The Variational Iteration Method to Solve the Korteweg-de Vries-Burgers Equation

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Abstract: To solve the Korteweg-de Vries-Burgers Equation equations the Variational Iteration Method (VIM) is used. Two case study problems of Korteweg-de Vries-Burgers equation are solved by using the VIM and the exact solutions are obtained. As a result, the capability and great potential of the VIM in solving the non-linear differential equations is proved by applying the method to the Korteweg-de Vries-Burgers equation.

Keywords: Korteweg-de Vries-Burgers equation, Variational Iteration Method, Nonlinear differential equations

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1. Introduction

Many phenomena in different scientific fields such as physics, chemistry, biology and engineering can be described through nonlinear partial differential which have attracted lots of attention among scientists. In order to analyze behavior of the modeled system, the solution of the differential equations should be obtained. Large classes of nonlinear equations do not have analytic solution which can be solved by semi-analytical method as the Variational Iteration Method (VIM).

The VIM can construct equations which are initially approximated with possible unknowns and have no specific requirements to nonlinear operators. A rapid convergence toward the exact solution can be achieved by using the VIM due to establishing correction functions by the general Lagrange multiplier.

The Variational Iteration Method (VIM) is pioneered by He [1]. Then, the VIM is applied by He [2,3] in order to solve autonomous ordinary differential equation as well as delay differential equation. He and Xh [4] presented new development and applications of the VIM to nonlinear wave equation, nonlinear fractional differential equations, nonlinear oscillations and nonlinear problems arising in various engineering applications.

Nourazar et al. [5] obtained the exact solution of the Burgers–Huxley equation by using the homotopy perturbation method. Also, application of the the homotopy perturbation method to the exact solution of nonlinear differential equations is presented by Nourazar et al. [6,7], Soori et al. [8] presented application of the Variational Iteration Method and the Homotopy Perturbation Method to the Fisher Type Equation.

Analysis and computation of a discrete KdV-Burgers type equation with fast dispersion and slow diffusion is presented by Artstein et al. [9].
El-Ajou et al. [10] used a new iterative algorithm in order to present approximate analytical solution of the nonlinear fractional KdV–Burgers equation. New exact solutions to the compound KdV-Burgers system with nonlinear terms of any order is presented by Wang et al. [11].

Many physical problems such as non-linear shallow water waves as well as wave motion in plasmas can be modeled by The Kdv equation. Also, the equation can be applied in modeling of waves generated by a wave maker in a channel and waves incoming from deep water into near shore zones.

The Korteweg-de Vries-Burgers equation is written as:

$$\frac{\partial u}{\partial t} + au^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^3 u}{\partial x^3} = 0,$$

(1.1)

Where $\alpha, \beta, \gamma$ are real constants and $n$ is a positive integer. When $\beta = 0$, the Eq. (1.1) is turned into the Korteweg-de Vries equation. Also, when $\gamma = 0$, the Eq. (1.1) is turned into the Burgers equation.

In the present research work, the Variational Iteration Method (VIM) is applied to obtain the closed form solution of the non-linear Korteweg-de Vries-Burgers equation. The main aim of the study is to present trend of rapid convergence of the sequences constructed by the VIM toward the exact solution of the equation. So, two case study problems of non-linear Korteweg-de Vries-Burgers equations are solved by using the VIM and the exact solution is obtained.

The ideas of variational iteration method is presented in the section 2. Application of the variational iteration method to the exact solution of Korteweg-de Vries-Burgers equation is presented in the section 3.

2. The idea of variational iteration method

The idea of the variational iteration method is based on constructing a correction functional by a general Lagrange multiplier. The multiplier is chosen in such a way that its correction solution is improved with respect to the initial approximation or to the trial function. To illustrate the basic idea of the variational iteration method, consider the following nonlinear equation:

$$Lu(t) + Nu(t) = g(t),$$

(2.1)

Where $L$ is a linear operator, $N$ is a nonlinear operator, and $g(t)$ is a known analytic function. According to the variational iteration method, we can construct the following correction functional:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^\infty \lambda(\xi) \left( Lu_n(\xi) + N \tilde{u}_n(\xi) - g(t) \right) d\xi,$$

(2.2)

Where $\lambda$ is a general Lagrange multiplier which can be identified optimally via variational theory and $\tilde{u}_n$ is considered as a restricted variation which means $\delta \tilde{u}_n = 0$.

