April 1, 2011

Radial standing and self-similar waves for the hyperbolic cubic NLS in 2D

Panos Kevrekidis, *UMASS, Amherst*
Andrea R. Nahmod
Chongchun Zeng

RADIAL STANDING AND SELF-SIMILAR WAVES FOR THE HYPERBOLIC CUBIC NLS IN 2D

PANAYOTIS KEVREKIDIS[1], ANDREA R. NAHMOD[2], AND CHONGCHUN ZENG[3]

Abstract. In this note we propose a new set of coordinates to study the hyperbolic or non-elliptic cubic nonlinear Schrödinger equation in two dimensions. Based on these coordinates, we study the existence of bounded and continuous hyperbolically radial standing waves, as well as hyperbolically radial self-similar solutions. Many of the arguments can easily be adapted to more general nonlinearities.

1. Introduction

The hyperbolic cubic NLS equation has been studied in [18, 19, 20] (see also [28, 12, 11] and references therein) and has received attention by both the physics and the applied mathematics communities. This model has been used for examining the evolution of optical pulses in normally dispersive (quasi-discrete) optical waveguide array structures [16, 24], as well as more generally in normally dispersive optical media [13, 14]. These studies have, in turn, motivated a number of works examining the type of coherent structures, such as the experimentally tractable X-waves [13, 14, 11], as well as more elaborate structures such as dark-bright [21] or vortex-bright solitary waves [17]. On the one hand, the spontaneous emergence of nonlinear structures such as the X-waves has been revealed by means of mechanisms such as the modulational instability [9] and their nonlinear dynamics has been considered [10], but also more recently studies have begun to systematically address their spatio-temporal interactions [25].

The equation arises when to the normally-dispersive (1+1)-NLS, effects due to the diffraction (in the quasi-discrete setting) or anomalous dispersion (in the continuum setting) are incorporated as an additional term. The $(2+1)$-dimensional quasi-discrete equation [16, 24] is described as

$$i\frac{\partial U_n}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 U_n}{\partial t^2} + C(U_{n+1} + U_{n-1}) + \gamma |U_n|^2 U_n = 0 \tag{1.1}$$

where U is the complex amplitude that represents an envelope of the relevant wave field, z is the longitudinal direction along which propagation occurs; and n and t are

[1] The first author is funded in part by NSF-DMS-0349023, NSF DMS-0806762, NSF-CMMI-1000337 and by the Alexander von Humboldt and Alexander S. Onassis Foundations. He also acknowledges numerous discussions on this theme with Kody J.H. Law.

[2] The second author is funded in part by NSF DMS 0803160 and a 2009-2010 Radcliffe Institute for Advanced Study Fellowship.

[3] The third author is funded in part by nsf-dms 0801319.

the two transverse axes (which represent space and time respectively). In that same vein, it is relevant to mention that very recently also (3+1)-dimensional generalizations of the above quasi-discrete setting have emerged, giving rise to the so-called discrete-continuous X-waves in photonic lattices [22].

The continuous version of the $(2+1)$-dimensional equation can be written as

$$iu_t + u_{xx} - u_{yy} + \gamma|u|^2 u = 0 \qquad \text{(3HNLS)}$$

and is related to both the Davey-Stewartson and Ishimori systems [19, 28]. In fact, in [18] Ghidaglia and Saut showed that (3HNLS) satisfies the same linear Strichartz estimates as its elliptic counterpart and relying on them, they proved the hyperbolic-elliptic Davey-Stewartson is locally well posed in $L^2(\mathbb{R}^2)$ with time of existence depending on the profile of the initial data and globally well posed for sufficiently small data. The same argument proves that (3HNLS) enjoys the same well posedness results. In [20], Ghidaglia and Saut showed that there are no nontrivial localized standing wave solutions to (3HNLS); more precisely they proved that there are no solutions of the form $e^{i\mu t}\phi(x)$ with $\phi \in H^1 \cap H^2_{loc}$ and any $\mu \in \mathbb{R}$ (regardless of dimension). From the physical point of view (3HNLS) or equation (2.1) below represents the canonical model for the study (experimentally and also numerically) of X-waves, as considered in [13, 14]; see also the earlier and more recent reviews of [12, 11]. It should also be mentioned that the linear counterpart of the equation has also been considered extensively [11, 8] and has given rise to interesting recent experimentally examined structures such as the linear alternative to light bullets proposed in [7] in the form of Airy-Bessel wavepackets.

The major differences between the regular Laplacian $\Delta = \partial_{xx} + \partial_{yy}$ and the hyperbolic Laplacian $\Box_{xy} = \partial_{xx} - \partial_{yy}$ include that a) the latter has nontrivial characteristics; b) the former is invariant under the Euclidean rotations on $\mathbb{R}^2$ while the latter is invariant under the hyperbolic rotations; and c) the former corresponds to the positive Dirichlet energy and the latter to the energy $E(u) := \int_{\mathbb{R}^2} |u_x|^2 - |u_y|^2 dxdy$ which is indefinite. Due to observation a), this PDE system may have singularities forming along the characteristics and thus we first find some explicit compatibility conditions along the discontinuities of weak solutions which are piecewise smooth. Observation b) naturally motivates us to consider the hyperbolic coordinates (a, α), given by $x = \pm a \cosh \alpha$ and $y = \pm a \sinh \alpha$. Since the hyperbolic NLS is invariant under the hyperbolic rotations $(a, \alpha) \to (a, \alpha + \alpha_0)$, it is natural to start our study with solutions with the corresponding symmetry – continuous and piecewise smooth hyperbolically radial (or equivariant) standing waves and hyperbolically radial (or equivariant) self-similar waves[1] – as our initial attempts to further our understanding of this PDE. They also usually play important roles in studying the dynamics of (3HNLS). In particular, states which are hyperbolically radial share some similarity with the profiles of the two-dimensional analogs of the so-called X-waves observed in numerics; as a relevant example see e.g. Figs. 18.6-18.7 of [12] (cf. also [11]). Since the measure $dxdy = |a| da d\alpha$, non-zero hyperbolic radial states do not belong to $L^2(\mathbb{R}^2)$. We do

[1]By self-similar waves we mean standing waves in the self-similar variables; c.f. (3.6) (3.7).

not view this as an extremely undesirable situation. On the one hand, some important elementary waves, like plane waves and kinks, do have infinite L^2 norm. On the other hand, since the hyperbolic Laplacian $\Box_{xy}$ does not correspond to a positive energy and has characteristics, it is reasonable, at least as an initial step, to relax the finite energy requirement and look for hyperbolically radial standing waves which may have asymptotic values other than zero at spatial infinity in each hyperbolic quadrant.

In constructing these hyperbolically radial (or equivariant) waves, we encounter an ODE with a regular singular point. Though for this specific equation there are many standard results concerning the existence of local solutions, we provide a different proof based on the invariant manifold theory in dynamical systems which is a generalization of a trick used in [15]. Unlike the power series method or variational method, this approach requires minimal smoothness and no extra structure. Our aim in the present paper is to use such techniques and, especially, the hyperbolic coordinate system in order to construct, for the first time to our knowledge, nontrivial standing wave solutions to the (3HNLS) equation. In addition, we also consider radial self-similar solutions in the same coordinate system. The introduction of the hyperbolic coordinates and the analysis on hyperbolically radial waves represent the first step in a longer term program to further our understanding of PDEs involving the hyperbolic Laplacian operator.

In the rest of the paper radial always means hyperbolically radial.

