Spatial solitons under competing linear and nonlinear diffractions

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We introduce a general model which augments the one-dimensional nonlinear Schrödinger (NLS) equation by nonlinear-diffraction terms competing with the linear diffraction. The new terms contain two irreducible parameters and admit a Hamiltonian representation in a form natural for optical media. The equation serves as a model for spatial solitons near the supercollimation point in nonlinear photonic crystals. In the framework of this model, a detailed analysis of the fundamental solitary waves is reported, including the variational approximation (VA), exact analytical results, and systematic numerical computations. The Vakhitov-Kolokolov (VK) criterion is used to precisely predict the stability border for the solitons, which is found in an exact analytical form, along with the largest total power (norm) that the waves may possess. Past a critical point, collapse effects are observed, caused by suitable perturbations. Interactions between two identical parallel solitary beams are explored by dint of direct numerical simulations. It is found that in-phase solitons merge into robust or collapsing pulses, depending on the strength of the nonlinear diffraction.

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I. INTRODUCTION

It is well known that bright spatial solitons in optical media are supported by the balance between the self-focusing nonlinearity and diffraction. A great deal of attention has also been recently attracted to more special cases, when the diffraction is subject to various forms of management, so that the diffraction coefficient gradually vanishes or periodically changes its sign [1]. In particular, it was shown in Ref. [2] that related to such a setting with a vanishing diffraction coefficient is another recently proposed class of models, in which stable bright solitons are supported by the interplay of the ordinary diffraction and self-defocusing nonlinearity, whose local strength grows at \( r \to \infty \) at any rate faster than \( r^p \), where \( D \) is the spatial dimension [3].

A physically important realization of the effectively weak diffraction, which may therefore be sensitive to corrections that are usually negligible, is the setting corresponding to the supercollimation in photonic crystals. This setting was demonstrated experimentally [4] and studied in detail theoretically [5–7]. Similar situations, and the possibility of the existence of solitary waves in them, were also analyzed in other photonic media, such as waveguiding arrays [8], cavities [9], and chains with the quadratic nonlinearity [10]. Furthermore, it was proposed that similar schemes may be implemented as well for acoustic waves in sonic crystals [11], and in Bose-Einstein condensates loaded into optical lattices [12].

The smallness of the diffraction coefficient near the supercollimation point suggests that the corresponding model should be extended by adding nonlinear corrections to the diffraction term in the usual nonlinear Schrödinger (NLS) equation. Such a generalization of the NLS equation was proposed in Ref. [7]. In the framework of the generalized model, new properties of spatial solitons were predicted—in particular, the existence of a maximum total power, above which solitary beams do not exist, as well as inelastic interactions between the beams. The objective of the present work is to extend the model proposed in Ref. [7], and study basic properties of solitary waves in the extended system. The main modifications, in comparison with the previously studied model, are that we propose the most general nonlinear-diffraction terms which agree with the Hamiltonian structure of the extended NLS equation (the term introduced in Ref. [7] cannot be derived from a natural Hamiltonian), and that these general terms depend on two irreducible parameters, rather than a single parameter, which was the case in Ref. [7].

The extended model is introduced in Sec. II, where we also present a variational approximation (VA) [13] for the study of fundamental solitary waves in this model, and additional analytical results. Numerical results for the shape and stability of the solitons, including the comparison with predictions of the VA and interactions between solitons forming out-of-phase and in-phase pairs, are reported in Sec. III. The paper is concluded by Sec. IV.

II. MODEL AND ANALYTICAL FRAMEWORK

Our aim is to derive a general NLS model including different contributions to the nonlinear diffraction. This is done by developing an extended version of the model proposed in Ref. [7] for the description of the supercollimation in photonic crystals. We aim to extend the model by using a Lagrangian formulation that produces terms similar to the one introduced in Ref. [7], along with additional terms invoked by the underlying Lagrangian and/or Hamiltonian structure.