$u_0(t)$ is an initial approximation with possible unknowns. We first determine the Lagrange multiplier $\lambda$ that will be identified optimally via integration by parts. With $\lambda$ determined, then several approximations $u_n(t), n \geq 0$ follow immediately. Consequently, the exact solution may be obtained as:

$$u(t) = \lim_{n \to \infty} u_n(t).$$

(2.3)

The correction functional of the Eq. (2.1) gives several approximations. Therefore, the exact solution can be obtained as the limit of resulting successive approximations.

3. Application of VIM to the Korteweg-de Vries-Burgers equation

To illustrate the capability and reliability of the method, three cases of nonlinear diffusion equations are presented.

Case I: In Eq. (1.1) for $\alpha = 1, \beta = 1, \gamma = 1$ the Korteweg-de Vries-Burgers equation is written as:

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^3} = 0,$$

(3.1)

Subject to a constant initial condition:

$$u(x, 0) = \frac{\sqrt{5}}{6} + \frac{\sqrt{5}(e^x - e^{-x})}{e^x + e^{-x}}.$$  

(3.2)
The correction functional for Eq. (3.1) is in the following form:

\[
\begin{align*}
    u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda(\xi) \left( \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi) \frac{\partial u_n(x,\xi)}{\partial x} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^3 u_n(x,\xi)}{\partial x^3} \right) d\xi,
\end{align*}
\]  

(3.3)

Where \( u_n \) is restricted variation \( \partial u_n = 0 \), \( \lambda \) is a Lagrange multiplier and \( u_0 \) is an initial approximation or trial function.

With the above correction functional stationary we have:

\[
\begin{align*}
    \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \left( \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi) \frac{\partial u_n(x,\xi)}{\partial x} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^3 u_n(x,\xi)}{\partial x^3} \right) d\xi, \\
    \delta u_{n+1}(x,t) &= \delta u_n(x,t) \delta \int_0^t \lambda(\xi) \left( \frac{\partial u_n(x,\xi)}{\partial \xi} \right) d\xi, \\
    \delta u_{n+1}(x,t) &= \delta u_n(x,t) \left( 1 + \lambda(\xi) \right) - \delta \int_0^t \lambda(\xi) u_n(x,\xi) d\xi,
\end{align*}
\]

(3.4)

By using the following stationary conditions:

\[
\begin{align*}
    \delta u_n : & \quad 1 + \lambda(\xi) = 0, \\
    \delta u_n : & \quad \lambda(\xi) = 0,
\end{align*}
\]

(3.5)

(3.6)

This gives the Lagrange multiplier \( \lambda(\xi) = -1 \), therefore the following iteration formula becomes as:

\[
\begin{align*}
    u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left( \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi) \frac{\partial u_n(x,\xi)}{\partial x} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^3 u_n(x,\xi)}{\partial x^3} \right) d\xi,
\end{align*}
\]

(3.7)

We can select \( u_0(x,y) = \frac{\sqrt{6}}{6} + \frac{\sqrt{6}e^{-x-e^{-x}}}{e^x-e^{-x}} \), from the given condition.

