_b, t_a = t_b)$

3 General representation results

In this section we provide a number of representation results for the primary ordering $\succ$. These are based on various assumptions on both the structure of the set of alternatives and the cognitive abilities of the decision maker. All the results represent preferences by means of two real valued functions $u$ and $\sigma$ on the set of alternatives.

The natural interpretation is that the $u$ function expresses the ‘pure’ time preferences of the individual. This function can be chosen to be monotonic in the natural direction in both outcome and time. On the other hand $\sigma$ is the ‘vagueness function’ which captures the imaginative difficulty the individual faces when appraising outcomes in the future. There are two notable aspects of the representations. First, the vagueness term enters additively. Second, the vagueness when appraising whether $a = (x, t_x)$ is better than $b = (y, t_y)$ is a function of the components of $b$ alone; it does not depend on $t_x$ or $x$.

In what follows we say that a preference ordering $\succ$ is $(u, \sigma)$-representable if there exist real valued functions $u$ and $\sigma$ on $A$ such that

$$(x, t_x) \succ (y, t_y) \iff u(x, t_x) > u(y, t_y) + \sigma(y, t_y) \forall (x, t_x), (y, t_y) \in A$$
It is worth comparing our class of representations with that of Ok and Masatlioglu [18]. They consider a complete binary preference relation $B$ over a set of outcome-date pairs and axiomatise the following representation class:

$$(x, t) B (y, s) \text{ if and only if } U(x) \geq U(y) + \varphi(s, t),$$

where $U$ is interpreted as an instantaneous utility function and $\varphi$ captures the effect of time delay. In this representation, which is not an interval order, the ‘contributions’ of outcome and time to the agent’s utility are separated, unlike in our model. Note that in Ok and Masatlioglu’s approach cycles can be accounted for without resorting to a secondary criterion. Our view is different: the primary criterion represents the ‘fully rational’ component of decision making, hence we assume it transitive. In our approach intransitivities are the byproduct of resorting to the rule of thumb invoked when ‘full rationality’ is not decisive.

### 3.1 Finite sets of alternatives

With vague time preferences, an individual is not always able to compare two alternatives $a$ and $b$. However we require that the individual has at least some discriminatory ability regarding the sets of alternative dominated by $a$ and $b$, respectively. Formally, let $L(a)$ denote the lower contour set of $a \in A$, that is $L(a) = \{b | a \succ b\}$, and let $V(a)$ denote the vague set of $a$, that is $V(a) = \{b | a \sim b\}$. Then we assume:

**Discrimination:** For every pair of alternatives $a, b \in A$ either $L(a) \subseteq L(b)$ or $L(b) \subseteq L(a)$.

Transitivity of $\succ$ ensures that $L(b) \subseteq L(a)$ whenever $a \succ b$, thus Discrimination has a bite only in the case of vague alternatives. Two alternatives $a$ and $b$ may be so vague that they also have the same set of worse alternatives. But suppose that the agent perceives that alternatives that are not worse than $a$ are worse than $b$. In this case, the agent perceives some kind of improvement in moving from $a$ to $b$ (though not strong enough to make $b$ preferred to $a$). The property of Discrimination requires that after the ‘improving’ move to $b$ no alternative that was dominated by $a$ ceases to be dominated (and it poses the converse requirement if the move to $b$ was perceived to be negative). In other words, moving from one alternative to the other may involve either an (weak) improvement or a (weak) worsening in the above sense, but not both.\(^8\)

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\(^8\)An alternative justification for Discrimination is the ‘sure-thing’ property if alternatives were allowed to have a risk dimension as well, i.e. lotteries of the type $(\frac{1}{2}, a; \frac{1}{2}, b)$ over the riskless alternatives $a$ and $b$ were included in the domain. Sure-thing implies that if $a \succ b$ and $c \succ d$ then $(\frac{1}{2}, a; \frac{1}{2}, c) \succ (\frac{1}{2}, b; \frac{1}{2}, d)$ and similarly for $\sim$. It is easily seen that Discrimination on the riskless alternative would follow.
Next, we consider some straightforward monotonicity properties:

**Time Monotonicity:** For every $x \in X$ and $t_1, t_2 \in T$ such that $t_2 \geq t_1$:

$L(x, t_2) \subseteq L(x, t_1)$.

The requirement of Time Monotonicity is that anything which is dominated by some alternative $a$ must also be dominated by an alternative with the same outcome as $a$ but at an earlier time. Note that these allow for two alternatives with the same outcome available at different times to be vague. A similar requirement can be imposed in the outcome dimension:

**Outcome Monotonicity:** For every $x_1, x_2 \in X$ and $t \in T$ such that $x_2 
Rightarrow x_1$:

$L(x_1, t) \subseteq L(x_2, t)$.

Our next representation result applies to the case where the number of alternatives is arbitrarily large but finite.

**Proposition 1** Let Time Monotonicity, Outcome Monotonicity and Discrimination hold. Let $L(a)$ be finite for all $a \in A$. Then $\succ$ is $(u, \sigma)$-representable with $\sigma(a) > 0$ for all $a \in A$ and $u$ non-decreasing in outcome and non-increasing in time.

**Proof.** For any $a \in A$, define $u(a) = \#L(a)$. By Time Monotonicity and Outcome Monotonicity the function $u$ is weakly monotonic in the desired direction with respect to time and outcomes. Next, define $\sigma(a)$ as follows. Choose

$$a^* \in \arg\max_{b \in V(a)} \#L(b)$$

Then let

$$\sigma(a) = \max \left\{ \frac{1}{2}, u(a^*) - u(a) \right\}$$

Observe that for any two alternatives $c, d$ with $c \succ d$ it must be $u(c) > u(d)$ (since $L(d) \subseteq L(c)$ by the transitivity of $\succ$, and since $d \in L(c)$ but $c \notin L(c)$, implying $\#L(c) > \#L(d)$).

