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Quantum Number Definition Problems

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Abstract

This study discusses scatters and stationary states of nonrelativistic quantum theory. Among other statements, the author shows that if \( V(r) = -a/r^l + b/r^{l-1} + \ldots + z/r + \alpha \beta + r\gamma + \ldots + \omega r \) n-dimensional "spherically symmetric" potentials, \( a, b, \ldots, z \in \mathbb{R}, a > 0 \) and \( \alpha, \beta, \ldots, \omega \in \mathbb{R} \), as well as \( i, j \in \mathbb{N} \cup \{0\} \) and \( i \geq 1 \), then the radial wave function \( \psi(r) \) has an infinite number of zeros in the interval \( 0 < r < b \); moreover, the amplitude of \( \psi \) is a strictly monotonous increasing function of \( r \).

Key words: investigation of the \( n \)-dimensional radial Schrödinger equation with the help of a qualitative theory, Sturm-Liouville form of the \( n \)-dimensional Schrödinger equation, Schrödinger operator and self-adjointness, basic theorems and definitions, physical examples and applications, new oscillation theorem and its proof, new stability theorem and its proof, example of stability theorem, application of theorems and corollaries for time-symmetric gravitational waves

1. INTRODUCTION

The central problem of the S-matrix theory of scattering processes\(^1\), and eigenstates\(^2\) is the solution of the \( n \)-dimensional radial Schrödinger equation\(^2\):

\[
\frac{d^2 \psi}{dr^2} + \frac{n-1}{r} \frac{d \psi}{dr} + \left[ \frac{2mE}{\hbar^2} - \frac{2m}{r^2} U(r) \right] \psi = 0. \tag{1}
\]

Generally, Eq. (1) is insolvable analytically. Therefore, (1) is solved by the Born approximation, the Wentzel-Kramers-Brillouin approximation (WKB method), etc.,\(^4\) and the approximate solution \( \psi_a \) is connected with the general solution by the \( Y(r, \cdots, \omega) \) function: \( \Psi(r, \theta, \ldots, \omega) = \psi_a(r) Y(\theta, \ldots, \omega) \).

The Born approximation can be used for high-energy particles only. Then, nearly everywhere in the scattering domain, the \( U(r) \) potential is small considering energy \( E \) of a particle, and the amplitude of the scattered wave is small considering the incident plane wave. The quasi-classical WKB approximation can be used for slowly changing \( U(r) \) potentials only:

\[
(mh/p)^3 |dU/dr| < 1. \tag{2}
\]

In a case of the frequently occurring \( U(r) = \alpha r^{-\beta}, \beta > 0 \), the power potential (2) has the form

\[
r \gg ((\alpha \beta / p)^{1/(\beta - 1)} \neq 0. \tag{3}
\]

It can be seen from (3) that in the case of \( U(r) = \alpha r^{-\beta} \), the solutions of Eq. 1 cannot be WKB approximated about \( r = 0 \). Furthermore, it can be seen from (2) that in the case of \( p \to 0 \) the approximation is not valid. The approximation is also not valid at the turning point \( E = U(r) \), so the WKB wave functions produce the approximate solution \( \psi_a \) of the Schrödinger equation only away from the turning points.

The approximate solutions are essential, but the determinations are difficult and they depend on \( U(r) \). In connection with the Schrödinger equation (1) this question is raised: Can the general nature of the solutions of (1) be investigated without knowledge of the analytical solutions with the help of qualitative theory so that the results are true with respect to classes of fast-changing potentials?

2. QUALITATIVE INVESTIGATION OF EQ. (1)

The Schrödinger equation (1) is a second-order linear differential equation with the form

\[
q'' + p(r)q' + q(r) = 0 \quad (a \leq r < \infty). \tag{4}
\]

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The qualitative theory of the second-order linear equations attempts to answer questions such as the following: Is a solution always bounded? Are there functions bounding a solution from above, or is a solution monotone from below? Does a solution have a finite or infinite number of zeros, or is a solution stable?