2. Weak solutions and Compatibility Conditions

In the following, we give the compatibility conditions satisfied by weak solutions following the standard argument. Let us denote by $\Box_{xy} := \partial_x^2 - \partial_y^2$ and for $(x, y) \in \mathbb{R}^2$ suppose $u(t, x, y)$ is a solution of (3HNLS) with $\gamma = 1$; ie. u satisfies:

$$i\partial_t u + \Box_{xy} u + |u|^2 u = 0, \tag{2.1}$$

in the distribution sense and is smooth except on a surface $\Gamma \subset \mathbb{R}^{1+2}$, let us denote by $\Omega_\pm$ the two domains separated by Γ and by $N = (N_0, N_1, N_2)$ the unit outward normal vector of Ω_+. Let $[f] = f_+ - f_-$ where $f_\pm$ is the one side limit of any quantity f on Γ. For any test function $\phi \in C_0^\infty(\mathbb{R}^{1+2})$ one may compute that,

$$\int_\Gamma ((0, \nabla\phi) \cdot RN)[u] - \phi RN \cdot (i[u], [\nabla u]) dS = 0 \tag{2.2}$$

where ∇ denotes the spatial gradient and the matrix $R := \mathrm{diag}(1, 1, -1)$ performs a reflection across the xt-plane. By taking ϕ identically zero on Γ, we obtain that, if u has a jump discontinuity along Γ, then

(C1) $N_1^2 = N_2^2$ or equivalently Γ is parallel to $x = y$ or $x = -y$ which are characteristics of $\Box_{xy}$.

Therefore, the surface takes the form of

$$\Gamma = \{(t, x_0(t) + \alpha_1 s, y_0(t) + \alpha_2 s \mid t,\, s \in \mathbb{R}\} \text{ and } N_0 = N_0(t)$$

where $\alpha_{1,2} = \pm 1$. For such smooth Γ, integrating by parts on the first integrand leads to

$$\int_\Gamma \phi R N \cdot (i[u], 2[\nabla u]) dS = 0 \tag{2.3}$$

and thus we have

(C2) $\frac{i}{2}[u]N_0 + [u_x]N_1 - [u_y]N_2 = 0$.

Condition (C2) is equivalent to that $[u_x]$, $[u_y]$, $i[u]$ are colinear in the complex plane. In fact, if u is piecewise C^1 and $[u] = 0$, then, on the one hand, (C2) implies that the vector $(N_1, -N_2)$ lies in the kernel of the 2×2 matrix $[\nabla u]$. On the other hand, if $[\nabla u] \neq 0$, then its kernel is given exactly by the tangent space of Γ, i.e. the one dimensional subspace spanned by $(N_2, -N_1)$, and thus $N_1^2 = N_2^2$ holds. This means that, even though (C1) follows from the assumption $[u] \neq 0$, it still follows from (C2) if $[u] = 0$ and $[\nabla u] \neq 0$. Therefore, (C1) and (C2) should be imposed on weak solutions as long as u or ∇u have a jump discontinuity along Γ. Condition (C2) can also be viewed as a necessary condition for the singularity to propagate as a surface in $\mathbb{R}^{1+2}$. If (C2) is not satisfied by some given initial data, then the singularity will not maintain a simple geometry for $t > 0$.

Notice that, according to (C1), the jump discontinuity can occur along lines in the directions of $(1, 1)$ or $(1, -1)$. To study the local behavior of the weak solutions at the intersection of its lines discontinuity, we consider the case where

$$\Gamma_* = \{(x - x_0(t))^2 = (y - y_0(t))^2\},$$

i.e. a cross in the xy-plane with a moving center $c(t) = (x_0(t), y_0(t))$. We will use $u^{\pm}_{\mathrm{h,v}}$ to denote the value of u in the cone either in the horizontal or vertical directions, opening in the positive or negative directions. In particular, $u^{1,2,3,4}_{\mathrm{h,v}}$ denote the limiting values of $u^{\pm}_{\mathrm{h,v}}$ on Γ in each quadrant. We use N to denote the unit outward normal of the horizontal cones $(x - x_0(t))^2 > (y - y_0(t))^2$. In this case, instead of (2.3), integrating by parts on the first integrand in (2.2) implies

$$\begin{aligned} 4 \int \phi(c(t)) \{ & u^1_{\mathrm{h}}(c(t)) - u^1_{\mathrm{v}}(c(t)) + u^2_{\mathrm{h}}(c(t)) - u^2_{\mathrm{v}}(c(t)) + u^3_{\mathrm{h}}(c(t)) - u^3_{\mathrm{v}}(c(t)) \\ & + u^4_{\mathrm{h}}(c(t)) - u^4_{\mathrm{v}}(c(t))\} dt - \int_\Gamma \phi R N \cdot (i[u], 2[\nabla u]) dS = 0. \end{aligned} \tag{2.4}$$

Therefore, in addition to (C2), we also obtain

(C3) $u^1_{\mathrm{h}} - u^1_{\mathrm{v}} + u^2_{\mathrm{h}} - u^2_{\mathrm{v}} + u^3_{\mathrm{h}} - u^3_{\mathrm{v}} + u^4_{\mathrm{h}} - u^4_{\mathrm{v}} = 0$ along $c(t)$, $t \in \mathbb{R}$.

Remark 2.1. *When a solution has singularity along a surface, the compatibility conditions (C1–C3) above are only necessary. In fact, one should be careful when talking about the propagation of the singularity as the system does not have finite propagation speed, which can be seen from the formula of the fundamental solution if the free equation* [18]. *For example, if the initial data has some simple jump discontinuity, this singularity might become unpredictable for $t > 0$ except for highly symmetric cases. In what follows Γ will describe the cross X in $\mathbb{R}^2$ defined by $\{\, |x| = |y| \,\}$. We will see*

in Section 4 that the simplest example of data satisfying the compatibility conditions described above are those of the form $(f, g, -f, -g)$ *given in each of the four cones of* $\mathbb{R}^2 \setminus X$.

3. Hyperbolic coordinates and radial solutions

Notice that the regular Euclidean rotation does not keep $\Box_{xy}$ invariant, and hence the usual polar coordinates may not be the most appropriate ones for this system. Instead, we introduce an alternative: we foliate $\mathbb{R}^2 \setminus X$ by hyperbolas and consider the following change of variables:

$$\begin{cases} x := a\cosh\alpha & y := a\sinh\alpha & a^2 := x^2 - y^2 & \text{when} \quad |x| > |y| \\ x := a\sinh\alpha & y := a\cosh\alpha & a^2 := y^2 - x^2 & \text{when} \quad |x| < |y| \end{cases} \tag{3.1}$$

where $a, \alpha \in \mathbb{R}, a \neq 0$.

Under this change of variables we have that the area form $dxdy = |a|dad\alpha$. Moreover for any $a \in \mathbb{R} \setminus \{0\}$ we have:

$$\nabla a = (\frac{x}{a}, -\frac{y}{a}) \quad \text{and} \quad \Box a = \frac{1}{a} \quad \text{for} \quad |x| > |y|,$$
$$\nabla a = (-\frac{x}{a}, \frac{y}{a}) \quad \text{and} \quad \Box a = -\frac{1}{a} \quad \text{for} \quad |x| < |y|.$$

For $\nabla\alpha$ we need to distinguish whether $a > 0$ or $a < 0$. We have:

$$\begin{aligned} &\nabla\alpha = (-\frac{y}{a^2}, \frac{x}{a^2}) \text{ for } \quad |x| > |y| \text{ and } a > 0 \\ &\nabla\alpha = (\frac{y}{a^2}, -\frac{x}{a^2}) \text{ for } \quad |x| > |y| \text{ and } a < 0 \\ &\nabla\alpha = (\frac{y}{a^2}, -\frac{x}{a^2}) \text{ for } \quad |x| < |y| \text{ and } a > 0 \\ &\nabla\alpha = (-\frac{y}{a^2}, \frac{x}{a^2}) \text{ for } \quad |x| < |y| \text{ and } a < 0 \\ &\Box\alpha = 0 \qquad \text{in all cases.} \end{aligned}$$

We moreover note that

$$a_x^2 - a_y^2 = 1 \quad \text{and} \quad \alpha_x^2 - \alpha_y^2 = -\frac{1}{a^2} \quad \text{for} \quad |x| > |y|,$$
$$a_x^2 - a_y^2 = -1 \quad \text{and} \quad \alpha_x^2 - \alpha_y^2 = \frac{1}{a^2} \quad \text{for} \quad |x| < |y|.$$

