Dark-Bright Ring Solitons in Bose-Einstein Condensates

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We study dark-bright ring solitons in two-component Bose-Einstein condensates. In the limit of large densities of the dark component, we describe the soliton dynamics by means of an equation of motion for the ring radius. The presence of the bright, “filling” species is demonstrated to have a stabilizing effect on the ring dark soliton. Near the linear limit, we discuss the symmetry-breaking bifurcations of dark-bright soliton stripes and vortex-bright soliton clusters from the dark-bright ring and relate the stabilizing effect of filling to changes in the bifurcation diagram. Finally, we show that stabilization by means of a second component is not limited to the radially symmetric structures, but can also be observed in a cross-like dark-bright soliton configuration.

PACS numbers: 03.75.-Lm, 67.90.+z

Introduction. The dramatic increase of interest in atomic Bose-Einstein condensates (BECs) and the parallel significant developments in the field of nonlinear optics have fueled studies of single- and multi-component nonlinear Schrödinger (NLS) equations. These operate as a mean-field (so-called Gross-Pitaevskii (GP)) near-zero-temperature model of the condensates in the former case, and as a suitable envelope equation for the electric field propagation in the latter.

In the following we will focus on the realm of multi-component atomic BECs which were realized in a series of pioneering experiments. Among many fundamental effects that have been observed, we highlight longitudinal spin waves, transitions between triangular and square vortex lattices, phase-separation-induced target patterns, tunability of interspecies interactions and miscibility-immiscibility transitions. Furthermore, classes of so-called symbiotic solitary waves (i.e., ones that could not be sustained without the nonlinear cross-component interactions) have been theoretically predicted and experimentally observed, such as dark-bright (DB) solitary waves and their vortex-bright analogs.

In this work we demonstrate the existence of a radially symmetric two-dimensional generalization of the DB soliton, namely the DB ring soliton whose density profile is shown in Fig. 1. In the first, “dark” component the nodal line without the nonlinear cross-component interactions) have been theoretically predicted and experimentally observed, and the latter.

Theoretical Analysis. The mean-field model of interest is set up by the two-dimensional (2D) coupled Gross-Pitaevskii equations

\[ i\partial_t \psi_1(x,y,t) = \left( -\frac{1}{2} \Delta + V(x,y) + g_1|\psi_1(x,y,t)|^2 + \sigma_{12}|\psi_2(x,y,t)|^2 \right) \psi_1(x,y,t) \]
\[ i\partial_t \psi_2(x,y,t) = \left( -\frac{1}{2} \Delta + V(x,y) + g_2|\psi_2(x,y,t)|^2 + \sigma_{12}|\psi_1(x,y,t)|^2 \right) \psi_2(x,y,t). \]

In this dimensionless form, time, length, energy and densities \(|\psi_j|^2, j \in \{1, 2\}\), are measured in units of \(\omega^{-1}, a_z, \hbar \omega_z\) and \((2\sqrt{2\pi}|a_{12}|a_z)^{-1}\), respectively. Here, \(\omega_z, a_z\) denote the oscillator frequency and length in the frozen \(z\)-direction, respectively (in the \(x-y\) plane these are \(\omega_r, a_r\)), whereas \(a_{11}, a_{22}, a_{12}\) refer to the intra- and intercomponent scattering lengths. This scaling leaves us with dimensionless coupling constants \(g_j = a_{jj}/|a_{12}|\), and \(\sigma_{12}\) denotes the sign of \(a_{12}\). Here, we focus on the experimentally relevant case of \(^{87}\text{Rb}\) hyperfine states \((F = 1, m_F = -1)\) and \((F = 2, m_F = 1)\), where all scattering lengths are positive and nearly equal, leading to dimensionless coupling constants of \(g_1 = 1.03\), \(g_2 = 0.97\), \(\sigma_{12} = +1\). Furthermore, we will assume an isotropic harmonic potential \(V(x,y) = \omega^2(x^2 + y^2)/2\), and for the numerics fix \(\omega \equiv \omega_r/\omega_z = 0.2\). As usual, stationary solutions are obtained by factorizing \(\psi_j(x,y,t) = \exp(-i\mu_j t)\phi_j(x,y)\), where \(\mu_1, \mu_2\) denote the two components’ chemical potentials.