Not enough has been made of the results of the qualitative theory of the second-order linear equations for Eq. (1).

Equation (4) has the trivial solution \( \varphi(r) \equiv 0 \) for \( a \leq r < \infty \). Further, we exclude this trivial solution consideration. One of the oldest results on the zeros of Eq. (4) is Sturm's separation theorem.\(^{(5),(6)}\)

**Theorem 1:** If \( \varphi_1 \) and \( \varphi_2 \) are linearly independent solutions of Eq. (4), then \( \varphi_2 \) has exactly one zero between any two successive zeros of \( \varphi_1 \).

It follows that if Eq. (4) has one solution that has infinitely many zeros, then so does every solution of the equation.

**Definition 1** (oscillatory equation): Equation 4 is said to be “oscillatory” if every solution has infinitely many zeros. Otherwise, it is said to be “nonoscillatory.”\(^{(5),(6)}\)

The oscillatory equation can be defined in other ways. For example, if a solution of Eq. (4) does not have more than one zero in \((a, b)\), then this solution is nonoscillatory in this interval; otherwise, this solution is oscillatory.\(^{(7)}\)

Again, using the Sturm separation theorem we can deduce that if Eq. (4) is nonoscillatory, then every solution has at most one zero in \( \alpha \leq r < \infty \), for some \( a \geq \alpha \).

The radial Schrödinger equation (1) with the Liouville transformation \( \varphi = \psi e^{-1/2 \int ((n-1)/r) dr} \) leads to the Sturm-Liouville form:

\[
\psi'' + Q(r)\psi = 0,
\]

\[
Q(r) = \eta + \sum_{k=0}^{n-1} \frac{(n-1)/2 - (n-1)/2}{r^2} - V(r).
\]

Because of the Liouville transformation \( \varphi = \psi e^{-1/2 \int ((n-1)/r) dr} \), the number of zeros is not influenced by multiplying (or dividing) by \( e^{1/2 \int ((n-1)/r) dr} \) and thus is the same for Eq. (5) and Eq. (4). In (5) \( \psi : \mathbb{R} \to \mathbb{R}, \psi \in \mathcal{H} \neq \{0\} \), \( \mathcal{H} = L^2(\mathbb{R}) \) is a separating Hilbert space, \( \eta = (\mathcal{N} \psi, \psi)_{\mathcal{H}} \) is the coordinate of center of mass, \( \eta \in \mathbb{R} \), \( \mathcal{N} \psi, \psi \) is the \( i \)th eigenvalue \( (k_i = 0) \) of the \( n \) dimensional Laplace-Beltrami operator\(^{(2)}\) and \( V: \mathbb{R}^n \to \mathbb{R} \) is a continuous, complex valued function. In scattering processes \( \eta \geq 0 \); in fixed states \( \eta < 0 \) and \( V(r) < 0 \). In [5] and below, differentiation with respect to \( r \) is denoted by the prime symbol.

**Definition 2** (spherically symmetric potential): The \( V \) potential is spherically symmetric if it satisfies the following conditions:

1. \( V(z) \) is analytic in the half-plane \( \text{Re } z > 0 \).
2. Along any ray \( z = x \exp[i\theta], \theta < \pi/2 \),
   \[
   \int_{z_0}^{\infty} \exp(i\theta z) \cdot V(z) \, dz
   \]
   converges for each \( x_0 > 0 \) and \( r < \mu \) (range of potential), and

\[
(3) \int_{\mu}^{\infty} r |V(r)| dr < \infty;
\]

i.e., the first absolute moment of \( V \) is finite, or

\[
(4) \lim_{r \to \infty} r^2 V(r) = V_+ > 0
\]

in such a way that

\[
\int_{\mu}^{\infty} r \left| V(r) - \frac{V_+}{r^2} \right| dr < \infty,
\]

or

\[
(5) \int_{\mu}^{\infty} \frac{1}{r^2} |V(r)|^{1/2} dr < \infty, \text{ or } \int_{\mu}^{\infty} r^2 |V(r)| dr < \infty,
\]

for any \( c > 0 \).