By changing coordinates $(x, y) \longrightarrow (a, \alpha)$ in $\mathbb{R}^2 \setminus X$ the (3HNLS) equation transforms itself into the system [2]

[2] Note that in the first quadrant $a > 0$ and $\alpha > 0$; in the second quadrant $a < 0$ and $\alpha < 0$; in the third $a < 0$ and $\alpha > 0$ and in the fourth quadrant $a > 0$ and $\alpha < 0$

$$\begin{cases} i\partial_t u + \partial_{aa}u + \frac{1}{a}\partial_a u - \frac{1}{a^2}\partial_{\alpha\alpha}u + |u|^2 u = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ i\partial_t u - \partial_{aa}u - \frac{1}{a}\partial_a u + \frac{1}{a^2}\partial_{\alpha\alpha}u + |u|^2 u = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases} \tag{3.2}$$

It is clear that (3.2) is invariant under the translation in α, or equivalently (3HNLS) is invariant under the rotation in the hyperbolic coordinates. Thus it motivates us to study radial or equivariant solutions. We will focus on radial solutions $u(t, a, \alpha) = u(t, a)$. (See Remark 3.9 for equivariant solutions.) In the rest of this note, we will consider radially symmetric solutions assuming $u^{\pm}_{\mathrm{h,v}}$ is *a priori* smooth for $a \in [0, \infty)$ except for some jump discontinuity across X. Note in what follows these facts will be used without any further mention. In other words we will be working with

$$\begin{cases} i\partial_t u + \partial_{aa}u + \frac{1}{a}\partial_a u + |u|^2 u = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ i\partial_t u - \partial_{aa}u - \frac{1}{a}\partial_a u + |u|^2 u = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases} \tag{3.3}$$

The compatibility condition (C3) implies that at the origin $(0, 0) \in \mathbb{R}^2$

$$u^+_{\mathrm{h}}(0) + u^-_{\mathrm{h}}(0) = u^+_{\mathrm{v}}(0) + u^-_{\mathrm{v}}(0). \tag{3.4}$$

Some obvious solutions of this type are $(u^-_{\mathrm{h}} = -u^+_{\mathrm{h}}, u^+_{\mathrm{v}} = u^-_{\mathrm{v}} = 0)$ or $(u^+_{\mathrm{h}} = u^-_{\mathrm{h}} = 0, u^-_{\mathrm{v}} = -u^+_{\mathrm{v}})$ where u^+_{h} or u^+_{v} solves the above (F) or (D) respectively.

We are interested in the existence (construction) of radial standing wave solutions $u(t, a) = e^{-i\mu t}v(a)$ such that as $a \to \pm\infty$ $v \to v_{\pm\infty}$ with $v_{\pm\infty}$ a constant.

Remark 3.1. *If $u(x, y)$ is C^1 and (hyperbolically) radially symmetric, then the radial symmetry implies $Du(0, 0) = 0$ and thus $v_a(0) = 0$. We assume the latter from now on.*

From (3.3) we then have that v satisfies:

$$\begin{cases} \mu v + v_{aa} + \frac{1}{a}v_a + |v|^2 v = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ \mu v - v_{aa} - \frac{1}{a}v_a + |v|^2 v = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases} \tag{3.5}$$

Remark 3.2. *We call v_d the solution to (D) and v_f the solution to (F) and $v = (v_d, v_f)$ the solution to* (3.5)*. Note that a radial solution to* (3.5) *is C^1 if and only if v_d and v_f have equal values at $a = 0$; ie. $v_d(0) = v_f(0)$.*

Remark 3.3. *Under the assumption $v_a(0) = 0$, without any loss of generality we consider v real valued only. In fact, suppose $v(0) = \omega_0 e^{i\theta_0}$ with $omega_0 \in \mathbb{R}$ and $\theta_0 \in \mathbb{R}$. Let $\tilde{v}(a) = v(a)e^{-i\theta_0}$, then $\tilde{v}(0)$, $\tilde{v}_a(0) \in \mathbb{R}$. Theorem 7.1 implies the solutions $\tilde{v}(a)$ exist and are real.*

Remark 3.4. *In the analysis below we only consider $a > 0$ but corresponding results hold for $a < 0$.*

Remark 3.5. *By considering another coordinate change $x' = a\cos\alpha$ and $y' = a\sin\alpha$,* (3.5) *(F)(D) simply become $\pm\Delta_{x'y'}v+\mu v+v^3 = 0$. Solutions of this equation in L^2 have been studied extensively (see, for example,* [1, 2, 3, 26, 27]*). Based on those results, it is obviously impossible to find solutions of (F) and (D) with L^2 decay at infinity so that they match at $a = 0$. Instead, we consider all possible bounded solutions here through a more careful analysis.*

Theorem 3.6 (Main Theorem 1)**.** *C^1 weak radial standing wave solutions $u(t, a, \alpha) = e^{-i\mu t}v(a)$ to* (2.1)*, which are C^2 except along $x^2 = y^2$ (or equivalently $a = 0$), exist only if $\mu < 0$. In fact, for any $v_0 \in (0, \sqrt{-\mu})$, there exists a unique C^1 such solution $(u_f, u_d) = (e^{-i\mu t}v_f(a), e^{-i\mu t}v_d(a))$ such that v_f and v_d are smooth on $a \in [0, \infty)$ and*

$$v_d(0) = v_f(0) = v_0 \text{ and } \partial_a v_d(0) = 0 = \partial_a v_f(0).$$

Moreover, for $a \to \infty$,

$$v_f(a) = \sqrt{-\mu} + \frac{r_f}{\sqrt{a}}\cos(\gamma a + \sigma\log a + \beta) + O(a^{-\frac{3}{2}})$$
$$v_d(a) = \frac{r_d}{\sqrt{a}}\cos(\gamma a + \sigma\log a + \beta) + O(a^{-\frac{3}{2}})$$

for some $\gamma, \beta, \sigma \in \mathbb{R}$ and $r_f \neq 0$ and $r_d \neq 0$.

For $v_0 \in (-\sqrt{-\mu}, 0)$, we only need to consider $-v$ and apply the above theorem.

Notice that equation (3HNLS) is invariant under the scaling transformation $u \to u_\lambda(t, x, y) = \lambda u(\lambda^2 t, \lambda x, \lambda y)$. Accordingly, in the hyperbolic coordinates, consider (3.2) in the usual similarity coordinates $u(t, a, \alpha) = t^{-\frac{1}{2}}\tilde{u}(\log|t|, t^{-\frac{1}{2}}a, \alpha)$, we obtain

(3.6)

$$\begin{cases} i\partial_t\tilde{u} - \frac{i}{2}\tilde{u} - \frac{i}{2}a\partial_a\tilde{u} + \partial_{aa}\tilde{u} + \frac{1}{a}\partial_a\tilde{u} - \frac{1}{a^2}\partial_{\alpha\alpha}\tilde{u} + |\tilde{u}|^2\tilde{u} = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ i\partial_t\tilde{u} - \frac{i}{2}\tilde{u} - \frac{i}{2}a\partial_a\tilde{u} - \partial_{aa}\tilde{u} - \frac{1}{a}\partial_a\tilde{u} + \frac{1}{a^2}\partial_{\alpha\alpha}\tilde{u} + |\tilde{u}|^2\tilde{u} = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases}$$

where now $\tilde{u} = \tilde{u}(t, a, \alpha)$. Next we look for radial standing waves in the self-similar variables of the form $\tilde{u} = e^{-i\mu t}v(a)$ which satisfies

$$\begin{cases} \mu v - \frac{i}{2}v - \frac{i}{2}av_a + v_{aa} + \frac{1}{a}v_a + |v|^2 v = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ \mu v - \frac{i}{2}v - \frac{i}{2}av_a - v_{aa} - \frac{1}{a}v_a + |v|^2 v = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases} \tag{3.7}$$

The remarks we made for the radial standing waves are valid for radial self-similar waves except here we can no longer assume v has only real values.