In our first approach, we generalize earlier notions from in order to construct an approximately conserved adiabatic invariant of the DB ring, from which a description of its radial dynamics can be inferred. For a highly localized DB ring soliton, i.e. for one whose width is much smaller than its radius \(R\), such an approximate constant
of the motion in the regime of large $\mu_1$ and small $N_2$ is given by

$$E_{DB}(R) = 2\pi R \left( \frac{3}{4} \left[ \mu_1 + \frac{N_2^2}{128\pi^2 R^2} - V(R) \right]^{3/2} + \frac{N_2}{\sqrt{2\pi 4\pi R}} V(R) \right) - 4\pi R \dot{R}^2 \frac{N_2}{\sqrt{128\pi R}} - V(R).$$

To construct the DB ring energy $E_{DB}$, we multiply the energy of a one-dimensional DB soliton [15] located at distance $R$ from the trap center by the circumference $2\pi R$ [20]. In this expression, we take into account a scaling correction and the rescaled atom numbers $N_{1,2} = \int dxdy |\psi_{1,2}(x, y)|^2$. Energy conservation results in the radial equation of motion which for a harmonic potential $V(R) = \omega^2 R^2 / 2$ and for low velocities reads

$$\ddot{R}(t) + \frac{\mu_1}{3R} - \frac{N_2^2}{192\pi^3 R^4} - \frac{2}{3} \omega^2 R + \frac{N_2\omega^2}{16\pi \sqrt{2\pi}} \frac{N_2^2}{128\pi^2 R^2} - \frac{\omega^2}{2} R^2$$

The validity of Eq. (4) is constrained within radii satisfying $\mu_1 - V(R) > 0$. This equation of motion predicts a stable (with respect to radial motions) DB ring equilibrium at $R = R_0$ (cf. panels (b)-(c) of Fig. 1) and a breathing radial motion around this equilibrium (cf. Fig. 2(a)).

FIG. 1. (a) Dark-bright ring soliton’s profile at $\omega = 0.2$, $\mu_1 = 5$, $\mu_2 = 4.07$, $N_1 = 1616$, $N_2 = 150$. (b) DB ring’s radius as a function of $N_1$ at $N_2 = 100$ fixed, as calculated from Eq. (3) (black) and numerical data (red; gray in the print version). (c) DB ring’s radius as a function of $N_2$ at $\mu_1 = 5$ fixed.

FIG. 2. (a) Radial oscillations of the DB ring when excited with the breathing mode at $\mu_1 = 2$, $N_2 = 100$, numerical data (red/gray) and harmonic oscillation at the predicted frequency (black). (b) Bogolyubov-de Gennes (BdG) spectrum as a function of $N_2$ at $\mu_1 = 5$ fixed, prediction for the breathing frequency in black. Anomalous modes in the real part shown in red/gray, background modes in blue/light gray; purely imaginary modes in red/gray. (c) BdG spectrum as a function of $N_1$ at $N_2 = 20$ fixed, exhibiting a stable parameter regime at small $N_1$.