Potentials satisfying 1), 2), and 3) are called “regular potentials,” those satisfying 1), 2), and 4) are called “centrifugal potentials,” and those satisfying 1), 2), and 5) are called “singular potentials.”\(^{(4)}\) Further, if the potential is real and positive (or negative) in some interval \([0, \infty)\), then it is called “repulsive” (or “attractive”). Below there are \( V: \mathbb{R}^n \to \mathbb{R} \) real spherically symmetric potentials only.

From the point of view of oscillatory behavior, it is important to answer this question: What sort of spherically symmetric \( V(r) \) potentials can be in Eqs. (5)? One of the fundamental postulates of quantum theory is that self-adjoint operators correspond to physical quantities (energy, impulse, etc.). A densely defined operator \( H \) on an \( \mathcal{H} \) Hilbert space is called “symmetric” (or “Hermitian”) if \( H \subset H^* \); that is, if \( \mathcal{D}(H) \subset \mathcal{D}(H^*) \) and \( H_\eta = H^*_\eta \) for all \( \eta \in \mathcal{D}(H) \). Equivalently, \( H \) is symmetric if and only if \( (H_\eta, \psi) = (\psi, H_\eta) \) for all \( \eta, \psi \in \mathcal{D}(H) \). The operator \( H \) is called “self-adjoint” if \( H = H^* \); that is, if and only if \( H \) is symmetric and \( \mathcal{D}(H) = \mathcal{D}(H^*) \). The spectrum \( \sigma(H) \) of self-adjoint operator \( H \) is on the real axis: \( \sigma(H) \subset \mathbb{R} \). In the perturbation theory for linear operators\(^{(3)}\) the fundamental postulate is extended in such a way that instead of self-adjoint operators, “essentially self-adjoint” operators are used. A symmetric operator \( H \) is called “essentially self-adjoint” if its closure \( \overline{H} \) is self-adjoint. Equivalently, \( H \) is essentially self-adjoint if and only if \( H^* = H^{**} \); that is, if \( \mathcal{D}(H) \subset \mathcal{D}(H^{**}) = \mathcal{D}(H^*) \). Because a self-adjoint operator is always closed, the symmetric operators can always become closed.\(^{(8)}\) If essentially self-adjoint operators are used, then the property \( \sigma(H) \subset \mathbb{R} \) is retained.

With the above notation Eq. (1) can be written in the form

\[
[-\Delta + V(r)]\psi = \lambda \psi,
\]

in which \( \Delta \) is an \( n \)-dimensional Laplace operator in polar coordinates, \( H : = -\Delta + V(r) \) is an \( n \)-dimensional radial Schrödinger operator, and \( H_\eta = H^*_\eta \) is \( \eta \in \sigma(H) \). There are many important results for self-adjointness of Schrödinger operators.\(^{(2),(3)}\)

One of the most general theorems for spherically symmetric potentials and essentially self-adjoint operators follows.

**Theorem 2:** Let \( V(r) \) be a continuous symmetric potential on \( \mathbb{R}^n - \{0\} \).

If \( V(r) \) satisfies the inequality

\[
V(r) + \frac{[(n-1)(n-3)/4](1/r^2)}{3/4r^2},
\]

(7)
then \( -\Delta + V(r) \) is essentially self-adjoint on \( C_c^\infty(R^n - \{0\}) \). If \( V(r) \) satisfies the inequality
\[
0 \leq V(r) + \frac{\{(n - 1)(n - 3)/4\}(1/r^2)}{c} \leq 3/4, \quad c < 3/4,
\]
then \( -\Delta + V(r) \) is not essentially self-adjoint on \( C_c^\infty(R^n - \{0\}) \).