Theorem 3.7 (Main Theorem 2)**.** *For any $\mu \in \mathbb{R}$, there exists $v_* > 0$ such that for any $v_0 \in (-v_*, v_*)$, there exists a unique C^1 weak radial self-similar wave solution (u_f, u_d) of* (2.1) *which is C^2 except at $x^2 = y^2$ (or equivalently $a = 0$) and*

$$u_f(t, 0) = u_d(t, 0) = v_0 t^{-\frac{1}{2}}e^{-i\mu\log|t|} \text{ and } \partial_a u(t, 0) = 0$$

As $\frac{a}{\sqrt{t}} \to \infty$*, there exist constants* $r_f \neq 0$*,* $r_d \neq 0$*,* θ_f*, and* θ_d *such that*

$$u_f(t,a) = \frac{r_f}{a} e^{i(\frac{a^2}{8t} - \mu \log|t|)} \cos(\theta_f - \frac{a^2}{8t} - 2\mu \log \frac{a}{\sqrt{t}}) + O(\frac{t^{\frac{3}{2}}}{a^3})$$

$$u_d(t,a) = \frac{r_d}{a} e^{i(-\frac{a^2}{8t} - \mu \log|t|)} \cos(\theta_d - \frac{a^2}{8t} + 2\mu \log \frac{a}{\sqrt{t}}) + O(\frac{t^{\frac{3}{2}}}{a^3}).$$

Remark 3.8. *From the asymptotic form of the above self-similar waves, for any* $a \neq 0$*, i.e. away from the cross* $|x| = |y|$*, as* t *approaches the singular value* 0 *(here* t *may be replaced by* $\pm(t - t_0)$*), the complex values of the solutions oscillate rapidly in both the angle and the magnitude. In particular, their magnitudes oscillate with an envelope proportional to* $\frac{1}{a}$*. At the cross* $|x| = |y|$*, the magnitude of the solutions blows up like* $\frac{1}{\sqrt{t}}$*. Therefore they strongly display a kind of profile of the X shape.*

To prove these theorems, we first prove existence of solutions near $a = 0$ and next we study their properties as $a \to +\infty$. Equations (3.5)(F)(D) and (3.7)(F)(D) have been extensively studied. To establish local existence of solutions near $a = 0$, well known methods include for example, power series or variational methods. Here we present (in the Appendix) another method based on invariant manifold theory to study solutions near a regular singular point of a class of ODE system of which (3.5)(F)(D) and (3.7)(F)(D) are special cases. The advantage of this method is that it does not require either much smoothness as in the power series approach or variational structures. More specifically, for any v_0, let $x_1 = v - v_0$ and $x_2 = av_a$ in these four ODEs. Each of them turns into a form for which we may apply Theorem 7.1. For example, for (3.5)(F), in the notation of Theorem 7.1 we have that $F(a, x, v_0) = (x_2, -\mu a^2(x_1 + v_0) - a^2(x_1 + v_0)^3)^T$ with the parameter v_0 and $F(0, 0, v_0) = 0$ for all v_0. Moreover $F_x(0, 0, v_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and thus $E_+ = \{0\}$. Therefore, for any v_0, there exists a unique solution $v(a)$ for small $a \geq 0$ so that v and av_a are Hölder continuous in both a and $v(0) = v_0$. Moreover, the statement (4) in Theorem 7.1 implies $x_2 = av_a = O(a^2)$ which implies $v_a(0) = 0$. The other equations are handled similarly.

Remark 3.9. *The same Theorem 7.1 can also be applied to study equivariant standing waves or equivariant self-similar solutions where the equivariance means* $u(t, a, \alpha) = e^{ic\alpha} e^{-i\mu t} v(a)$*,* $c, \mu \in \mathbb{R}$*. In this case, for equivariant standing waves,* (3.2) *implies*

$$\begin{cases} \mu v + v_{aa} + \frac{1}{a} v_a + \frac{c^2}{a^2} v + |v|^2 v = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ \mu v - v_{aa} - \frac{1}{a} v_a - \frac{c^2}{a^2} v + |v|^2 v = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases}$$

For $c \neq 0$*, Theorem 7.1 immediately implies that the only continuous solution on* $[0, a_0)$*,* $a_0 > 0$*, of either of the above has to be trivial. Therefore, there are no nontrivial continuous equivariant standing waves. The same argument applies to the self-similar case as well.*

To finish the proof of the theorems, we will study the asymptotic properties of these systems as $a \to \infty$ in the next two sections.

4. Radial Standing Wave Solutions

Note that by considering $a_1 = |\mu|^{\frac{1}{2}} a$ and $v = |\mu|^{\frac{1}{2}} v_1$, we only need to consider $\mu = \pm 1, 0$ for (3.5)(F)(D). To study the asymptotic properties of the solutions as $a \to +\infty$ we proceed according to the sign of μ.

Case $\mu > 0$. First we show that all solutions v_f to (3.5) (F) go to zero like $\frac{1}{\sqrt{a}} \cos(\gamma a + \sigma \log a + \beta)$ as $a \to \infty$. Indeed, for nontrivial v satisfying $v_{aa} + \frac{1}{a} v_a + \mu v + |v|^2 v = 0$ and $a > 0$ define $v =: \frac{1}{\sqrt{a}} w$. Then w satisfies:

$$(\frac{1}{4a^2} + \mu) \frac{1}{\sqrt{a}} w + \frac{1}{\sqrt{a}} w_{aa} + \frac{1}{a} \frac{1}{\sqrt{a}} |w|^2 w = 0. \tag{4.1}$$

We want to first show that w is bounded. Multiply (4.1) by $\sqrt{a}$ to get

$$w_{aa} + (\frac{1}{4a^2} + \mu) w + \frac{1}{a} |w|^2 w = 0. \tag{4.2}$$

Multiplying (4.2) by w_a and rewriting the resulting expression using 'perfect derivatives' we get

$$\partial_a \left[\frac{1}{2}(w_a^2) + \frac{1}{8a^2} w^2 + \frac{\mu}{2} w^2 + \frac{1}{4a} w^4 \right] = -\frac{1}{4a^3} w^2 - \frac{1}{4a^2} w^4. \tag{4.3}$$

Define

$$E(a) := \frac{1}{2}(w_a^2) + \frac{1}{8a^2} w^2 + \frac{\mu}{2} w^2 + \frac{1}{4a} w^4 \geq 0. \tag{4.4}$$

Then we have

$$\partial_a E(a) = -\frac{1}{4a^3} w^2 - \frac{1}{4a^2} w^4 \leq 0 \tag{4.5}$$

Hence $E(a)$ is decreasing, which implies w and w_a are bounded and $E_0 := \lim_{a \to \infty} E(a)$ exists. The boundedness of w and (4.5) again imply

$$0 \geq \partial_a E \geq -\frac{C}{a^2} E$$

for some $C > 0$ independent of a. Therefore $E_0 > 0$ and $E(a) - E_0 = O(\frac{1}{a})$. Moreover, let us change the variables

$$w = r \cos \theta, \quad w_a = r \sin \theta.$$

Recall $\mu = 1$, we have $r^2 - 2E_0 = O(\frac{1}{a})$ and thus

$$r = \sqrt{2E_0} + O(\frac{1}{a}) \tag{4.6}$$

as $a \to \infty$. One may compute

$$\begin{aligned} r_a =& -\frac{1}{4a^2} r\cos\theta\sin\theta - \frac{1}{a} r^3\cos^3\theta\sin\theta \\ \theta_a =& -1 - \frac{1}{4a^2}\cos^2\theta - \frac{1}{a} r^2\cos^4\theta \\ =& -1 - \frac{3E_0}{8a} + \frac{E_0}{a}(\frac{3}{8} - \cos^4\theta) + \frac{E_0 - r^2}{a}\cos^4\theta. \end{aligned}$$

Integrating the θ equation by parts and using $\theta_a = -1 + O(\frac{1}{a})$, one may show,

$$\theta = \theta_0 + a + \frac{3E_0}{8}\log a + O(\frac{1}{a}). \tag{4.7}$$

We next consider solutions v_d to (3.5) (D). Multiplying (D) by av and integrating on $[0, a]$, we obtain

$$(v^2)_a = \frac{2}{a}\int_0^a \tilde{a}\left((v_a)^2 + v^2 + v^4\right) d\tilde{a}$$

which implies $v^2 \geq O(\log a)$ as $a \to \infty$ unless $v \equiv 0$. Therefore there are no nontrivial bounded solutions v_d to (3.5)(D).