We show that $u$ and $\sigma$ as defined above represent $\succ$.

Let $a \succ b$. Then $u(a) > u(b)$ by the above observation. Now suppose in negation that

$$u(a) \leq u(b) + \sigma(b)$$

By definition $\sigma(b) = \max \left\{ \frac{1}{2}, u(b^*) - u(b) \right\}$. If $\sigma(b) = \frac{1}{2}$ we get an immediate contradiction (since $u(a) - u(b) > \frac{1}{2}$). So let $\sigma(b) = u(b^*) - u(b)$ and therefore

$$u(a) \leq u(b) + \sigma(b) = u(b^*)$$
By the above inequality and the definition of \( u, \# L(b^*) \geq \# L(a) \). Then by Discrimination \( L(b^*) \supseteq L(a) \). By the fact that \( b \in L(a) \) this implies \( b \in L(b^*) \), contradicting \( b \in V(b^*) \).

Consider now \( a \sim b \). We need to show that both \( u(a) \leq u(b) + \sigma(b) \) and \( u(b) \leq u(a) + \sigma(a) \). If \( b = a^* \) then \( u(b) = u(a^*) \leq u(a) + \sigma(a) \), and if \( b \neq a^* \) then, by the definition of \( a^* \), \( u(b) \leq u(a^*) \leq u(a) + \sigma(a) \). Therefore in all cases \( u(b) \leq u(a) + \sigma(a) \). The same argument applies interchanging \( a \) and \( b \), so that \( u \) and \( \sigma \) as defined above represent \( \sim \).

Finally assume \( u(a) > u(b) + \sigma(b) \). Then by Discrimination \( L(a) \supseteq L(b) \). So it cannot be \( b > a \), since then we would have \( a \in L(b) \) but \( a \not\in L(a) \), a contradiction. It cannot be \( b \in V(a) \) either, since then \( u(a) \leq u(b) + \sigma(b) \).

The case of a finite set of alternatives \( A \) is a natural setting for this result. Observe however that all that is required is that \( L(a) \) is finite.

The assumptions of Discrimination and Time and Outcome Monotonicity are too weak to allow for a constant \( \sigma \), even with simple finite sets of alternatives. We illustrate this with an example (see figure 1, in which the arrows denote dominance and the dashed lines denote vagueness).

**Example 2** \( X = \{x, y\} \) with \( xPy \) and \( T = \{0, 1, 2\}, A = X \times T \) and \( (x, 0) \succ (x, 1) \succ (x, 2), (x, 0) \succ (y, 2) \) with all other relations not implied by the transitivity of \( \succ \) being ones of vagueness. So, \( L(x, 0) = \{(x, 1), (x, 2), (y, 2)\} \), \( L(x, 1) = \{(x, 2)\} \) and for all other \( a \in A \), \( L(a) = \emptyset \). Then it is easily checked that these preferences satisfy Discrimination, Time Monotonicity and Outcome Monotonicity, so that there exists a \((u, \sigma)\) representation. However, the relations \( (x, 0) \succ (x, 1), (x, 1) \succ (x, 2), (y, 1) \sim (x, 2) \) and \( (x, 0) \sim (y, 1) \) with \( \sigma \) constant would be equivalent to:

\[
\begin{align*}
u(x, 0) & > u(x, 1) + \sigma \\
u(x, 1) & > u(x, 2) + \sigma \\
u(y, 1) & \leq u(x, 2) + \sigma \text{ and } u(x, 2) \leq u(y, 1) + \sigma \\
u(x, 0) & \leq u(y, 1) + \sigma \text{ and } u(y, 1) \leq u(x, 0) + \sigma
\end{align*}
\]

\(^9\)This result is related but not the same as Theorem 2.7 in Fishburn [6]. There are several technical differences between Fishburn’s result and ours. First, Fishburn’s theorem applies to the case where the number of vagueness classes is countable (i.e. the set \( X/\sim \) is countable), whereas ours applies to the case where the lower contour sets of the strict preference relation are finite. Second, as a consequence, we can give a short proof of our result based on a simple direct construction of the utility and vagueness functions. Fishburn uses an indirect argument based on the construction of a set of artificial alternatives, the use of the Axiom of Choice, and the use of several intermediate representation theorems. Finally, we exploit the structure of the alternative set to characterize the monotonicity properties, which are not present in the more abstract setting of Fishburn’s.
From the last two lines it follows that

$$ u(x, 0) - \sigma \leq u(x, 2) + \sigma $$

whereas from the first two inequalities we have

$$ u(x, 0) - \sigma > u(x, 2) + \sigma $$

a contradiction.

**Remark.** We conclude by noting that, given our assumptions, the projection of $\succeq^*$ on the set $X$ of ‘pure’ alternatives coincides with the complete order $R$ whenever the partial order $\succ$ is $(u, \sigma)$ representable with $u$ non-decreasing in outcome and non-increasing in time. To see this, take the comparison between two outcomes $x, y \in X$ available at time $t$. Assume first that $xPy$. Then by Outcome monotonicity $u(x, t) \geq u(y, t)$, so that it cannot be that $(y, t) \succ (x, t)$. Now there can be two possibilities: if $u(x, t) > u(y, t) + \sigma (y, t)$, then $(x, t) \succ^* (y, t)$; if instead $u(x, t) \leq u(y, t) + \sigma (y, t)$, then $(x, t) \sim (y, t)$, so that by the secondary criterion $xPy$ implies $(x, t) \succ^* (y, t)$. In both cases, $\succ^*$ and $P$ agree.

Next, suppose that $(x, t) \succ^* (y, t)$. This can be the case either because $(x, t) \not\succ (y, t)$ and $(y, t) \not\succ (x, t)$, but $xPy$, in which case $\succ^*$ and $P$ agree; or because $(x, t) \succ (y, t)$. If the latter, since $\succ$ is $(u, \sigma)$ representable, it must be
Because $u$ is non-decreasing in outcome, it cannot be that $yRx$, since that would require $u(y, t) \geq u(x, t)$, so that it must be $xPy$.