Our starting point is the Lagrangian density for an extended NLS equation including the nonlinear diffraction in a general form, \( \mathcal{L} = (i/2) (q^\ast q_t - q q_t^\ast) - \mathcal{H} \), where \( q \) is the local wave amplitude, the asterisk stands for the complex conjugation, \( \xi \) is the propagation distance, \( n \) is the transverse coordinate in the planar waveguide, and the Hamiltonian density is

\[
\mathcal{H} = |q_n|^2 - \frac{1}{2}|q|^4 - \frac{\beta}{2}|q|^2|q_n|^2 - \frac{\gamma}{4} (q^\ast q_n q_n^\ast q^\ast q_n q_n^\ast q^\ast q_n q_n^\ast).
\]

(1)
with real coefficients $\beta$ and $\gamma$. Varying the Lagrangian, $L = \int_{-\infty}^{\infty} L dq$, gives rise to the generalized NLS equation,

$$\begin{align*}
 iq_t + q_{yy} + |q|^2 q - \frac{\beta}{2} (i|q|^2 q_{yy} + q^* q_y) \\
 + \frac{\gamma}{2} (q^* q_y - 2q|q|^2 - q^2 q_{yy}) = 0. 
\end{align*} \tag{2}$$

This constitutes an extension of Eq. (3) from Ref. [7]. In particular, the term $- (\beta/2) |q|^2 q_{yy}$, which did not appear in Ref. [7], emerges along with its counterpart, $- (\beta/2) |q|^2 q_{yy}$, that accounted for the nonlinear diffraction in Ref. [7], in the framework of the self-consistent Lagrangian/Hamiltonian formulation. Additionally, the terms proportional to $\gamma$, which were absent in the equation adopted in Ref. [7], represent contributions to the nonlinear diffraction at the same order as the terms $\sim \beta$, hence they should be included too. The other terms in Eq. (2) correspond to the ordinary NLS equation. Dynamical invariants of Eq. (2) are the Hamiltonian, $H = \int_{-\infty}^{\infty} \mathcal{H} dq$ and the soliton stability region according to the Vakhitov-Kolokolov criterion, $dP/dk = 0$, is determined by the equation $P + 2 (\beta + \gamma) k = 0$ (the prime stands here for the derivative with respect to the bracketed argument).

Rescaling the field amplitude and coordinate as $Q \equiv \sqrt{k} \tilde{Q}$, $x \equiv \sqrt{k} \eta$, casts Eq. (7) into a normal form,

$$-\tilde{Q} + \frac{d^2 \tilde{Q}}{dx^2} - \frac{1}{2} (\beta + \gamma) k \left[ \left( \frac{d \tilde{Q}}{dx} \right)^2 + \tilde{Q}_0^2 \frac{d^2 \tilde{Q}}{dx^2} \right] + \tilde{Q}^3 = 0, \tag{9}$$

where $\tilde{Q}_0$ is considered as a function of $(\beta + \gamma) k$.

$k_{co}$, which is a border of the soliton stability region according to the Vakhitov-Kolokolov (VK) criterion, $dP/dk = 0$, is determined by the equation $P + 2 (\beta + \gamma) k_{co} = 0$ (the prime stands here for the derivative with respect to the bracketed argument). It follows from here that the cutoff value of the power, $P_{max}$, above which the solitons do not exist (see the top right panel in Fig. 2 below), together with the corresponding soliton’s peak power and width (that represent, respectively, the largest peak power and smallest width that stable solitons may have), depend on the nonlinear-diffraction parameters as follows:

$$k_{co} = C_k (\beta + \gamma)^{-1}, \quad P_{max} = C_P (\beta + \gamma)^{-1/2}, \quad A_{max}^2 = C_A (\beta + \gamma)^{-1}, \quad W_{min} = C_W (\beta + \gamma)^{1/2}, \tag{11}$$

where numerical found constants are $C_k \approx 0.66, C_P \approx 2.35, C_A \approx 1.28$, and $C_W \approx 2.42$. For comparison, it is relevant to mention that in the model based on Eq. (4) the cutoff value of the propagation constant, $k_{co}$, is bounded by the existence, rather than stability, of the solitons [14]. Note that Eq. (12) reveals the same scaling of $k_{co}$ as $k_{co}$ as $\beta$ vs $\beta$ as Eq. (11) demonstrates for $k_{co}$ vs $\beta + \gamma$.