All in all we conclude then that for the case $\mu > 0$ there are no bounded solutions other than $(v_f, 0, -v_f, 0)$, which can not be in C^0 unless trivial.

Case $\mu = 0$. For (3.5) (F), let $v =: a^{-\frac{1}{3}} w$ one may compute

$$w_{aa} + \frac{1}{3a} w_a + \frac{1}{9a^2} w + a^{-\frac{2}{3}} w^3 = 0.$$

Multiplying it by $a^{\frac{2}{3}} w_a$, we have

$$\partial_a(\frac{1}{2} a^{\frac{2}{3}}(w_a)^2 + \frac{1}{4} w^4 + \frac{1}{18} a^{-\frac{4}{3}} w^2) = -\frac{2}{27} a^{-\frac{7}{3}} w^2 \leq 0.$$

Therefore $w = O(1)$ and $w_a = O(a^{-\frac{1}{3}})$ as $a \to \infty$, which implies

$$v_a = O(a^{-\frac{2}{3}}), \quad v = O(a^{-\frac{1}{3}}), \text{ as } a \to \infty.$$

In fact, for different values of $v(0)$, the solution $v(a)$ only differs by a scaling change.

Finally, the same argument as in the case of $\mu > 0$ shows that the only bounded solution v_d to (3.5)(D) is the trivial one.

We thus also conclude in the case $\mu = 0$ that there are no bounded solutions other than $(v_f, 0, -v_f, 0)$, which can not be in C^0 unless is trivial.

Case $\mu < 0$ ($\mu = -1$). From the uniqueness given in (4) of Theorem 7.1 of solutions which are Hölder continuous at 0, clearly $v_d, v_f \equiv 0, \pm 1$ if $v_d(0), v_f(0) = 0, \pm 1$.

We first consider solutions v_d to (3.5) (D) and claim that if $|v_d(0)| > 1$, then $|v_d(a)| > 1$ and $|v_d(a)| \to \infty$ as $a \to \infty$. Indeed, without any loss of generality assume $v_d(0) > 1$ and $v_d(a) = 1 + z(a)$. One may compute

$$z_{aa} + \frac{1}{a} z_a - 2z - 3z^2 - z^3 = 0. \tag{4.8}$$

Multiplying it by z_a, we have

$$\partial_a(\frac{1}{2}(z_a)^2 - (z^2 + z^3 + \frac{1}{4}z^4)) = -\frac{1}{a}(z_a)^2,$$

which implies

$$0 > -(z(0)^2 + z(0)^3 + \frac{1}{4}z(0)^4) \geq (\frac{1}{2}(z_a(a))^2 - (z(a)^2 + z(a)^3 + \frac{1}{4}z(a)^4)).$$

Therefore, $z(a) \neq 0$, for all $a > 0$ which yields $z(a) > 0$. Multiplying (4.8) by $-az$ implies

$$\partial_a(z^2) = \frac{2}{a}\int_0^a \tilde{a}\,((z_a)^2 + 2z^2 + 3z^3 + z^4)\,d\tilde{a}$$

which implies $z^2 \geq c \log a$.

Thus we assume $v_d(0) \in (-1, 1)$ and claim $v_d \to 0$ like $\frac{1}{\sqrt{a}}\cos(\gamma a + \sigma \log a + \beta)$ as $a \to +\infty$. Indeed, multiplying (3.5) (D) by v_a we obtain

$$\partial_a(\frac{1}{2}(v_a)^2 + \frac{1}{2}v^2 - \frac{1}{4}v^4) = -\frac{1}{a}(v_a)^2 \leq 0.$$

Since $\frac{1}{2}v^2 - \frac{1}{4}v^4$ is strictly increasing in v^2 for $v^2 \in [0, 1]$, we have

$$v(0) \in (-1, 1) \implies v(a)^2 \leq v(0)^2 < 1, \quad a > 0. \tag{4.9}$$

We will follow closely the procedure which we used to handle solutions to (3.5) (F) in the case of $\mu > 0$. Define $v =: \frac{1}{\sqrt{a}}w$. Then w satisfies:

$$w_{aa} + (\frac{1}{4a^2} + 1)w - \frac{1}{a}w^3 = 0 \tag{4.10}$$

Multiplying (4.10) by w_a and rewriting the resulting expression as 'perfect derivatives' we get

$$\partial_a\left[\frac{1}{2}(w_a^2) + \frac{1}{8a^2}w^2 + \frac{1}{2}w^2 - \frac{1}{4a}w^4\right] = -\frac{1}{4a^3}w^2 + \frac{1}{4a^2}w^4. \tag{4.11}$$

Define now

$$E(a) := \frac{1}{2}(w_a^2) + \frac{1}{8a^2}w^2 + \frac{1}{2}w^2 - \frac{1}{4a}w^4 \geq \frac{1}{2}(w_a^2) + (\frac{1}{2} - \frac{|v(0)|^2}{4})w^2, \tag{4.12}$$

where we used $|a^{-\frac{1}{2}}w| = |v| \leq |v(0)|$. From (4.10) we have that

$$\partial_a E(a) = -\frac{1}{4a^3}w^2 + \frac{1}{4a^2}w^4 \leq (\frac{w^2}{4a^2})w^2. \tag{4.13}$$

Using $|a^{-\frac{1}{2}}w| = |v| \le |v(0)|$ and (4.9) again, we obtain from (4.13)

$$\partial_a E(a) \le \frac{c}{a}E, \quad c = \frac{|v(0)|^2}{2-|v(0)|^2} < 1. \tag{4.14}$$

The differential inequality (4.14) implies $E \le O(a^c)$ and thus $w \le O(a^{\frac{c}{2}})$ as $a \to \infty$. Now from (4.12) and (4.9) we obtain that

$$E(a) \ge \kappa_0 w^2 \quad \text{where} \quad \frac{1}{2} > \kappa_0 := \frac{1}{2} - \frac{|v(0)|^2}{4} > 0. \tag{4.15}$$

Using (4.13) and (4.15) we conclude that

$$\partial_a E(a) \le O(a^{c-2})w^2 \le O(a^{c-2})E. \tag{4.16}$$

Hence $E = O(1)$, which gives $w = O(1)$ and $w_a = O(1)$ from (4.12). The oscillatory asymptotic form of v then follows from the exactly same argument used in the case $\mu > 0$.

Next, we consider solutions v_f to (3.5) (F). We will show that for $v_f(0) \in (0,2)$, solutions v_f to (3.5)(F) behave like $1 + \frac{1}{\sqrt{a}}\cos(\gamma a + \sigma \log a + \beta)$ as $a \to +\infty$. By the odd symmetry of the equation, the case when $v_f(0) \in (-2,0)$ is the same. First, multiplying (3.5) (F) by v_a, we have

$$\partial_a(\frac{1}{2}(v_a)^2 + \frac{1}{4}(v^2-1)^2) = -\frac{1}{a}(v_a)^2$$

which, along with $v_a(0) = 0$ implies that

$$|v(a)^2 - 1| \le |v(0)^2 - 1|. \tag{4.17}$$

In particular, (4.17) yields $v(a) > 0$. Again, let $v =: 1 + a^{-\frac{1}{2}}w$; then w satistfies

$$w_{aa} + (2 + \frac{1}{4a^2})w + 3a^{-\frac{1}{2}}w^2 + \frac{1}{a}w^3 = 0. \tag{4.18}$$