Using this selection into the Eq. (3.7) the following successive approximation can be obtained as:

\[
\begin{align*}
    u_0(x,t) &= -\frac{\sqrt{6}(-7e^x + 5e^{-x})}{6(e^x - e^{-x})}, \\
    u_1(x,t) &= -\frac{\sqrt{6}(-7e^x + 5e^{-x})}{6(e^x - e^{-x})} + \int_0^t \left( \frac{\partial u_0(x,\xi)}{\partial \xi} + u_0^2(x,\xi) \frac{\partial u_0(x,\xi)}{\partial x} + \frac{\partial^2 u_0(x,\xi)}{\partial x^2} + \frac{\partial^3 u_0(x,\xi)}{\partial x^3} \right) d\xi
    \\
    &= -\frac{\sqrt{6}(-7e^x + 5e^{-x})}{6(e^x - e^{-x})} - \frac{2}{3} \frac{\sqrt{6}e^{-x}}{(e^x - e^{-x})^4} \left( 61(e^x)^2 - 166e^x e^{-x} + 61(e^{-x})^2 \right) t,
\end{align*}
\]
As a result, the series of the exact solution of the Eq. (3.1) can be constructed as:

\[
\begin{equation}
\frac{u_n(x, t)}{6(e^x - e^{-x})} = -2 \frac{\sqrt{6}e^x e^{-x}}{3(e^x - e^{-x})^2} \left(61(e^x)^2 - 166e^x e^{-x} + 61(e^{-x})^2\right) t
\end{equation}
\]

Using the identity:

\[
\begin{equation}
u(x, t) = \lim_{n \to \infty} u_n(x, t),
\end{equation}
\]

We can write Eq. (3.9) in the closed form as:

\[
\begin{equation}
u(x, t) = \frac{\sqrt{6}}{6} + \frac{\sqrt{6} \left(\frac{e^{-12t} - e^{-x+12t}}{e^{-12t} + e^{-x-12t}}\right)}{e^{-12t} + e^{-x-12t}},
\end{equation}
\]

This is the exact solution of the problem, Eq. (3.1).

**Case II:** In Eq. (1.1) for \(a = 1, \beta = 1, \gamma = 2\) the Korteweg-de Vries-Burgers equation is written as:

\[
\begin{equation}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^3 u}{\partial x^3} = 0,
\end{equation}
\]

Subject to initial condition:

\[
\begin{equation}u(x, 0) = \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}(e^x - e^{-x})}{e^x + e^{-x}},
\end{equation}
\]
The correction functional for Eq. (3.12) is in the following form:

\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left( \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi) \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 2 \frac{\partial^3 u_n(x,\xi)}{\partial x^3} \right) d\xi, \quad (3.14) \]

Where \( u_n \) is restricted variation \( \partial u_n = 0 \) , \( \lambda \) is a Lagrange multiplier and \( u_0 \) is an initial approximation or trial function.

With the above correction functional stationary we have:

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \left( \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi) \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 2 \frac{\partial^3 u_n(x,\xi)}{\partial x^3} \right) d\xi, \]

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) \delta \int_0^t \lambda(\xi) \left( \frac{\partial u_n(x,\xi)}{\partial \xi} \right) d\xi, \quad (3.15) \]

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) (1 + \lambda(\xi)) - \delta \int_0^t \lambda(\xi) u_n(x,\xi) d\xi, \]

By using the following stationary conditions:

\[ \delta u_n : 1 + \lambda(\xi) = 0, \quad (3.16) \]

\[ \delta u_n : \lambda(\xi) = 0, \quad (3.17) \]

This gives the Lagrange multiplier \( \lambda(\xi) = -1 \), therefore the following iteration formula becomes as:

\[ u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left( \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi) \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 2 \frac{\partial^3 u_n(x,\xi)}{\partial x^3} \right) d\xi, \quad (3.18) \]

We can select \( u_0(x,y) = \frac{\sqrt{\pi}}{2} + \frac{2\sqrt{3}(e^{x} - e^{-x})}{e^{x} - e^{-x}}, \) from the given condition.