### 3.2 Infinite sets of alternatives

This section is a little more technical, its main thrust being to show that our representation result is valid beyond finite sets of alternatives, and can be skipped on a first reading without missing any substantial ingredient. Consider the case where the set of alternatives is allowed to be infinite, and in particular let $X = T = [0, 1]$. A technique similar to the one used for proposition 1 can be applied, provided only that the sets $L(a)$ are sufficiently regular as to be Lebesgue-measurable (this will be the case in virtually every conceivable application), and that the following hold:

**Continuity:** If $a = (x, s) \succ b = (y, t)$ then there exists $\varepsilon > 0$ such that (i) $(x', s') \succ b$ for all $x', s' \in [0, 1]$ with $|x' - x| < \varepsilon$ and $|s' - s| < \varepsilon$, and (ii) $a \succ (y', t')$ for all $y', t' \in [0, 1]$ with $|y' - y| < \varepsilon$ and $|t' - t| < \varepsilon$.

**Betweenness:** If $a \succ b$ then there exists $(x, s) \in L(a) \setminus L(b)$ and $\varepsilon$ such that $(x', s') \in L(a) \setminus L(b)$ for all $x', s' \in [0, 1]$ with $|x' - x| < \varepsilon$ and $|s' - s| < \varepsilon$.

Continuity says that a strict preference between two alternatives should hold for all other alternatives sufficiently close the original two. Betweenness says that there exists a (non-negligible) set of alternatives which lies ‘between’ (in terms of preference) any two strictly ranked alternatives. ‘Between’ in this context means: worse that the better alternative and not worse than the worse alternative. Such assumptions would be satisfied, for example, in a model with ‘thick’ vagueness curves through an alternative, that separate the upper and lower contour sets. However, there may be other patterns of cognitive ‘holes’, for example one where vagueness occurs very close to an alternative (because the individual does not distinguish such close alternatives), it occurs again at far away alternatives (for example because of the delay), and does not occur in the intermediate region.

**Proposition 3** Let $X = T = [0, 1]$. Let Discrimination, Time Monotonicity, Outcome Monotonicity, Continuity and Betweenness hold. Then $\succ$ is $(u, \sigma)$-representable with $\sigma(a) \geq 0$ for all $a \in A$ and $u$ non-decreasing in outcome and non-increasing in time.

**Proof.** The proof follows the same steps as the proof of proposition 1, taking care of some additional technical details. It is given for completeness in the Appendix.
We can obtain a similar representation in the case of a countable set of alternatives, using an alternative notion of continuity:

**Proposition 4** Let $X$ and $T$ be countable sets. Let Discrimination, Time Monotonicity (i), Outcome Monotonicity (i) hold. Suppose that $(\lim a_n) \in V(a)$ for any sequence $\{a_n\}$ with $a_n \in V(a)$. Then $\succ$ is $(u, \sigma)$-representable with $\sigma(a) \geq 0$ for all $a \in A$ and $u$ non-decreasing in outcome and non-increasing in time.

**Proof.** In the Appendix.

Finally, we provide a result which does not make any structural assumption on the set of alternatives but is formulated directly in terms of a representation. We need two additional straightforward monotonicity properties.

**Time Monotonicity**: For every $x, y \in X$ and $t, t_1, t_2 \in T$ such that $t_2 > t_1$: $(x, t_2) \in V(y, t)$ implies $(x, t_1) \notin L(y, t)$.

**Outcome Monotonicity**: For every $y, x_1, x_2 \in X$ and $t, t' \in T$ such that $x_2 P x_1$: $(x_1, t) \in V(y, t')$ implies $(x_2, t) \notin L(y, t')$.

Time Monotonicity says that, given a relationship of vagueness between two contemporaneous alternatives, it cannot be that anticipating one of them makes it dominated by the other one. Outcome Monotonicity makes an analogous statement for the outcome dimension.

**Proposition 5** Let $\succ$ be $(u, \sigma)$-representable with $\sigma(a) \geq 0$ for all $a \in A$ and let Time Monotonicity, Time Monotonicity*, Outcome Monotonicity and Outcome Monotonicity* hold. If $\sigma$ can be chosen to be non-decreasing in outcome, then $u$ can be chosen to be non-decreasing in outcome at an arbitrarily large number of alternatives, and if $\sigma$ can be chosen to be non-decreasing in time then $u$ can be chosen to be non-decreasing in time at an arbitrarily large number of alternatives.

**Proof.** In the Appendix.

Proposition 5 shows that the utility function can be chosen to display the expected monotonicity properties (on an arbitrarily large number of alternatives) with respect to the delaying and improvement of outcomes, provided that vagueness does not increase as the outcome improves and it does not decrease as delay increases. This latter condition in particular accords well with our motivating intuition, that the distance in time has a blurring effect on
the comparisons between alternatives. Of course, this result is not completely satisfactory because we would like to have an assumption on the primitives, namely the preferences, implying the desired conditions on the function \( \sigma \).

In the next section we concentrate on one element of the class of representations obtained, which incorporates the feature of (relative) vagueness increasing with delay.

## 4 The \((\sigma, \delta)\) model

From now on we consider \( X \) as a real interval, e.g. money amounts. For this setting we propose our \((\sigma, \delta)\) model, a specialisation of the OPM and TPM. In doing so we follow the hyperbolic discounting tradition whereby out of the entire hyperbolic family attention has been focused on the application friendly \((\beta, \delta)\) version. There are several motives for studying our chosen specialisation. First, it constitutes in a sense the minimal possible departure from the standard EDM; yet its implications are still widely different from that of the EDM. Second, it is parsimonious, in that it depends only on two parameters, which can be potentially estimated (see section 4.3). Third, the parameters have a natural interpretation, and in particular they incorporate the psychological feature at the root of our approach, that as choices are pushed further into the future the ‘vagueness’ associated with the outcomes increases.