Numerical solutions obtained with the help of Eq. (9) are produced in the next section. Before turning to that, we present the VA for soliton solutions. This approach is well known to produce relevant predictions for many dynamical effects exhibited by solitons [13], such as the influence of phase...
modulations on the formation of solitons [18]. To this end, we adopt the Gaussian ansatz with amplitude $A$ and width $W$,
\[
Q(\eta) = A \exp[-\eta^2/(2W^2)],
\]
which is relevant for small values of the propagation constant $k$ (see Fig. 1 below). With the increase of $k$, the numerically found solution develops a peakonlike shape, hence we expect the VA to fail at larger $k$.

Inserting the ansatz (13) into the Lagrangian density corresponding to Eq. (7),
\[
\mathcal{L}_Q = -kQ^3 - \left(\frac{dQ}{d\eta}\right)^2 + \frac{\beta - \gamma}{2} Q^2 \left(\frac{dQ}{d\eta}\right)^2 + \frac{1}{2} Q^4,
\]
we calculate the effective Lagrangian,
\[
L_{\text{eff}} = \int_{-\infty}^{\infty} \mathcal{L}_Q d\eta = -\sqrt{\pi} k A^2 W - \frac{\sqrt{\pi \alpha}}{2} A^2 W + (\beta - \gamma) A^3 \frac{\sqrt{2\pi}}{16 W} + \frac{1}{2 \sqrt{2}} A^4 W,
\]
(15)
where $A^2$ may be replaced by the total power (3) corresponding to ansatz (13), which is $P = A^4 W$. This substitution yields
\[
4 \frac{\beta - \gamma}{\sqrt{\pi}} L_{\text{eff}} = -4k P - \frac{2P^2}{W^2} + \frac{\beta - \gamma}{\sqrt{2}} \frac{P^2}{W^3} + \frac{2\sqrt{2}}{3} \frac{P^2}{W}.
\]
(16)

Then, the Euler-Lagrange equations associated with stationary solutions within the framework of the VA are given by
\[
\frac{\partial L_{\text{eff}}}{\partial P} = 0 \Rightarrow -4k - \frac{2}{W^2} + \frac{\beta - \gamma}{\sqrt{2}} \frac{P}{W^3} + \frac{2\sqrt{2}}{3} \frac{P}{W} = 0,
\]
(17)
\[
\frac{\partial L_{\text{eff}}}{\partial W} = 0 \Rightarrow \frac{1}{W} - \frac{3(\beta - \gamma)}{8 \sqrt{2}} \frac{P}{W^2} - \frac{1}{2 \sqrt{2}} \frac{P}{W} = 0.
\]
(18)
Eliminating $P$ here, we arrive at an equation for the width,
\[
8k W^4 + 6(k(\beta - \gamma) - 2)W^2 - (\beta - \gamma) = 0.
\]
(19)

Obviously, this equation can be solved analytically. For instance, the physical solution for $\beta = \gamma$ is $W^2 = 3/(2k)$.