Multiplying (4.18) by w_a, we have

$$\partial_a E = -\frac{1}{4}a^{-3}w^2 - \frac{1}{2}a^{-\frac{3}{2}}w^3 - \frac{1}{4}a^{-2}w^4 = -\frac{1}{4}a^{-1}w^2(a^{-2} + 2a^{-\frac{1}{2}}w + a^{-1}w^2) \tag{4.19}$$

where

$$E = \frac{1}{2}(w_a)^2 + (1+\frac{1}{8}a^{-2})w^2 + a^{-\frac{1}{2}}w^3 + \frac{1}{4}a^{-1}w^4 \ge \frac{1}{2}(w_a)^2 + \frac{1}{4}(1+v)^2w^2 > \frac{1}{2}(w_a)^2 + \frac{1}{4}w^2. \tag{4.20}$$

In the last two inequalities, we used the fact $v(a) > 0$. For the same reason, we can estimate the right hand side of of (4.19) as

$$\partial_a E \le \frac{w^2}{4a}(1-v^2) \le \frac{c}{a}E, \quad c = |v(0)^2 - 1| < 1.$$

It implies $E = O(a^c)$ and thus $w = O(a^{\frac{c}{2}})$ as $a \to \infty$. Substituting this into (4.19) we obtain

$$\partial_a E = O(a^{-\frac{3-c}{2}})E$$

and thus $E = O(1)$ and in fact $E \to E_0 > 0$ as $a \to \infty$ unless $w \equiv 0$. The oscillatory asymptotic form of w can be obtained much as in the previous cases.

In fact, by a more careful analysis, one may show the behavior of v_f at $a \to \infty$ is always one of oscillatory convergence to ± 1 except maybe for a sequence of data $v_f(0) = v_j$, $|v_j| > 2$ which make v_f converge to 0 exponentially. Since we seek bounded continuous solutions to (3.5) and we have already showed that solutions v_d to (3.5)(D) are unbounded if $|v_d(0)| > 1$, the above result is sufficient for us.

From the above we conclude then that for the case $\mu < 0$ there exist smooth v_d and v_f so that $v_d(0) = v_f(0) \in (-1, 1)$, $\partial_a v_d(0) = 0 = \partial_a v_f(0)$, $v_f(\infty) = \pm 1$ and $v_d(\infty) = 0$ where the convergence is like $a^{-\frac{1}{2}} \cos(\gamma a + \beta \log a + \sigma + O(\frac{1}{a}))$.

5. Radial self-similar waves

To analyze the system (3.7) for self-similar solutions, we first change the independent variable to $s = \frac{a^2}{2}$ and then change the unknown to $w = s^{\frac{1}{2}} e^{\mp \frac{i}{4} s} v$. Consequently, (3.7) becomes

$$\begin{cases} w_{ss} + (\frac{1}{16} + \frac{1}{4s^2} + \frac{\mu}{2s})w + \frac{1}{2s^2}|w|^2 w = 0 & \text{for } |x| > |y| \quad \text{(F)} \\ w_{ss} + (\frac{1}{16} + \frac{1}{4s^2} - \frac{\mu}{2s})w - \frac{1}{2s^2}|w|^2 w = 0 & \text{for } |x| < |y| \quad \text{(D)} \end{cases} \tag{5.1}$$

Without loss of generality (see Remark 3.1 and 3.3), we may assume w is real-valued. It is clear that, as long as we obtain the boundedness of the solutions to the ODEs, their oscillatory asymptotic forms follow from the exactly same change of variables as in the previous section $w = r\cos\theta$ and $w_s = \frac{1}{4} r \sin\theta$ and similar arguments. So we focus only on the boundedness.

Multiplying (5.1) by w_s and letting

$$E(s) = \frac{1}{2} w_s^2 + (\frac{1}{32} + \frac{1}{8s^2} \pm \frac{\mu}{4s})w^2 \pm \frac{1}{8s^2} w^4 \tag{5.2}$$

we obtain

$$\partial_s E(s) = -(\frac{1}{4s^3} \pm \frac{\mu}{4s^2})w^2 \mp \frac{1}{4s^3} w^4, \tag{5.3}$$

which we would like to control in terms of (5.2).

Focusing equation (F). In this case, for $s > s_0 := 16(|\mu| + 1)$ we have $E > 0$ and

$$\partial_s E(s) \le \frac{|\mu|}{4s^2} w^2 \le \frac{16|\mu|}{s^2} E(s).$$

Therefore $E(s) \to E_0 > 0$ as $s \to \infty$ which implies that w and w_s are bounded as desired.

Defocusing equation (D). In this case, we follow a standard bootstrap argument. For those solution $w(s)$ such that $|w(s_0)| \leq 1$, let

$$s_1 := \sup\{s \geq s_0 \mid |w(s')| \leq 1 \, \forall s' \in [s_0, s]\}.$$

For $s \in [s_0, s_1]$, we have from (5.2)

$$E(s) \geq \frac{1}{2} w_s^2 + \frac{1}{64} w^2 \tag{5.4}$$

and from (5.3) and (5.4) we obtain that

$$\partial_s E(s) \leq \frac{|\mu|}{4s^2} w^2 + \frac{1}{4s^3} w^4 \leq \frac{|\mu|+1}{4s^2} w^2 \leq \frac{16(|\mu|+1)}{s^2} E(s).$$

Therefore, if $|w(s_0)|$ and $|w_s(s_0)|$ are sufficiently small -which by the continuity in Theorem 7.1 is guaranteed by taking $w(0)$ sufficiently small- we have that $s_1 = \infty$. It follows $w(s)$ and $w_s(s)$ are bounded.

6. Conclusions & Future Directions

In conclusion, in the present work, we have considered the two-dimensional hyperbolic NLS equation. We have illustrated the relevance of the use of the so-called hyperbolic variables in the context of this PDE model (relevant to optical media combining normal dispersion –typically in the time variable– to anomalous one –typically in the continuous spatial variables). We have discussed the context of weak solutions within the model and how relevant compatibility conditions naturally arise along the characteristic lines of the hyperbolic operator. We have subsequently elucidated the prototypical weak standing wave solutions that the model supports establishing that they only exist for negative values of the propagation constant μ and characterizing their weak modulated power law decay on the basis of energy-type methods (in the radial hyperbolic variable). Also, similar techniques but in appropriately rescaled variables have been used to discuss the existence of radial, self-similar wave solutions to the PDE for all $\mu \in \mathbb{R}$.

It would be particularly interesting to revisit the numerical computations of earlier works [11, 13, 14, 16, 17] in the context of weak solutions and to examine whether for appropriately crafted initial conditions the relevant direct numerical computations support the decay rates analytically obtained herein. Preliminary computations with radial initial data such as $u(x, y, 0) = \tanh(a)$ for $|x| > |y|$ (and zero otherwise) seem to support the radial evolution of the solution profile and the development of oscillations around the unit asymptotic state, as suggested by our Main Theorem (see Fig. 1). In particular, this initial profile provides a vanishing solution in the (D) region, as well as a radial solution asymptoting to unity in the (F) region. The observation that can be made based on these preliminary results is that in the latter region the data indeed remains radial in nature (since the hyperbolae forming in the middle panel are equi-a lines) yet it also develops an oscillatory dynamics associated with the decay to the stationary state, which is reminiscent of the oscillatory convergence suggested by our

Main Theorem. It should also be noted that the numerical method used here was a 4th order Runge-Kutta for marching the system in time, combined with a centered-difference in space scheme in a sufficiently fine spatial grid. However, detailed numerical investigations are relevant to establish the relevant decay; these will be deferred to a future publication. From the point of view of analytical considerations, it would be relevant to generalize the results obtained herein to the more experimentally tractable directions of 3-dimensional media (with two focusing and one defocusing direction) as in [13, 14, 11, 12]. In that setting, it would be relevant to connect generalizations of the present results to the rapidly evolving literature on the nonlinear X-waves of the above works. Another highly promising direction is that of the quasi-discrete waveguide array setting of equation (1.1) in $2+1$-dimensions or of [22] in $3+1$-dimensions. Notice, however, that even the setting of equation (3HNLS) considered herein may be of relevance to physical applications of light propagation within planar waveguides in glass membrane fibers, where already some interesting phenomena such as linear and nonlinear guidance with ultralow optical attenuation have been observed; see e.g. [23].