Using this selection into the Eq. (3.18) the following successive approximation can be obtained as:

\[ u_0(x,t) = -\frac{\sqrt{3}(-5e^x + 3e^{-x})}{2(e^x - e^{-x})}, \]

\[ u_1(x,t) = -\frac{\sqrt{3}(-5e^x + 3e^{-x})}{2(e^x - e^{-x})} - \int_0^t \left( \frac{\partial u_0(x,\xi)}{\partial \xi} + u_0^2(x,\xi) \frac{\partial^2 u_0(x,\xi)}{\partial x^2} + 2 \frac{\partial^3 u_0(x,\xi)}{\partial x^3} \right) d\xi \]

\[ = -\frac{\sqrt{3}(-5e^x + 3e^{-x})}{2(e^x - e^{-x})} - \frac{2\sqrt{3}}{(e^x - e^{-x})^2} \left( 99(e^x)^2 - 218e^x e^{-x} + 67(e^{-x})^2 \right)t, \]

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As a result, the series of the exact solution of the Eq. (3.1) can be constructed as:

\[
\begin{align*}
  u_n(x, t) &= -\frac{\sqrt{3}(-5e^x + 3e^{-x})}{2(e^x - e^{-x})} - \frac{2\sqrt{3}}{(e^x - e^{-x})^4}(99(e^x)^2 - 218e^xe^{-x} + 67(e^{-x})^2)t \\
  &\quad - \frac{\sqrt{3}}{2}(e^x e^{-x})^2 (9801(e^x)^5 - 508549(e^x)^4(e^{-x}) + 797906(e^x)^3(e^{-x})^2) \\
  &\quad - 742098(e^x)^2(e^{-x})^3 + 154085(e^x)(e^{-x})^4 - 4489(e^{-x})^5t^2 \\
  &\quad + \frac{8\sqrt{3}(e^x)^2(e^{-x})^3}{(e^x - e^{-x})^5}(99(e^x)^2 - 218e^xe^{-x} + 67(e^{-x})^2)(495(e^x)^4 - 4358(e^x)^3(e^{-x})^3 \\
  &\quad + 6256(e^x)^2(e^{-x})^2 - 2514(e^x)(e^{-x})^3 + 201(e^{-x})^4)t^3 \\
  &\quad - \frac{12\sqrt{3}(e^x)^2(e^{-x})^3}{(e^x - e^{-x})^{113}}((99(e^x)^3 - 733(e^x)^2(e^{-x}) + 637(e^{-x})^2)^2 \\
  &\quad - 67(e^{-x})^3)(99(e^x)^2 - 218e^xe^{-x} + 67(e^{-x})^2)^2)t^4, \\
  &= \cdots 
\end{align*}
\] (3.19)

Using the identity:

\[
u(x, t) = \lim_{\alpha \to \infty} u_n(x, t),
\] (3.21)

We can write Eq. (3.20) in the closed form as:

\[
\begin{align*}
  \nu(x, t) &= \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}(e^{x \frac{4\gamma}{12}} - e^{-x \frac{4\gamma}{12}})}{e^{x \frac{4\gamma}{12}} + e^{-x \frac{4\gamma}{12}}},
\end{align*}
\] (3.22)

This is the exact solution of the problem, Eq. (3.12).

The exact solution of Eq. (1.1) is presented by Zheng et al. [12] as:

\[
u(x, t) = \pm \frac{\alpha}{\sqrt{6\alpha \gamma}} \pm \frac{6\gamma}{\beta} \lambda \tanh \left[ \lambda \left( x - \frac{12\gamma \lambda^2 + \beta^2}{6\gamma} t + C_0 \right) \right], \quad \alpha \gamma > 0,
\] (3.23)

Where \( \lambda \) and \( C_0 \) are constant.
This is in full agreement of the closed form solutions of Eq. (3.1) and Eq. (3.12) obtained by the VIM for selected value of parameters $\alpha, \beta, \gamma$.

4. Conclusion

In the present research work, the exact solution of the Korteweg-de Vries-Burgers nonlinear diffusion equation is obtained by using the VIM. The validity and effectiveness of the VIM is shown by solving two non-homogenous non-linear differential equations. So, great potential of the method in solving the Korteweg-de Vries-Burgers equation is proved by presenting the convergence of sequences obtained by using the VIM towards the exact solution. Moreover, applying the VIM to the the Korteweg-de Vries-Burgers equation can indicate the capability and efficiency of the method in obtaining the exact solution of linear and nonlinear differential equations.

References