In what follows below, the linear stability of the solitary-wave solutions is analyzed by means of numerical methods, with the aim to identify parameter regions where the solutions are dynamically stable or unstable. To this end, we consider perturbations of the stationary solutions in the form of $q(\xi, \eta) = \{Q(\eta) + \varepsilon [v(\xi, \eta) + i w(\xi, \eta)]e^{i k \xi}\}$, with an infinitesimally small perturbation amplitude $\varepsilon$. Upon the substitution of this into Eq. (2) and linearization, we obtain the following equations:
\[
v_\xi = k w - w_{\eta\eta} - \beta(-Q^2 w_{\eta\eta} + Q^2 w - 2Q Q_{\eta} w_{\eta})
- \gamma(2Q Q_{\eta} w - 3Q^2 w_{\eta} + 2Q^2 w_{\eta\eta} - 2Q Q_{\eta\eta} w) - Q^2 w,
\]
\[
w_\xi = \tilde{L}_1 v + v_{\eta\eta}
+ \beta(-Q^2 v_{\eta\eta} - 2Q Q_{\eta} v - Q^2 v - 2Q Q_{\eta} v_{\eta})
+ \gamma(-2Q Q_{\eta} v_{\eta} - Q^2 v_{\eta} - Q^2 v_{\eta\eta} - 2Q Q_{\eta\eta} v) + 3Q^2 v,
\]
(20)
which can be written in a symbolic form as $v_\xi = -\tilde{L}_1 v$, $w_\xi = \tilde{L}_2 v$, with accordingly defined operators $\tilde{L}_{1,2}$. Looking for the stability eigenvalues $\lambda$, so that $v(\xi, \eta) = e^{i \lambda \xi} \tilde{v}(\eta)$, $w(\xi, \eta) = e^{i \lambda \xi} \tilde{w}(\eta)$, we arrive at the linear-stability problem in the form of $\lambda^2 \tilde{v} = -L_1 \tilde{L}_2 \tilde{v}$, and $\lambda^2 \tilde{w} = -L_2 \tilde{L}_1 \tilde{w}$. The solution of the linear stability problem is reported in the following section.

A well-known necessary stability condition relevant to fundamental solitary waves in models with the self-focusing nonlinearity is given by the VK criterion [17–19], $dP/dk > 0$. It is shown below that this criterion applies to the present model, being also sufficient for the stability of the solitons.

III. NUMERICAL RESULTS

We now turn to numerical solutions, and their comparison with results of the VA. Figure 1 presents the numerically obtained solutions of Eq. (9) as a function of the propagation constant $k$ for $\beta = 1$ and $\gamma = 1$ in Eqs. (7) and (9). To obtain these solutions, Newton’s method was used for solving the respective boundary-value problem, with vanishing (i.e., homogeneous Dirichlet) boundary conditions and the derivatives calculated by means of the centered-finite-difference approximation. Our initial guess for small $k$, i.e., small values of the coefficient in front of the nonlinear-diffraction terms in

FIG. 1. (Color online) The top panel shows a sequence of solution profiles ($Q$) obtained with the increase of $k$ (recall that $\eta$ is the spatial coordinate). The middle and bottom panels display the profile of $Q$, and the corresponding squared eigenvalues of the linear stability problem, for $k = 0.1$ and $k = 0.5$, respectively (accordingly, the former soliton is stable, while the latter one is not). In this figure and below, the results are displayed for $\beta = 1, \gamma = 1$, unless otherwise noted.
Eq. (9), was $\tilde{Q}(x) = \sqrt{2}\text{sech} x$, i.e., the stationary solution of the NLS equation, and we used the parametric continuation in $k$ (for each subsequent step in $k$, the previous solution was used as the initial guess). Naturally, it is observed that, as $k$ increases, the nonlinearity plays a more significant role and the relevant solution becomes narrower (i.e., with a smaller width) and taller (with a larger amplitude).

Figure 2 presents basic properties of the numerically obtained solutions, and the comparison with their counterparts produced by the VA. Particular characteristics of the soliton solutions that are displayed here are the amplitude, total power, and full width at half maximum (FWHM) (the total power is defined according to Eq. (3), cf. Eq. (10)). It is observed that, as expected, the VA is accurate for small values of $k$; however, it does not provide an adequate description for sharply peaked solutions at larger $k$, and it does not predict either the existence border for the solitons, that corresponds to the largest possible value of the total power, $P = P_{\text{max}}$, nor the stability-cutoff value of the propagation constant, $k = k_{\text{co}}$, which are evident in the top right panel of Fig. 2. However, as shown above, these values can be found in the exact analytical form given by Eq. (11), which is unrelated to the VA. In fact, the wave packets suffer collapse at $P > P_{\text{max}}$. It is relevant to mention that curves $A(k)$ and $P(k)$, which are displayed in the left and right top panels of Fig. 2, are qualitatively similar to the respective dependences reported in the inset of Figs. 2(d) and 2(b) of Ref. [7], for the equation which is tantamount to Eq. (4).