It is obvious, but still worth emphasising, that many other members of the class of representations obtained in the previous sections would explain any of the anomalies to be considered below. For an analogy, consider the popular \((\beta, \delta)\) version of hyperbolic discounting: it has been chosen in most applications not because the particular discounting function associated with it has an axiomatic foundation, but rather because it is simple and it departs from exponential discounting in a minimal way, while being able to explain some EDM anomalies.\(^{10}\)

The \((\sigma, \delta)\) model is a representation of the individual’s (primary) preferences that depends on a discount parameter \( \delta \in (0, 1) \) and a vagueness parameter \( \sigma > 0 \). The individual values alternatives using exponential discounting with factor \( \delta \) and linear utility. However, in order to regard an alternative better than another one, its discounted utility must exceed that of a competing alternative by at least \( \sigma \) (Primary Criterion). That is, loosely speaking,

\(^{10}\)For another analogy, the von Neumann-Morgenstern axioms say nothing about the convexity or otherwise of the utility function. However an ‘inflated’ von Neumann-Morgenstern utility function has been postulated in the classic paper by Friedman and Savage [9] to explain the puzzle of people’s willingness both to gamble and to buy insurance.
δ measures the way the individual trades off outcomes across time, while σ measures the precision with which such trade-offs can be made (see section 4.3 below for a more precise interpretation). When the individual cannot compare alternatives on the basis of present discounted value, his preferences conform to the Secondary Criterion, that is: in the OPM he prefers the alternative corresponding to the higher amount of money, and if the monetary amounts are the same for both alternatives, then he prefers the alternative corresponding to the earlier date; while in the TPM he prefers the alternative delivering the money at the earlier date, and if the delivery dates are the same for both alternatives, then he prefers the alternative yielding the higher amount of money. Formally: in the (σ, δ) OPM we have that \((x, t_x)\) is chosen over \((y, t_y)\) whenever either

\[
xδ^{t_x} > yδ^{t_y} + σ
\]

or

\[
xδ^{t_x} \leq yδ^{t_y} + σ,
\]

\[
yδ^{t_y} \leq xδ^{t_x} + σ
\]

\[
x > y
\]

or

\[
xδ^{t_x} \leq yδ^{t_y} + σ
\]

\[
yδ^{t_y} \leq xδ^{t_x} + σ
\]

\[
x = y \text{ and } t_x < t_y
\]

In the (σ, δ) TPM instead \((x, t_x)\) is chosen over \((y, t_y)\) whenever either

\[
xδ^{t_x} > yδ^{t_y} + σ
\]

or

\[
xδ^{t_x} \leq yδ^{t_y} + σ,
\]

\[
yδ^{t_y} \leq xδ^{t_x} + σ
\]

\[
t_x < t_y
\]

or

\[
xδ^{t_x} \leq yδ^{t_y} + σ
\]

\[
yδ^{t_y} \leq xδ^{t_x} + σ
\]

\[
t_x = t_y \text{ and } x > y
\]
4.1 Cycles

Several contributions in decision theory have pointed out on both normative and descriptive grounds that human decision procedures may generate non transitivity in choice (see e.g. Tversky [34]; Rubinstein [25]; Tversky et al. [35]; Vilà [38]; Roelofsma and Read’s [23]; Read [22]; Ok and Masatlioglu [18]). To understand this phenomenon, consider the following non time-related dietary example. You like your coffee sweet to the point of neglecting dietary considerations, so that one sugar spoon is preferred to half a sugar-spoon. However, the sweetening impact of a quarter spoon is negligible to your tongue: in this case dietary considerations prevail and you prefer half a spoon to three quarters and three quarters to one!11

With time preferences, we have argued for an analogous ‘switch of criterion’ when vagueness occurs. So the preference relation $\geq^*$ generated by the OPM and TPM is not necessarily transitive, even if both the primary and secondary criteria are, and this is true for the $(\sigma, \delta)$ specialisation too.

Let us study the sufficient conditions for a cycle to occur with three alternatives $(x, t_x), (y, t_y)$ and $(z, t_z)$, with $x > y > z$.

In the OPM we need

$$
z \delta^{t_z} > x \delta^{t_x} + \sigma$$
$$y \delta^{t_y} \leq x \delta^{t_x} + \sigma \text{ and } x \delta^{t_z} \leq y \delta^{t_y} + \sigma$$
$$y \delta^{t_y} \leq z \delta^{t_z} + \sigma \text{ and } z \delta^{t_z} \leq y \delta^{t_y} + \sigma$$

so that $(z, t_z) \succ^* (x, t_x)$ but $(x, t_x) \sim (y, t_y)$ and $(y, t_y) \sim (z, t_z)$ by the Primary Criterion. By the Secondary Criterion $(x, t_x) \succ^* (y, t_y)$ and $(y, t_y) \succ^* (z, t_z)$. To see that these inequalities are compatible, let for instance $(x, t_x) = (10, 3)$, $(y, t_y) = (9, 2)$ and $(z, t_z) = (8, 1)$. Then

$$\sigma < 8\delta - 10\delta^3$$

and

$$\sigma \geq \max \{ |9\delta^2 - 10\delta^3|, |9\delta^2 - 8\delta| \}$$

These inequalities can be represented in the $(\sigma, \delta)$ space, with $\delta$ is measured on the horizontal axis, as the gray and kinked black line in figure 2: any combination of $\delta$ and $\sigma$ lying below the gray line (representing the locus $\sigma = 8\delta - 10\delta^3$) and above the black one (locus $\sigma = \max \{ |9\delta^2 - 10\delta^3|, |9\delta^2 - 8\delta| \}$) satisfies the inequalities above.