It is clear that even if an \( n \)-dimensional spherically symmetric potential \( V(r) \) does not satisfy inequality (7), \( H \) can be essentially self-adjoint. (Note that, independently of the dimension number, the potential \( V(r) = (2m\hbar^2)c^2/r^2 \) for the hydrogen atom does satisfy inequality (7).) However, potentials \( V(r) \) satisfying (8) have to be excluded from consideration. [Obviously, \( \psi \in C_c^\infty(R^n - \{0\}) \).]

If in the case \( r \to \infty \), \( V(r) \) tends to zero, then the asymptotic form of Eq. (5) is the following:
\[
\psi'' + \frac{n}{r} \psi = 0 \quad (0 \leq r < \infty).
\]
In scattering processes \( \eta = |\eta| \), so (9) has the general solution
\[
\psi(r) = \alpha \cos(\sqrt{\eta} r + \beta),
\]
a function with infinitely many zeros, which remains bounded as \( r \) tends to infinity. In eigenstates \( \eta = -|\eta| \), so (9) has the general solution
\[
\psi(r) = \alpha_1 \exp(\sqrt{|\eta|} r) + \alpha_2 \exp(-\sqrt{|\eta|} r),
\]
a function with at most one zero. If \( \alpha_1 = 0 \) in (11), then \( \psi \in L^2(R^n) \).

It is obvious from Definition 1 that in the case \( \eta = |\eta| \), Eq. (9) is oscillatory, and in the case \( \eta = -|\eta| \), Eq. (9) is not oscillatory. Roughly speaking, if \( Q(r) \) is always positive, we should expect bounded oscillating solutions, and if \( Q(r) \) is always negative, we should expect monotone solutions, some of which may be unbounded.

**Theorem 3:** If \( Q(r) \leq 0 \) for \( r \geq a \), then Eq. (5) is nonoscillatory and every solution has at most one zero in \( a < r < \infty \).

Example 1: The properties 1), 2), and 5) of real singular potentials of Definition 2 are restricted by
\[
\int_a^\infty r |V(r)|dr < \infty \quad \text{and} \quad \int_a^\infty r^2 |V(r)|dr < \infty
\]
for any given \( a, b > 0 \). (These two conditions ensure that the phase shift \( \delta \) is defined for \( \eta = 0 \).) Aly et al. \(^\text{(1)}\) considered the three-dimensional radial Schrödinger equation (5) in the case of repulsive potentials
\[
V(r) = g(e^{2r} + r^2)/r^4, \quad g > 0, \quad v_1 > 0
\]
satisfying (12) and having essential singularity. [At this time (13) does not satisfy inequality (8).] For \( \eta = 0 \), \( n = 1 \), or \( n = 3 \) and \( k_1 = 0 \) (s waves), Eq. (5) has the form
\[
\psi'' - \frac{\gamma_1^2 \psi}{r} = 0.
\]
Since \( Q(r) \leq 0 \) for \( r > 0 \), (14) is not oscillatory and every solution has at one zero in \( 0 < r < \infty \).

If \( Q(r) > 0 \) for \( r > a \) we expect Eq. (5) to be oscillatory. However, this is not always so, as the following example shows.

Example 2: In the case of \( \eta \neq 0 \) the potential \( V(r) = \eta + \frac{k_1}{r} + \frac{(n - 1/2 - (n - 1)/2)}{3} - k\) does not satisfy inequality (8). [For example, \( V(r) \) is centrifugal if \( \eta = 0, \quad k_1 = 0, \quad \text{and} \quad n = 2 \).] Then Eq. (5) has this form:
\[
\psi'' + \frac{(k/r^2)\psi}{r} = 0 \quad (0 < r < \infty).
\]

The Euler equation (15) is oscillatory if \( k > 1/4 \) and nonoscillatory if \( k < 1/4 \). If \( \omega = (k^2 - 1/4)^{1/2} \), the general solution \( \psi(k, r) \) is
\[
\gamma_1 \sqrt{r} \cos(\omega r + \beta), \quad \text{if} \quad k > 1/4,
\]
\[
\gamma_1 \sqrt{r} \cos(\omega r - \beta), \quad \text{if} \quad k < 1/4,
\]
\[
\sqrt{r} \gamma_1 + \gamma_2 \log r, \quad \text{if} \quad k = 1/4.
\]

We can see from (16), (17), and (18) that the solutions of (15) are unbounded in \( 0 < r < \infty \). The next theorem generalizes the properties of solutions (17) and (18).