Two squared eigenvalues which, for sufficiently large $k$, give rise to instabilities are also shown in Fig. 2, as a function of $k$. It should be kept in mind that there is always a vanishing eigenvalue in the system due to the phase (gauge) invariance. At $k > k_{\text{co}} = 0.33$, when the monotonicity of the power of the solution changes, two eigenvalue pairs turn positive (i.e., $\lambda^2 > 0$), leading to instability, in accordance with the VK criterion and with analytical results (11). Thus the sharp-peaked solutions found at large $k$ are unstable.

Predictions of the stability analysis are confirmed by direct numerical simulations of the evolution of perturbed solitary waves. In the top left panel of Fig. 3, one observes that, despite adding perturbations, the solution remains intact for small $k$ (where the solitary waves are linearly stable). On the contrary, the self-focusing instability, leading to the above-mentioned weak collapse, which involves a small part of the total power at the center of the beam, and splits the rest into diverging jets, is evident for a narrower solution with a larger amplitude (i.e., a larger $k$) in the top right panel of Fig. 3. Strictly speaking, the collapse per se is not correctly described by Eq. (2), as it does not include the fourth-order linear diffraction (which would ultimately prevent it); however, the equation is adequate to describe the evolution of the beam toward the collapse.

As mentioned before there are two unstable eigenvalues when $k > 0.33$. One of them corresponds to an odd eigenmode, and the other one to an even eigenmode with a positive amplitude at the center. Numerical simulations show that, for $k > 0.33$, the stationary soliton perturbed by the odd eigenmode will end up in the weak collapse; see the top right panel in Fig. 3. On the other hand, with the even perturbation, the dynamics differ for different signs of the perturbations. The stationary soliton with the even-eigenmode perturbation added to it will collapse; see the middle left panel in Fig. 3. On the other hand, under the action of the perturbation with the opposite sign, the solitary wave tends to lower its amplitude by emitting small bursts of radiation and ultimately relaxing into a breather, as illustrated by the middle right panel of Fig. 3. Finally, the observation of the effects of the different parameter variations in the bottom panels of Fig. 3 leads to the conclusion that a higher value of $\gamma$ induces earlier collapse, while the opposite effect is observed with the increase of $\beta$.

The above simulations started with the input in the form of sech beams. On the other hand, the VA was based on the Gaussian ansatz (13), therefore it is relevant to test the evolution of solitons generated by the Gaussian input with parameters predicted by the VA. A typical example, for
parameters belonging to the stability region, is displayed in Fig. 4. One can observe that small (radiation) jets are shed at the initial stage of the evolution—obviously, with the intention to adjust the Gaussian to the exact solitonic shape—followed thereafter by the stable propagation of the resulting wave.

Finally, we consider the interaction of two solitary beams, for the basic cases of in-phase and π-out-of-phase soliton pairs (i.e., mutually attracting and repelling ones). The initial in-phase and out-of-phase pairs without a relative tilt (i.e., the solitons with zero relative velocity) were built using two well-separated copies of a stationary soliton produced by the Newton’s method for \( k = 1 \), while parameters \( \beta \) and \( \gamma \), which account for the nonlinear-diffraction terms in Eq. (2), were varied within boundaries of the stability regime. We observe that, in the presence of these terms, the repulsive interaction in the out-of-phase configuration does not change dramatically. However, the two parameters affect the interaction in different ways. In particular, the increase of \( \beta \) leads to a decrease in the strength of the repulsion (or attraction) between the solitons, while the increase of \( \gamma \) produces the opposite effect; see Figs. 5 and 6.