11Similarly, our friend Faye likes a sweet at the end of a meal but nonetheless opts for a synthetic sweetener with her coffee.
Alternatively, in the TPM we need

\[ x \delta_{t_x} > z \delta_{t_z} + \sigma \]
\[ y \delta_{t_y} \leq x \delta_{t_x} + \sigma \text{ and } x \delta_{t_x} \leq y \delta_{t_y} + \sigma \]
\[ y \delta_{t_y} \leq z \delta_{t_z} + \sigma \text{ and } z \delta_{t_z} \leq y \delta_{t_y} + \sigma \]

with \( t_x > t_y > t_z \), so that \((x, t_x) \succ^* (z, t_z)\) but \((x, t_x) \sim (y, t_y)\) and \((y, t_y) \sim (z, t_z)\) by the Primary Criterion. By the Secondary Criterion \((z, t_z) \succ^* (y, t_y)\) and \((y, t_y) \succ^* (x, t_x)\).

Using the same alternatives as in the example above, the inequalities are shown in figure 3 below, where the locus \( \sigma = x \delta_{t_x} - z \delta_{t_z} = 10\delta^3 - 8\delta \) is represented by the gray line and \( \sigma = \max \left\{ |9\delta^2 - 10\delta^3|, |9\delta^2 - 8\delta| \right\} \) is represented by the black line as in the previous example.

For sufficiently high values of \( \delta \) there exists a range of values of \( \sigma \) compatible with the TPM cycle.
4.2 Preference reversal

We now study under what conditions the \((\sigma, \delta)\) OPM generates preference reversals in a discrete model. Recall that preference reversal at time \(t'\) is the shorthand for the following situation

\[
\begin{align*}
(x, t') & \succ^* (y, t') \\
(y, t' + t'') & \succ^* (x, t + t'')
\end{align*}
\]

where \(t' \geq t\). Then if preferences can be \((u, \sigma)\) represented and OPM holds, conditions for preference reversal as expressed above are

\[
\begin{align*}
u(x, t) & > u(y, t') + \sigma(y, t') \\
u(x, t + t'') & \leq u(y, t' + t'') + \sigma(y, t' + t'') \\
u(y, t' + t'') & \leq u(x, t + t'') + \sigma(x, t + t'')
\end{align*}
\]

with \(xPy\). Note that, given that the \(\sigma(,)\) function is positive, if the first inequality holds so does the third one. Turning to the case of exponential discounting and constant \(\sigma\), let \(u(x, t) = \delta^t x\), with \(\delta \in (0, 1)\). System (1) above simplifies to

\[
\begin{align*}
x\delta^t & > y\delta^{t'} + \sigma \\
x\delta^{t + t''} & \leq y\delta^{t' + t''} + \sigma
\end{align*}
\]

or:

\[
\sigma \in \left[ (x\delta^t - y\delta^{t'}) \delta^{t''}, x\delta^t - y\delta^{t'} \right] \equiv I(x, y, \delta, t, t', t'')
\]
So the vagueness factor $\sigma$ must be sufficiently small to make the smaller amount initially more attractive than the larger, later one; and sufficiently large so that the same comparison is resolved in the opposite direction when all alternatives are pushed forward by $t''$. When $\sigma$ lies in the interval $I(x, y, \delta, t, t', t'')$ the present discounted value criterion is not sufficiently precise to rank $(x, t + t'')$ and $(y, t' + t'')$. The individual resorts to the Secondary Criterion, and prefers $(y, t' + t'')$.

Given $\sigma$ and the preference at the initial date, preference reversal will eventually occur after a sufficiently long time. This is due to the fact that although $\sigma$ is constant in absolute terms, the vagueness relative to the utilities involved is increasing with time, as the discounted values of the two amounts (and so their difference) tends to 0 as time tends to infinity. Note that in our setting preference reversal is possible whatever the date for the earliest alternative, whereas the $(\beta, \delta)$ model can explain preference reversal only if the earliest alternative occurs at time 0. Preference reversal may persist even when $\delta$ tends to one.

With the $(\sigma, \delta)$ TPM, too, one can obtain preference reversal (as can be easily checked by performing a set of calculations symmetric to the one above). However, in that case the pairwise choices are inverted, and therefore contrary to the direction of preference reversal which is commonly observed experimentally.

Finally, although in our model the primary ordering is not assumed to be a ‘time preference’ in the technical sense of Masatlioglu and Ok [18], our justification of preference reversal is not due to this feature: preference reversal can occur even when within each period primary preferences are complete, transitive and unchanging through time, and satisfy additional continuity conditions as required in that paper.

### 4.3 Interpretation of $\delta$ as a SPOt ratio

We stated before that $\delta$ measures the way the individual trades off outcomes across time. This can be made formally more precise. For brevity we focus on the OPM. For any alternative $(x, t_x)$ we are not able to define to define a ‘present value’ at time $t < t_x$ in the standard way, as an alternative at time $t$ which is indifferent to the later alternative $(x, t_x)$. This is because in the $(\sigma, \delta)$ model no two distinct alternatives are indifferent, as $\succ^*$ is a strict order. However, we can define a closely related concept, the Smallest Preferred Outcome at time $t$, or SPOt. This is the outcome $s(x, t_x, t)$ which solves

$$
\delta^t s(x, t_x, t) = \delta^{t_x} x + \sigma
$$
Note that \( s(x, t_x, t) \) is not exactly a present value in the sense of being indifferent to \((x, t_x)\). However, \( s(x, t_x, t) \) is the infimum of the outcomes available at time \( t \) which are preferred to \( x \). To see this, assume that \( \sigma \) is sufficiently small that \( s(x, t_x, t) < x \). Then all outcomes at time \( t \) which are greater than \( s(x, t_x, t) \) are preferred to \((x, t_x)\) by the Primary Criterion, and \((x, t_x)\) is preferred to all outcomes at time \( t \) which are not greater than \( s(x, t_x, t) \) by either the Primary or the Secondary Criterion. So a SPOt expresses an individual’s ‘willingness to pay’ to anticipate an alternative to time \( t \). Then for \( 0 \leq t - 1, t < t_x \) we have \( \delta^t s(x, t_x, t) = \delta^t x + \sigma = \delta^{t-1} s(x, t_x, t - 1) \), so that \( \delta \) can be written as a ratio of SPOt’s:

\[
\delta = \frac{s(x, t_x, t - 1)}{s(x, t_x, t)}
\]

Therefore \( \delta \) can be interpreted as the (subjective) price to anticipate a given alternative \((x, t_x)\) by one additional period from \( t + 1 \) to \( t \).