**Theorem 4:** Every nonoscillatory equation (5) has two solutions, \( u \) and \( v \) such that
\[
\int_0^\infty \psi u^2 \psi dr < \infty \quad \text{and} \quad \int_0^\infty \psi u\psi^2 dr = \infty.
\]

**Definition 3 (large and small solutions):** The \( u(r) \) solution is called the "large solution," and the \( v(r) \) solution is called the "small solution" of the nonoscillatory Eq. (5), if \( \lim_{r \to 0} v(r)/u(r) = 0 \).

Example 3: The case of \( V(r) = \eta + \frac{k_1}{r} + \frac{(n - 1/2 - (n - 1)/2)}{3} - k\) is interesting. Then Eq. (5) has the form:
\[
\psi'' = \frac{\gamma_1}{r} \psi + \frac{\gamma_2}{r} \psi + \frac{\gamma_3}{r^2} \psi = 0.
\]
"Small" and "large" solutions of (19) are
\[
v(r) = 1, \quad u(r) = r.
\]

Remark 1: In eigenstates \( \eta = -|\eta|, \quad V(r) < 0 \), and \( k_1 = 0 \). Let \( V(r) \) be a potential that does not satisfy the inequality (8).

\[
\psi'' + \frac{\gamma_1}{r} \psi + \frac{\gamma_2}{r} \psi + \frac{\gamma_3}{r^2} \psi = 0
\]
is given for Eq. (5) if we investigate such states in two-dimension. If in the case \( r \to \infty \), \( V(r) \) tends to zero, then the asymptotic form and solution of (21) is equivalent to (9) and (11). Let \( V(r) \) be a potential. If \( V(r) \sim r^{-\lambda} \) with \( 0 < \lambda < 2 \), then in the case of \( r \to 0 \) the asymptotic form and solution of (21) is equivalent to (15) and (18). In (11) the boundary condition \( \lim_{r \to 0} \psi(r) = 0 \) is equivalent to choosing \( \alpha_1 = 0 \). If \( \alpha_1 = 0 \), then the integration constants \( \gamma_1 \) and \( \gamma_2 \) cannot be arbitrary (18). That is, choosing \( \alpha_1 = 0 \) according to the general solution of (21) determines \( \gamma_1, \gamma_2 \) as well as the relation of the small solution \( u(r) \) and the large solution \( v(r) \). It is necessary to the regularity of the general solution of (21) that \( \psi \in L^2(R^n) \) in (18). This condition is realized because
\[
\int_0^\infty \gamma_1^2 r^2 + 2\gamma_1 \gamma_2 r^2 \log r + \gamma_2^2 r^2 \log^2 r dr < \infty.
\]
The graph of solution (18) can be seen in Fig. 1. We can see from Fig. 1 and (18) that in this case, independently from \( \eta \) and \( V(r) \), the general solution of (21) has at least two zeros: one at zero, the other at infinity.
atom in s state, too. The number of zeros $N$ of the wave function $u(r)$ in stationary states of three-dimensional hydrogen atoms is called the “principal quantum number”; then $E = -(m^2e^4/2h^2N^2)$, and $n$ depends on the principal quantum number: $n = -(m^2e^4/2h^2N^2)$.