7. Appendix: An ODE blow-up theorem near a regular singular point

Consider

$$x' = \frac{1}{t}F(t,x), \quad t>0,\ x\in\mathbb{R}^n, \quad F\in C^k(\mathbb{R}^{n+1},\mathbb{R}^n). \tag{7.1}$$

Here $t=0$ is a so called regular singular point of the system and we are interested in solutions continuous or smooth at $t=0$. A subcategory of this system is a scalar valued equation

$$y^{(n)} = \frac{1}{t^n}H(t,y,ty',\dots,t^{n-1}y^{(n-1)})$$

which include (3.5)(F)(D) and (3.7)(F)(D) as special cases. In fact, by setting $x_j = t^{j-1}(y^{(j-1)}-c_j)$ for $j=1,\dots,n$ and arbitrary constants $c_1,\dots,c_n$, the above equation turns into the form of (7.1).

Let $B(X,r)$ denote the ball centered at 0 and with the radius r in a Banach space X.

Theorem 7.1. *Assume $k\geq 1$.*

1. *There exists a solution $x(t)$ to (7.1) which is continuous on $t\in[0,\varepsilon]$ for some $\varepsilon>0$ and $x(0)=x_0$ if and only if $F(0,x_0)=0$;*
2. *Suppose $F(0,x_0)=0$. Let E_+ be the generalized eigenspace of $F_x(0,x_0)$ corresponding to all eigenvalues whose real parts are greater than 0. Then there exist $\varepsilon>0$ and a C^k mapping $h: B(E_+,\varepsilon)\to B((E_+)^\perp,\varepsilon)$ so that: $x(t)$ is a solution of (7.1) on $(0,\varepsilon]$ with $x(\varepsilon)\in B(E_+,\varepsilon)+B((E_+)^\perp,\varepsilon)$ and Hölder continuous for $t\in[0,\varepsilon]$ and $x(0)=x_0$ if and only if $x(\varepsilon)\in graph(h)$.*
3. *The above solutions are of order $O(t^\beta)$ where $\beta\in(0,\beta_0)$ and*

$$\beta_0 = \min\{1, \inf\{\mathrm{Re}\,\lambda \mid \lambda \textit{ is an eigenvalue of } \mathrm{F_x}(0,\mathrm{x_0}) \textit{ and } \mathrm{Re}\,\lambda>0\}\}.$$

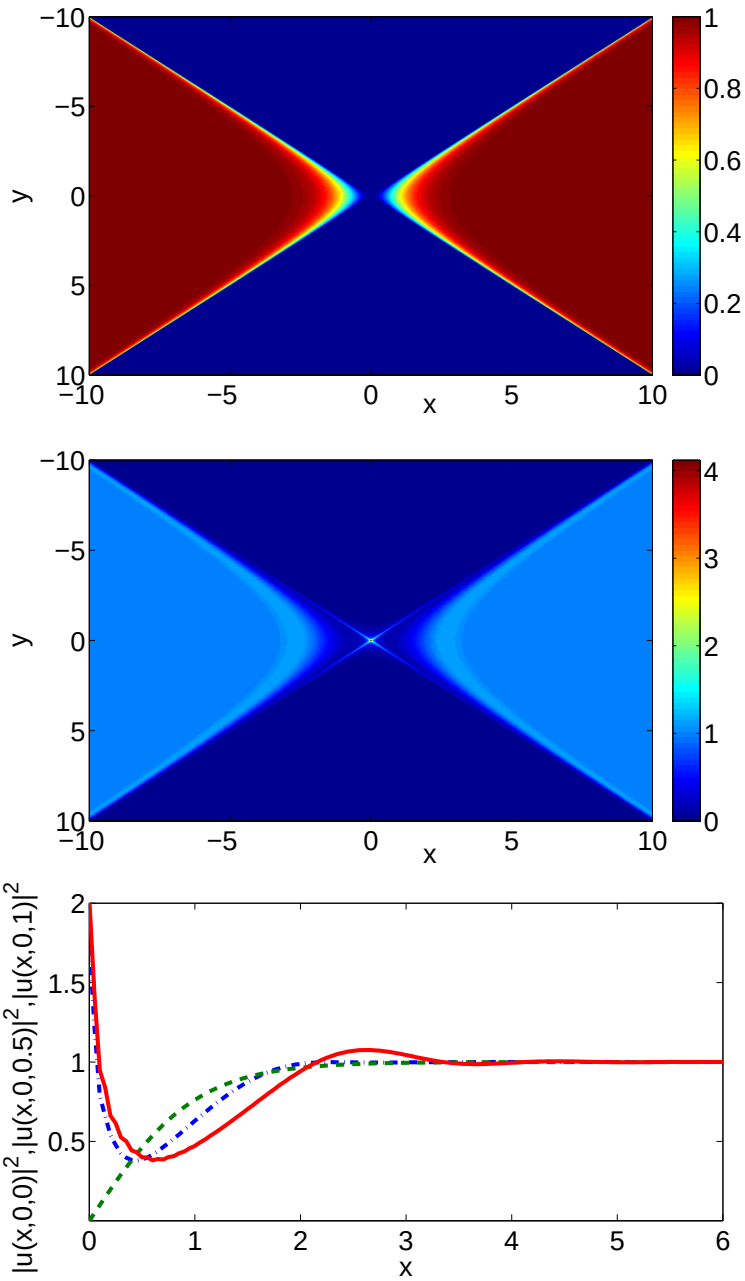

FIGURE 1. Evolution of Eq. (2.1) for $\gamma = 1$ (a factor of $1/2$ is also used in front of the $\Box_{xy}$ as is often done in the physics literature). The top panel shows the initial condition and the middle panel shows the result of the evolution at $t = 1$. Both panels show contour plots of $|u|^2$. Notice the hyperbolic equi-a contours forming in the middle panel. The bottom panel shows a radial cross-section (at $y = 0$) of the dynamics, illustrating the oscillations around the asymptotic state that spontaneously develop in the dynamics (as suggested by our Main Theorem). The dashed line shows the initial density profile, the dash-dotted one shows its evolution at $t = 0.5$ and the solid one its evolution at $t = 1$.

(4) *If $k \geq 2$ and $E_+ = \{0\}$, there exists only a unique solution $x(t)$ on $(0, \varepsilon]$ so that it is Hölder continuous in t on $[0, \varepsilon]$ and $x(0) = x_0$. Moreover, if $F_t(0, x_0) = 0$, $x(t) = O(t^2)$ as $t \to 0$.*

(5) *If $F = F(t, x, \alpha)$ with a external parameter α, $F \in C^k$, and $F(0, x_0, \cdot) \equiv 0$, the mapping h in item (2) is also C^{k-1} in α.*

Proof. To blow up the system to analyze solutions near $t = 0$, we create an auxiliary independent variable τ so that $\frac{dt}{d\tau} = t$, use $\dot{x}$ to denote the differentiation in τ, and rewrite (7.1) as an autonomous system

$$\begin{cases} \dot{t} = t \\ \dot{x} = F(t, x) \end{cases} \quad \text{or } \dot{\tilde{x}} = \tilde{F}(\tilde{x}) \tag{7.2}$$