An analogous definition can be given if \( s(x, t_x, t) > x \).

These comments on interpretation are also useful for an empirical estimation of the parameters \( \delta \) and \( \sigma \). What the above shows is that they can be elicited by asking subjects questions about willingness to pay in a way similar to the standard procedures used to elicit discount factors.

## 5 Discussion and concluding remarks

Recent evidence has posed a challenge to the traditional model of exponential discounting used to formalise time preferences. With the few notable exceptions discussed earlier, economists have mainly faced this challenge by modifying the exponential discounting function to a hyperbolic one. We have proposed that, instead, the evidence calls for a framework that is adherent to the cognitive procedures human decision makers adopt when making choices involving time. We have highlighted one particular cognitive shortcoming in evaluating future events (vagueness) and suggested one decision heuristics to resolve situations of vagueness. Even focussing on the simplest possible model in the class of our representations, and the one with the smallest departure from exponential discounting, we can explain several apparent paradoxes of decision making. Our modelling approach suggests in a sense a compromise between an ‘economic’ view based on rational orderings, and a ‘psychological’ view, based on heuristics.

Among heuristics based theories, the closest to our approach is Tversky’s [34] ‘lexicographic semiorder’, according to which agents rely on their ranking.
of the attributes of a (time dependent) alternative in a lexicographic way when choosing between different alternatives. The first attribute of each alternative is compared, and if the difference exceeds some threshold value then a choice is made accordingly. If the threshold value is not exceeded, then the agent compares the second attribute of each alternative, and so on.

Both this approach and our theory are close in spirit to Rubinstein’s [27] suggestion to extend his own similarity-based approach from risk to time preferences. He argues that, when evaluating alternatives, agents may perceive these alternatives to be ‘similar’ in either the time or the outcome dimension. At the formal level, this notion of similarity is related to our notion of vagueness. The crucial difference is that in our model vagueness applies to alternatives (i.e. outcome-dates pairs), not only to each date or outcome dimension separately. There are two aspects to this. First, vagueness (or similarity) in one dimension may not be defined independently of the other dimension. For example, 1 Million today may be clearly distinct from 1.01 Million today; but from today’s perspective the two amounts in thirty years time may well be considered similar. Second, ‘two-dimensional’ similarities between alternatives (outcome-date pairs) may be independent of similarities along each dimension: you may be able to distinguish between 1 Million and 1.01 Million, and also between the dates ‘1 year from now’ and ‘2 years from now’, yet be unable to express a preference between 1.01 Million 2 years from now and 1 Million 1 year from now.

We want to highlight that even when Rubinstein’s (incomplete) procedure is ‘completed’ along the very same lines of our secondary criterion, the prediction between that model and ours can still be quite different. Consider an individual exhibiting preference reversal with the following two choices:

- Choice A: choose 1,000 now over 1,100 tomorrow.
- Choice B: choose 1,100 in ten years and one day over 1,000 in ten years.

How can his choice be explained with Rubinstein’s procedure? Assume the natural time and outcome monotonicity properties (more/earlier is preferred to
less/later). Moreover, since the procedure contains an unspecified step, assume the latter corresponds to our secondary criterion. So, if both dimensions are similar, or if both dimensions are dissimilar and there is no Pareto dominance, the decision maker picks the alternative yielding the larger prize. It is easy to check that the only possibility for explaining the preference reversal by means of this ‘extended Rubinstein procedure’ is that 1,000 is similar to 1,100, ‘now’ is distinct from ‘tomorrow’, and ‘ten years’ is similar to ‘ten years and one day’. In choice A, the decision maker finds that the only dimension with distinguishable entities is time, and he picks the earlier prize. In choice B, there is no similarity in either dimension; he activates the additional ‘outcome criterion’ and picks the alternative with the later prize. As explained in section 4, this same pattern of choice can be accommodated in our $(\sigma, \delta)$ with suitable choices of $\delta$ and $\sigma$, with the decision maker relying on standard trade-off between outcome and time (using exponential discounting) for choice A, and using the secondary (outcome) criterion for choice B, where there is vagueness.

Now note that in the similarity model, the similarity between 1,000 and 1,100 entails the similarity between $1,000 + z$ and 1,100 for any $z \in [0, 100]$. Therefore, that model would predict the following choice:

- Choice C: choose $1,000 + z$ in ten years and one day over 1,000 in ten years, for any $z \in [0, 100]$.

No such constraint exists for our model (although it is compatible with choice C). While it is not impossible that choices as C are made, it is not unreasonable either that somebody does not see the point to wait one extra day, even in ten years time, to gain an extra cent. Lest there is any misunderstanding, let us emphasize again that the extended similarity model sketched above is not Rubinstein’s model, nor do we even claim it is the most compelling completion of his partial procedure: the point we are making is that our vagueness model is quite different, conceptually and formally, from a ‘similarity plus secondary criterion’ model.