It can be seen from (5) that using the qualitative method is advantageous in the cases of dimensions $n \neq 1$ and $n \neq 3$, since Sommerfeld’s polynomial method is unusable; i.e., only integer powers of $r$ can occur in the polynomial method, but in the cases $n \neq 1$ and $n \neq 3$, $Q(r) = (n - 1)((n - 2)/2) - (n - 1)/2$ generally is not equal to $-(l(l + 1))$, and so $Q$ cannot depend explicitly on the principal quantum number $N$. (In the cases $n \neq 1$ and $n \neq 3$, the well-known connections of the principal quantum number $N$, the secondary quantum number $l$, and the magnetic quantum number $m$ are generally invalid.) They can play a part, for example, in some “grand unified theories” or in the five-dimensional unified theory of gravitation and electromagnetic interaction or in the projective field theory in the case of “quanting the fields” as well as in the theory of strong gravitational waves and weak time-symmetric gravitational waves, and in some applications of “surface physics.”

Example 2 suggests that if Eq. (5) is oscillatory, then $Q(r)$ must not tend to zero too rapidly as $r$ tends to infinity. One such criterion is used in the next theorem.

**Theorem 5 (Wintner):** If $Q(r) > 0$ for $r \geq a$ and if $\int_a^\infty Q(r)dr = 0$, then Eq. (5) is oscillatory.\(^{(5)}\)

Example 4: The potentials $V(r) = \eta + \{k_j + [(n - 1)/2] - \[(n - 1)/2\]^2\}/r^2 - a^2r$ for $a, \alpha > 0$ and $V(r) = \eta + \{k_j + [(n - 1)/2] - \[(n - 1)/2\]^2\}/r^2 - b^2r - \beta^2$ for $b > 0, 0 < \beta < 1$ do not satisfy inequality (8). Then Eq. (5) has the following forms:

$$\psi'' + ar^2\psi = 0 \quad (0 < r < \infty) \quad \text{and} \quad \psi'' + br^2\psi = 0 \quad (0 < r < \infty). \quad (23)$$

Both equations of (23) are oscillatory since both

$$a\int_0^\infty r^2dr = a((1/\alpha + 1)r^3 + 1)_{r=0}^\infty \quad \text{and} \quad b\int_0^\infty r^2dr = b((1/\beta)r^3 - 1)_{r=0}^\infty \quad (24)$$

tend to infinity with $r$.

**Theorem 6 (Sonin-Pólya):** Let $\psi = \psi(r)$ satisfy the differential Eq. (5), where $Q(r)$ is a positive function having a continuous derivative of a constant sign in $a < r < b$, and $Q(r)'$ as $r$ increases from $a$ to $b$, from an increasing or decreasing sequence according as $Q(r)$ decreases or increases.\(^{(14)}\)

Example 5: Wu discussed\(^{(15)}\) scattering processes by a potential field with a branch point type singularity

$$V(r) = -g|ln r/\alpha_1|, \quad g > 0. \quad (25)$$

Equation (25) satisfies (12) but does not satisfy inequality (8). Equation (5) has the form

$$\psi'' + [\eta] + g|ln r/\alpha_1|\psi = 0 \quad (26)$$

considering $s$ waves (where $k_j = 0$) in the $n = 1$ or $n = 3$ dimensions.

Conditions of Theorem 6 are true if $1 < r < \infty$. Discussing the function, it can be seen that $Q(r) = |\eta| + g|ln r/\alpha_1|$ has its maximum at $r = e^{1/4}$. In the case of sufficiently large $|\eta|$, successive maxima of $|\psi|$ strictly monotonously decrease if $1 < r < e^{1/4}$ and strictly monotonously increase if $e^{1/4} < r < \infty$.

There remains the case $Q(r) > 0$ for $r \geq a$ and $\int_a^\infty Q(r)dr < \infty$. The results of Example 2 can be extended to a wide class of equations by using Sturm’s comparison theorem.\(^{(5)}\)

**Theorem 7:** Let $\psi_1$ and $\psi_2$ be solutions of the equations

$$\psi''_1 + Q_1(r)\psi_1 = 0 \quad (27)$$

and

$$\psi''_2 + Q_2(r)\psi_2 = 0, \quad (28)$$

respectively. If $Q_1$ and $Q_2$ are continuous functions and if

$$Q_2(r) \geq Q_1(r) > 0 \quad (29)$$

for $a \leq r \leq b$, then $\psi_2$ has at least one zero between any two successive zeros of $\psi_1$ in $a \leq r \leq b$.