\[1\] In Eq. (3), we have used the values $g_1 = g_2 = \sigma_{12} = 1$, for mathematical simplicity. We have checked that our numerical results are essentially unchanged, if these values are used instead of the experimentally relevant parameters.
To complement this approach valid in the regime of large $N_1$, we also study the DB ring near the linear limit of $N_1 \to 0$, $N_2 \to 0$, where the coupled GPEs reduce to two independent harmonic oscillator Schrödinger equations. Stationary solutions of the GPEs have to reduce to harmonic oscillator eigenstates in this limit. To discuss the DB ring, we therefore consider linear combinations of the form
\[
\psi(x, y, t) = c_0(t)\varphi_0(x, y) + c_1(t)\varphi_1(x, y)
\]
for the dark component, where the linear modes $\varphi_0 = |2, 0\rangle$, $\varphi_1 = |0, 2\rangle$. Here, $|n_x, n_y\rangle$ denote the 2D harmonic oscillator eigenstates and $c_{0,1}(t)$ are complex time-dependent prefactors. This two-mode approximation has been shown to be useful in describing one-component ring dark solitons in [23]. For the bright component of the DB ring, we numerically find that it reduces to the oscillator ground state $|0, 0\rangle$ in the linear limit, and therefore for small enough atom numbers we tentatively approximate the order parameter as $\psi_2 = \sqrt{N_2}|0, 0\rangle$. Note that in this near-linear regime the bright component does not yet “fill” the dark ring’s zero density region; the DB ring’s characteristic density profile only emerges in the nonlinear regime of larger atom numbers. Following an approach similar to [23], we substitute the above harmonic oscillator ansätze into Eq. (1), and use the amplitude-phase decomposition
\[
c_k = \beta_k e^{i\alpha_k},
\]
to obtain the consistency condition
\[
0 = \dot{\alpha} = -g_1(\beta_1^2 A_{1111} - \beta_0^2 A_{0000} + 3A_{0011}(\beta_0^2 - \beta_1^2)) - \frac{g_1}{\beta_0\beta_1}(A_{0111}\beta_1^2(3\beta_0^2 - \beta_1^2) + A_{0001}\beta_0^2(\beta_0^2 - 3\beta_1^2)) + A_{0022} - A_{1122} + \frac{1}{\beta_0\beta_1}A_{0122}(\beta_0^2 - \beta_1^2),
\]
where $\alpha = \alpha_0 - \alpha_1$, and $A_{0011} = \int dxdy\varphi_0^2\varphi_1^2$, $A_{0122} = \int dxdy\varphi_0\varphi_1|\psi_2|^2$ etc. denote the nonlinear overlap integrals. This algebraic equation for small $N_1 = \beta_0^2 + \beta_1^2$ has a single root at $\beta_0 = \beta_1 = \sqrt{N_1/2}$, corresponding to the DB ring approximately given by $\psi_1 \propto |0, 2\rangle + |2, 0\rangle$, $\psi_2 \propto |0, 0\rangle$ in this regime. Beyond a critical $N_1^c$, two more roots emerge, which quickly tend to $\beta_0 \approx 0$, $\beta_1 \approx \sqrt{N_1}$ and $\beta_1 \approx 0$, $\beta_0 \approx \sqrt{N_1}$, respectively, as $N_1$ is further increased. These new solutions correspond to branches of states exhibiting two parallel DB soliton stripes that bifurcate from the DB ring branch (cf. Fig. 3). The effect of this symmetry-breaking bifurcation on the stability of the DB ring and its parametric dependence on the bright soliton’s atom number $N_2$ will be explored below.

![Diagram](image)

**FIG. 3.** (a) $N_1(\mu_1)$ bifurcation diagram at $N_2 = 20$. Solid (dashed) lines denote stable (unstable) branches. (b,c,d) Density profiles of the relevant branches (two DB soliton stripes (b), vortex-bright hexagon (c), filled-empty six vortex state (d)) at $N_1 = 100$, $N_2 = 20$. (e) Critical particle numbers at which the two DB soliton branches bifurcate from the DB ring branch as a function of $N_2$: two-mode approach predictions (black) and numerical data (red/gray). The error bars indicate the stepsize of the branches’ numerical continuation in $N_1$.

**Numerical Computations.** We now turn to a comparison of our theoretical predictions with the numerical results. In Figs. 1(b)-(c) the dependence of the DB ring’s equilibrium radius $R_0$ is shown as a function of $N_1$ and $N_2$, respectively. It can be seen that the relevant prediction obtained from Eq. (11) is accurate in the proper regime and has the right functional dependence on $N_{1,2}$. Inaccuracies in the regime of small $N_1$ can be attributed to a breakdown of the Thomas-Fermi approximation. For large $N_2$, we expect repulsive interaction effects between the elementary bright solitonic components to become important which are not accounted for in our expression for $E_{DB}$. The radial breathing
of the DB ring is shown in Fig. 2(a) in fair agreement with the theoretical prediction for the relevant frequency of linearization around $R = R_0$, in Eq. (1). Figures 2(b)-(c) illustrate the dependence on $N_{1,2}$ of the frequencies of the Bogolyubov-de Gennes (BdG) analysis around a stationary DB ring. When one of these frequencies possesses a non-vanishing imaginary part, the corresponding DB rings are dynamically unstable. A key observation here is that for fixed $N_2$, the critical $N_1$ for which imaginary eigenfrequencies arise is larger, the larger $N_2$ is. In fact, for $N_2 = 0$, the dark ring is always unstable [21], while the presence of the second component yields a stabilization window at small $N_1$; see Fig. 2(c). This effect can also be observed in Fig. 2(b), showing the BdG frequency spectrum’s $N_2$ dependence, where a weakening of the dynamical instabilities (i.e., decrease of the corresponding growth rates $\text{Im}(\omega)$) and stabilization of a number of unstable modes is seen as $N_2$ is increasing. In these BdG plots, the radial breathing is represented by the “anomalous mode” of the DB ring whose frequency is in fairly good agreement with the corresponding analytical prediction.