Our theory is different from the standard exponential discounting one not so much because of a functional form change, but because it does not rely on the maximisation of a transitive ordering. In the second part of the paper we have focussed on the $(\sigma, \delta)$ specialisation of the theory because it is the simplest departure from the standard model which is still able to accommodate major anomalies. Clearly, however, any two-parameter model is valid only in a restricted context$^{14}$. For instance, the $(\sigma, \delta)$ OPM cannot account for ‘size

---

$^{14}$The same applies to the $\beta - \delta$ specialisation of the the hyperbolic model, which for instance can account for preference reversal only when the initial time 0 is involved.
effects’ of the following type: suppose that $\delta = .7$ and $\sigma = 1$, so that $(9, 0)$ is preferred to $(10, 1)$ but $(10, 3)$ is preferred to $(9, 2)$, so that preference reversal occurs\(^{15}\). Suppose now we double the stakes, and thus look at comparisons between $(18, 0)$ and $(20, 1)$, and between $(18, 2)$ and $(20, 3)$. Now it is easy to verify that the $(\sigma, \delta)$ OPM predicts a preference for the smaller, earlier amount in both comparisons, and preference reversal no longer occurs\(^{16}\). To some this might appear contrary to introspection: if for small stakes one is willing to wait for a future monetary gain (so that $(10, 3)$ is preferred to $(9, 2)$), then he should be even more willing if this gain is doubled. First of all one could argue that in the above example the misprediction is due to a misspecification of the taste parameters $\delta$ and $\sigma$: preference reversal exhibited in both comparisons would for instance be compatible\(^{17}\) with $\sigma \in [1.96, 2)$. More importantly, the apparent misprediction of the example is based on the implicit assumption that differences matter, and that some sort of internal consistency based on similarity is relied upon, so that if a unitary gain calls for preference reversal, so must a gain which is double in size\(^{18}\). Still, there is nothing particularly unreasonable in finding it more difficult to make comparisons when high stakes are involved. So for instance the vagueness term $\sigma(y, t)$ can be modelled as a proportion of the outcome, that is $\sigma(y, t) = \sigma y$ for some percentage $\sigma$. Then we have that $x > \delta^t y + \sigma y$ if and only if $\lambda x > \delta^t (\lambda y) + \sigma (\lambda y)$ for any positive scaling factor $\lambda$, so that the ‘misprediction’ of the numerical example above disappears\(^{19}\).

Note that this specialisation belongs to our general class of representations. The crucial aspect is that the vagueness term enters additively. Therefore any form of discounting in the utility function $u$ is compatible with increasing relative vagueness whenever the vagueness term $\sigma(y, t)$ is time-independent (or even when, if utility is discounted exponentially, $\sigma(y, t)$ increases less than exponentially with time). It is not necessary that $\sigma(y, t)$ is a constant, which is instead our assumption in the $(\sigma, \delta)$ model.

\(^{15}\)Note that $u(9, 0) = 9 > (.7) 10 + 1 = 8 = u(10, 1) + \sigma$, while $u(9, 2) = 9 (.7)^2 = 4.41 < 10 (.7)^3 + 1 = 4.43 = u(10, 3) + \sigma$.

\(^{16}\)Now $u(18, 0) = 18 > (.7) 20 + 1 = 15 = u(20, 1) + \sigma$, and also $u(18, 2) = 18 (.7)^2 = 8.82 > 20 (.7)^3 + 1 = 7.86 = u(20, 3) + \sigma$.

\(^{17}\)Now the required inequalities are $9 - (.7) 10 = 2 > \sigma$ and $9 (.7)^2 - 10 (.7)^3 = .98 \leq \sigma$ for the ‘small stakes’ comparison; and $18 - (.7) 20 = 4 > \sigma$ and $18 (.7)^2 - 20 (.7)^3 = 1.96 \leq \sigma$ for the ‘high stakes’ comparison.

\(^{18}\)Incidentally, this also seem to rely on a ‘similarity’ heuristic based on differences in one component - our initial premise was the rejection of this assumption as a starting point.

\(^{19}\)Preference reversal (between times 0 and $k$) is obtained when $y > x$ and $x > \delta^t y + \sigma y$ holds simultaneously with $\delta^k x \leq \delta^{t+k} y + \sigma y$, or $\sigma + \delta^t < \frac{\xi}{\sigma} \leq \frac{\xi}{\delta^t} + \delta^t$. 

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To sum up, our theory is more general than the \((\sigma, \delta)\) specialisation. The exponential discounting part of it is not central - indeed, it may well be that hyperbolic discounting is the ‘psychologically correct’ way of dealing with preference for anticipation. In this light, our core ideas may be viewed as complementary, rather than in contrast, to the class of hyperbolic discounting models. We believe they bring to the fore some interesting cognitive aspects of choice over time that the existing theories do not consider.

Our model is limited in that so far it does not deal with utility streams. Although the existing models of discounting have been readily applied to streams, we believe that this topic requires some special considerations. An important contribution in the same spirit as ours is that by Jehiel and Lilico [12]. They model the decision maker’s bounded rationality as limited foresight, so in this sense their agents are also ‘vague’ about the future. We leave the extension of our own framework to utility streams to future research.

6 Appendix

**Proposition 3** Let \(X = T = [0, 1]\). Let Discrimination, Time Monotonicity, Outcome Monotonicity, Continuity and Betweenness hold. Then \(\succ\) is \((u, \sigma)\)-representable with \(\sigma (a) \geq 0\) for all \(a \in A\) and \(u\) non-decreasing in outcome and non-increasing in time.