where $\tilde{x} = (t, x)^T$. A solution of (7.1) is continuous at $t = 0$ is equivalent to that its corresponding solution of (7.2) converges as $\tau \to -\infty$. On the one hand, it is clear that $x_0 = \lim_{\tau \to -\infty} x(\tau)$ only if $\tilde{x}_0 = (0, x_0)^T$ is a fixed point of (7.2). On the other hand, suppose $F(0, x_0) = 0$, we notice 1 is an eigenvalue of $D\tilde{F}(\tilde{x}_0)$ with a generalized eigenvector $\mathbf{T} = (1, w)^T$, for some $w \in \mathbb{R}^n$, and $\tilde{E}_+ = E_+ \oplus \text{span}\{\mathbf{T}\}$ is the generalized eigenspace of $D\tilde{F}(\tilde{x}_0)$ corresponding to all eigenvalues whose real parts are greater than 0. From the standard unstable manifold theorem (c.f for example [4, 5, 6]), there exists a C^k manifold $W^u \subset \mathbb{R}^{n+1}$ in a neighborhood of $\tilde{x}_0$ so that (a) $\tilde{x}_0 \in W^u$; (b) $T_{\tilde{x}_0} W^u = \tilde{E}_+$; (c) it is locally invariant under the flow of (7.2); and (d) $\tilde{x} \in W^u$ if and only the solution $\tilde{x}(\tau)$ of (7.2) with $\tilde{x}(0) = \tilde{x}$ stay close to $\tilde{x}_0$ for all $\tau \in (-\infty, 0]$ and $\tilde{x}(\tau) \to \tilde{x}_0$ exponentially as $\tau \to -\infty$. Changing the variable back from τ to t, the exponential rate $O(e^{r\tau})$ as $\tau \to -\infty$ becomes $O(t^r)$. Therefore, for equation (7.1), solutions on $W^u \backslash \{t = 0\}$ exactly correspond to solutions which are Hölder in t and converge to x_0 as $t \to 0$. This proves statement (2). Statement (3) is obvious when one notices that $t = ce^{\tau}$. Statement (4) is a consequence of the tangency of W^u to $\tilde{E}_+$, which is equal to $\mathbb{R}\mathbf{T} = \mathbb{R}(1, 0)^T$ in this case, at $\tilde{x}_0$. Statement (5) follows directly from the the standard result that unstable manifolds are smooth in the external parameters [6]. □

References

[1] H. Berestycki and P.L. Lions, *Une méthode locale pour l'existence de solutions positives de problèmes semi-linéaires elliptiques dans R^N,* J. Analyse Math. **38**, pp. 144-187 [1980] .

[2] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state,* Arch. Rat. Mech. Anal. **82** no. 4, pp. 313–345 [1983].

[3] H. Berestycki, P.L. Lions and L.A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in R^N,* Indiana Univ. Math. J. **30** no. 1, pp.141-157 [1981].

[4] S.-N. Chow, C. Li and D. Wang, *Normal forms and bifurcation of planar vector fields,* Reprint of the 1994 original. Cambridge University Press, Cambridge, 2008. viii+472 pp. ISBN: 978-0-521-10223-0.

[5] S.-N. Chow and K. Lu, *Invariant manifolds for flows in Banach spaces,* Journal of Differential Equations **74** no 2, pp. 285C317 [1988].

[6] S.-N. Chow, X.-B. Lin and K. Lu, *Smooth invariant foliations in infinite-dimensional spaces,* J. Differential Equations **94** no. 2, 266C291 [1991].

[7] A. Chong, W.H. Renninger, D.N. Christodoulides and F.W. Wise, *Airy-Bessel wave packets as versatile linear light bullets*, Nature Photonics **4**, 103-106 [2010].

[8] D.N. Christodoulides, N.K. Efremidis, P. Di Trapani and B.A. Malomed, *Bessel X-waves in two and three dimensional bidispersive optical systems* Opt. Lett. **29** (13), 1445-1448 [2004].

[9] C. Conti, *X-wave-mediated instability of plane waves in Kerr media*, Phys. Rev. E **68**, 016606 [2003].

[10] C. Conti, *Generation and nonlinear dynamics of X waves of the Schrödinger equation,* Phys. Rev. E **70**, 046613 [2004].

[11] C. Conti and S. Trillo, *Nonlinear X Waves* in *Localized Waves*, H.E. Hernandez-Figueroa, M. Zamboni-Rached and E. Recomi (Eds.), pp. 243-272 [2007].

[12] C. Conti, P. Di Trapani and S. Trillo, *X-Waves in Self-Focusing of Ultra-Short Pulses*, pp. 439-456 in *Self-Focusing: Past and Present - Fundamentals and Prospects*, Topics in Applied Physics **114**, Springer-Verlag, Berlin [2009].

[13] C. Conti, S. Trillo, P. Di Trapani, A. Piskarkas, O. Jedrkiewicz and J. Trull, *Nonlinear Electromagnetic X Waves*, Phys. Rev. Lett. **90**, 170406, pp. 1-4 [2003].

[14] P. Di Trapani, G. Valiulis, A. Piskarkas, O. Jedrkiewiecz, J. Trull, C. Conti and S. Trillo, *Spontaneously Generated X-Shaped Light Bullets*, Phys. Rev. Lett. **91**, 093904 [2003].

[15] W. Ding, H. Tang, C. Zeng, *Self-similar solutions of Schrödinger flows,* Calc. Var. Partial Differential Equations, **34**, no. 2, 267–277 [2009].

[16] S. Droulias, K. Hizanidis, J. Meier, and D.N Christodoulides, *X-Waves in nonlinear normally dispersive waveguide arrays* Optics Express **13** no. 6, pp. 1827-1832 [2005].

[17] N. Efremidis, K. Hizanidis, B. Malomed and P. Di Trapani *Three Dimensional Vortex Solitons in Self-Defocusing Media* Physical Rev. Letters **98** 113901, pp. 1-4 [2007].

[18] J.M. Ghidaglia and J.C. Saut, *On the initial value problem for the Davey-Stewartson systems.* Nonlinerity **3** no.2, pp. 475-506 [1990].

[19] J.M. Ghidaglia and J.C. Saut, *Nonelliptic Schrödinger Equations* J. Nonlinear Sci. **3** no.2, pp. 169-195 [1993].

[20] J.M. Ghidaglia and J.C. Saut, *Nonexistence of traveling wave solutions to nonelliptic Schrödinger equations* J. Nonlinear Sci. **6** no. 2, pp. 139-145 [1996].

[21] K. Hayata and M. Koshiba, *Bright-dark solitary wave solutions of a multidimensional nonlinear Schrödinger equation* Physical Rev. E **48**, no. 3, pp. 2312-2315 [1993]

[22] M. Heinrich, A. Szameit, F. Dreisow, R. Keil, S. Minardi, T. Pertsch and F. Lederer, *Observation of three-dimensional discrete-continuous X waves in photonic lattices,* Phys. Rev. Lett. **103**, 113903 [2009].

[23] N.Y. Joly, T.A. Birks, A. Yulin, J.C. Knight and P.St.J. Russel, *Linear and nonlinear guidance in an ultralow loss planar glass membrane*, Opt. Lett. **30**, 2469-2471 [2005].

[24] Y. Lahini, E. Frumker, Y. Silberberg, S. Droulias, K. Hizanidis , R. Morandotti and D. Christodoulides, *Discrete X-Wave Formation in Nonlinear Waveguide Arrays,* Physical Rev Letters **98**, 023901, pp. 1-4 [2007].

[25] N. Moshonas, Y. Kominis, P. Papagiannis, K. Hizanidis and D.N. Christodoulides, *Spatiotemporal interaction of optical beams in bidispersive media,* J. Opt. Soc. Am. B **26**, 1479–1483 [2009].

[26] S. I. Pohozaev, *Eigenfunctions of the equations* $\Delta u+\lambda f(u) = 0$*,* Sov. Math. Dokl. **5**, pp. 1408-1411 [1965].

[27] W. Strauss, *Existence of solitary waves in higher dimensions,* Comm. Math. Phys. **55**, pp. 149-162 [1977].

[28] C. Sulem and J.P. Sulem, *Nonlinear Schrödinger Equations: Self-Focusing And Wave Collapse,* Applied Mathematical Sciences **139**, Springer [1999].

[1] Department of Mathematics, University of Massachusetts, 710 N. Pleasant Street, Amherst MA 01003

E-mail address: kevrekid@math.umass.edu

[2] Radcliffe Institute for Advanced Study, Harvard University, Byerly Hall, 8 Garden Street, Cambridge, MA 02138 and Department of Mathematics, University of Massachusetts, 710 N. Pleasant Street, Amherst MA 01003

E-mail address: nahmod@math.umass.edu

[3] School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332

E-mail address: zengch@math.gatech.edu