For small atom numbers, we can relate the imaginary modes arising in the DB ring’s BdG spectrum, as $N_1$ is increased, to symmetry-breaking bifurcations leading to new states. A typical $N_1(\mu_1)$ bifurcation diagram for a relatively small bright atom number of $N_2 = 20$ is shown in Fig. 3(a). As the dark component’s atom number increases, branches of two DB soliton stripes bifurcate from the DB ring, destabilizing it. This class of bifurcations can be understood within the near-linear framework of the two-mode approximation introduced above. Both in this simplified model and in full numerical simulations of the GPE we find that the critical value $N_1^{cr}$ where the DB soliton stripes’ bifurcation occurs increases as a function of $N_2$, see the very good agreement of these predictions in Fig. 3(c). This critical point marks (for small $N_2$) the termination of the window of stability of the DB ring due to the second component referred to above and is associated with the emergence of imaginary modes in its BdG spectrum. Subsequent bifurcations from the ring lead to a sequence of polygonal vortex-bright clusters, starting with a hexagonal necklace state shown in Fig. 3(c). These configurations are natural two-component generalizations of the polygonal states discussed in [23]. For small $N_2$, they are found to be unstable, as they bifurcate from the destabilized DB ring branch. The DB soliton stripes also lose their stability to a bifurcating vortex cluster, namely the six vortex state shown in Fig. 3(d). Interestingly, in this cluster the bright component only fills four of the six vortices, while two of the cores remain empty. For $N_2 \gtrsim 42$ we observe a change in the bifurcation order, with the vortex-bright hexagon branch now bifurcating before the DB soliton stripes and thus inheriting the ring’s initial stability.

The above observations show that the presence of the second component parametrically “delays” the instability inducing bifurcations in the case of the ring dark soliton. This type of phenomenology is not only present in the case of the ring-like state, but can also be seen to persist for other states such as the crossed (diagonal) dark-bright solitons of Fig. 4. This can be thought of as a two-component generalization of the always unstable diagonal dark solitons of [23]. For this configuration, too, a BdG analysis reveals that for fixed $N_2 > 0$ there exists a range of $N_1$ for which it is dynamically stable, while beyond the first critical point $N_1^{cr}$, it becomes unstable towards the formation of a vortex-bright quadrupole cluster; see Fig. 4(c).

![Figure 4](image_url)

**FIG. 4.** (a) BdG spectrum of the crossed DB soliton branch as a function of $N_1$ at $N_2 = 30$. Same color coding as in Fig. 2, oscillatory unstable modes shown as green/light gray circles. (b) The state’s density profile in the stable region ($N_1 = 10$, $N_2 = 30$). (c) An example of a vortex-bright quadrupole cluster for $N_1 = 70$, $N_2 = 30$. 
Conclusions. In our above exposition, we have illustrated the existence of a series of states in quasi-2D, two-component BECs including dark-bright rings, dark-bright multi-stripes, vortex-bright polygons and cross-like dark-bright states. For the DB rings and the crossed DB state, we have shown that for a fixed atom number in the bright component, the immediate instability of the one-component state is partially inhibited by the presence of the second component. The eventual destabilization is caused by the bifurcation of “daughter” states (such as the ones containing DB stripes or vortex-bright polygons). For the DB ring it is possible to complement this picture with a (large $\mu_1$, small $N_2$) “particle” model which accounts for the existence of an equilibrium ring and its potentially observable breathing oscillations. We expect that stabilization by filling should not be limited to 2D realms (see also [17] for an example of such stabilization) but could be equally well applicable to one- or three-dimensional contexts. In one dimension, a potentially relevant context to consider is the instability of bubbles [26], while in three-dimensions, the study of spherical solitons and of multi-vortex-ring states would be of primary interest.

[17] V.A. Brazhnyi et al., Chaos, Solitons and Fractals 44, 381 (2011)