**Proof.** Define \(u(a) = \lambda (L(a))\), where \(\lambda\) denotes the Lebesgue measure. By Time Monotonicity and Outcome Monotonicity, the function \(u\) is weakly monotonic in the desired direction with respect to time and outcomes. Let

\[
L^* (a) = \lim_{b \in V(a)} L (b)
\]

This limit is well defined since by Discrimination the sets \(L (b)\) in the formula are nested. Denote \(\lambda^* (a) = \lambda (L^* (a))\), and define

\[
\sigma (a) = \max \{0, \lambda^* (a) - u (a)\}
\]

We show that \(u\) and \(\sigma\) as defined above represent \(\succ\). Suppose \(a \succ b\). We have \(L (b) \subset L (a)\) by the transitivity and irreflexivity of \(\succ\), and by Betweenness \(\lambda (L (a)) > \lambda (L (b))\), that is \(u (a) > u (b)\). Now suppose in negation that

\[
u (a) \leq u (b) + \sigma (b)
\]

By construction \(\sigma (b) = \max \{0, \lambda^* (b) - u (b)\}\). If \(\sigma (b) = 0\) we get an immediate contradiction with \(u (a) > u (b)\). So let instead \(\sigma (b) = \lambda^* (b) - u (b)\) and
therefore

\[ u(a) \leq u(b) + \sigma(b) = \lambda^*(b) \]

Suppose there exists \( b^* \in V(b) \) such that \( \lambda(L(b^*)) = \lambda^*(b) \). Then we have \( u(a) \leq u(b^*) \). By Discrimination \( L(a) \subseteq L(b^*) \), so that \( b \in L(a) \) contradicts \( b \in V(b^*) \). We show finally that such a \( b^* \) exists. If not, then \( L^*(b) \notin \{L(b') | b' \in V(b)\} \) and there must exist a sequence \( \{b_n\} \), with \( b_n \in V(b) \) for all \( n \), and with \( \lim b_n = b^* \notin V(b) \). If \( b^* \succ b \) then (i) of Continuity is violated, and if \( b \succ b^* \) then (ii) of Continuity is violated.

The proof concludes as the proof of proposition 1.

**Proposition 4** Let \( X \) and \( T \) be countable sets. Let Discrimination, Time Monotonicity, Outcome Monotonicity hold. Suppose that \( (\lim \{a_n\}) \in V(a) \) for any sequence \( \{a_n\} \) with \( a_n \in V(a) \). Then \( \succ \) is \( (u, \sigma) \)-representable with \( \sigma(a) \geq 0 \) for all \( a \in A \) and \( u \) non-decreasing in outcome and non-increasing in time.

**Proof.** The proof is the same as the proof of proposition 3, where \( \lambda(a) \) defines a numerical representation of the partial order of strict set inclusion instead of the Lebesgue measure (see e.g. Fishburn [6] for the existence of such a numerical representation). Note that the limit property in the statement implies the existence of a \( b^* \in V(b) \) such that \( \lambda(L(b^*)) = \lambda^*(b) \).

**Proposition 5** Let \( \succ \) be \( (u, \sigma) \)-representable with \( \sigma(a) \geq 0 \) for all \( a \in A \) and let Time Monotonicity, Time Monotonicity*, Outcome Monotonicity and Outcome Monotonicity* hold. If \( \sigma \) can be chosen to be non-increasing in outcome, then \( u \) can be chosen to be non-decreasing in outcome at an arbitrarily large number of alternatives, and if \( \sigma \) can be chosen to be non-decreasing in time then \( u \) can be chosen to be non-increasing in time at an arbitrarily large number of alternatives.

**Proof.** Fix a representation \( u \) and \( \sigma \), and suppose that \( u \) is decreasing in outcome at some point, that is there exist \( x_2 P x_1 \) and \( t \) such that \( u(x_1, t) > u(x_2, t) \). Consider the transformation \( v \) of \( u \) given by

\[
v(x, t') = \begin{cases} 
  u(x_2, t) & \text{if } x = x_1 \text{ and } t' = t \\
  u(x, t') & \text{otherwise}
\end{cases}
\]

This represents the same preferences as \( u \). To see this, note first that, for any \( (y, t') \succ (x_1, t) \), since \( u \) and \( \sigma \) represent \( \succ \) we have \( u(y, t') > u(x_1, t) + \sigma(x_1, t) \), and therefore

\[
v(y, t') = u(y, t') > u(x_1, t) + \sigma(x_1, t) > u(x_2, t) + \sigma(x_1, t) = v(x_1, t) + \sigma(x_1, t)
\]
so that \( v \) and \( \sigma \) still represent \( \succ \) for all alternatives which are preferred to \((x_1, t)\).

Next consider \((x_1, t) \succ (y, t')\). Suppose by contradiction that \( v(x_1, t) \leq u(y, t') + \sigma(y, t') \). Then by construction \( u(x_2, t) \leq u(y, t') + \sigma(y, t') \), so that \((x_2, t) \not\succ (y, t')\). This together with \((x_1, t) \succ (y, t')\) contradicts Outcome Monotonicity.

Consider now \((x_1, t) \sim (y, t')\). Suppose by contradiction that \( u(y, t') > v(x_1, t) + \sigma(x_1, t) \). Then by construction \( u(y, t') > u(x_2, t) + \sigma(x_1, t) \geq u(x_2, t) + \sigma(x_2, t) \), where the second inequality follows from the monotonicity assumption on \( \sigma \) in the statement. Then \((y, t') \succeq (x_2, t)\), which together with \((x_1, t) \sim (y, t')\) contradicts Outcome Monotonicity*.

This procedure can be repeated for an arbitrarily large number of pairs of alternatives where non-monotonicity occurs.

Finally, to prove that if \( \sigma \) can be chosen to be non-decreasing in time then \( u \) can be chosen to be non-increasing in time arguments essentially identical to the ones above apply by using Time Monotonicity in place of Outcome Monotonicity and the following transformation

\[
v(x', t) = \begin{cases} 
  u(x, t_1) & \text{if } x' = x \text{ and } t = t_2 \\
  u(x', t) & \text{otherwise}
\end{cases}
\]

at all points such that \( u(x, t_2) > u(x, t_1) \) with \( t_2 > t_1 \).

\* This is a placeholder for the actual reference.

References


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