While the repulsion, leading to the separation of the solitons in Fig. 5, does not reveal any surprising findings, the attractive case is far more interesting. The nonlinear-diffraction terms lead to the loss of elasticity of the collisions between NLS solitons and their eventual merger into a single pulsating beam. Furthermore, it is possible to identify two different outcomes of the interaction in this case: The merger may lead to a subcritical scenario, i.e., the appearance of a noncollapsing robust pulson, that may slowly relax into a stable stationary soliton through weak radiation losses, or a supercritical one, which ends up collapsing. These two cases are represented, respectively, by the middle and bottom panels in Fig. 6. In the subcritical case, it is observed that the merger into the stable pulson is delayed by the terms \( \propto \beta \), or accelerated by the ones \( \propto \gamma \), but the merger happens, eventually. On the other hand, in the case shown in the bottom row of Fig. 6, the supercritical merger, leading to the collapse, occurs at \( \beta = 0.12, \gamma = 0 \), while no collapse happens at \( \beta = 0, \gamma = 0.12 \). In the latter case, more radiation is observed to be emitted in the course of the merger, which, in turn, reduces the total power of the emerging pulson.
the soliton pair in the integrable NLS equation, with a self-consistent Lagrangian and/or Hamiltonian formulation. The resulting model contains two independent parameters, $\beta$ and $\gamma$ in Eq. (2). The immediate motivation for the development of the model is the application to nonlinear photonic crystals in a vicinity of the supercollimation point, as suggested by Ref. [7]. In the framework of the proposed general equation, we have developed a systematic analysis of the fundamental solitary waves, using the VA (variational approximation), exact analytical results (wherever possible), and numerical methods, which corroborate our analytical observations for individual solitons and produce interesting results for their pairwise interactions. In particular, it has been demonstrated that the VK stability criterion is valid for the solitons in the present model, and the solitons exist only up to a maximum value of the power. The solitary-wave family consists of stable and unstable parts, the border between which, $k_{c0}$, and the corresponding largest value of the total power of the solitons, $P_{\text{max}}$, along with the largest peak power and smallest width that the stable solitons attain, were found in the exact form, as written in Eq. (11). Interactions between pairs of identical solitons were studied by means of direct simulations. It has been demonstrated that the merger of in-phase solitons may lead to the formation of a robust pulson, if the nonlinear diffraction is not too strong; otherwise, the emerging beam collapses. On the other hand, the out-of-phase waves always repel each other and separate, with the interaction time altered by the nonlinear diffraction effects.

We expect that such a generalized model may hold promise for future explorations, especially if tuned to experimental data for relevant photonic crystal structures, so that the relative strength of the parameters considered herein can be evaluated in realistic settings. On the other hand, it would be particularly relevant to extend the present considerations to other regimes including the ones pertaining to self-defocusing nonlinearities, or to higher-dimensional photonic crystals (possibly with either sign of the nonlinearity). Such studies will be deferred to future publications.

IV. CONCLUSION

The objective of this work was to introduce a generalized model which accounts for phenomena of nonlinear diffraction within the one-dimensional NLS equation from the perspective of the self-consistent Lagrangian and/or Hamiltonian formulation. The resulting model contains two independent parameters, $\beta$ and $\gamma$ in Eq. (2). The immediate motivation for the development of the model is the application to nonlinear photonic crystals in a vicinity of the supercollimation point, as suggested by Ref. [7]. In the framework of the proposed general equation, we have developed a systematic analysis of the fundamental solitary waves, using the VA (variational approximation), exact analytical results (wherever possible), and numerical methods, which corroborate our analytical observations for individual solitons and produce interesting results for their pairwise interactions. In particular, it has been demonstrated that the VK stability criterion is valid for the solitons in the present model, and the solitons exist only up to a maximum value of the power. The solitary-wave family consists of stable and unstable parts, the border between which, $k_{c0}$, and the corresponding largest value of the total power of the solitons, $P_{\text{max}}$, along with the largest peak power and smallest width that the stable solitons attain, were found in the exact form, as written in Eq. (11). Interactions between pairs of identical solitons were studied by means of direct simulations. It has been demonstrated that the merger of in-phase solitons may lead to the formation of a robust pulson, if the nonlinear diffraction is not too strong; otherwise, the emerging beam collapses. On the other hand, the out-of-phase waves always repel each other and separate, with the interaction time altered by the nonlinear diffraction effects.

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