**Corollary 1:** If Sturm’s minorant equation (27) is oscillatory in $a \leq r < \infty$ and if inequalities (29) hold for $a \leq r < \infty$, then Sturm’s majorant equation (28) is oscillatory.

Example 6: In the case of central collisions $\eta = |\eta|$ and $k_j = 0$. The power potential $V(r) = V_{1}(r) + V_{2}(r) = -(2m_{e}h^{2}(c^{2}r^{2}) + G(m_{0}/r)^{2} = -a/r, \quad a > 0$ in the case of scatterings of proton-electron, etc. If we investigate scattering processes in two-dimension, the equation

$$\psi'' + [\eta] + (1/4r^{2}) + a/r\psi = 0 \quad (0 < r < \infty) \quad (30)$$

describes $s$ waves instead of (5). Fischbach’s potential\(^{(16)}\) of “fifth force” has the form $V_{5}(r) = -G(m_{0}/r^{2}) \cdot \exp(-r/\lambda), \quad \lambda = (7.2 \pm 3.6) \times 10^{-3}, \quad \lambda = 200 \pm 50$ m. Considering this potential among colliding particles we get $V_{5}(r) = (2m_{e}h^{2}[G(m_{0}/r^{2})\exp(-r/\lambda)] = \epsilon(r)/r$, where $\epsilon(r) > 0$. Then the particles in the potential field $V_{5}(r) = V_{2}(r) + V_{5}(r)$ move according to equation of state

$$\psi''_1 + \{\eta\} + (1/4r^{2}) + a/r\psi_1 = 0 \quad (0 < r < \infty). \quad (31)$$
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[Naturally, $V(r)$ does not satisfy inequality (8).] Since $|\dot{\psi}(r)| = \frac{\dot{\psi}(r)}{r} < \alpha(r)$ for all $0 < r < \infty$, we can see on the basis (30) and (31) that $Q_2(r) = Q_1(r) + \frac{\alpha(r)}{r}$, and so because of Sturm’s separation theorem, $\psi_2$ has at least one zero between any two successive zeros of $\psi_1$ in $0 < r < \infty$. Because Wintner’s theorem, (31) is oscillatory and (29) is valid in $0 < r < \infty$, so (30) is oscillatory too. Both $Q_1(r)$ and $Q_2(r)$ are continuously differentiable and strictly monotonically decreasing on $0 < r < \infty$, successively maximums of $|\psi_1|$ and $|\psi_2|$ are strictly monotonically increasing on $0 < r < \infty$.

Combining Corollary 1 with Example 2, we can state a useful test for oscillatory behavior.

**Theorem 8:** If

$$Q(r) > k/r^2 \ (a \leq r < \infty) \quad (32)$$

with $k > 1/4$, then Eq. (5) is oscillatory. If

$$Q(r) < 1/(4r^2) \ (a \leq r < \infty), \quad (33)$$

then Eq. (5) is nonoscillatory. (5)

**Corollary 2:** The potentials $V(r) \leq \eta + \left(\kappa + \left((n - 1)/2\right)/r^2 - k/r^2\right)$ with $k > 1/4$ and $V(r) > \eta + \left(\kappa + \left((n - 1)/2\right)/r^2 - 1/4r^2\right)$ in the case $\eta > 0$ do not satisfy inequality (8). These potentials can be regulars, singulars, centrifugals, etc. Equation (5) is oscillatory in $0 < r < \infty$ with potentials given in the first inequality and is nonoscillatory with potentials given in the second one.

By considering special cases of Eqs. (27) and (28) in Sturm’s comparison theorem, we can deduce interesting estimates of the distance between successive zeros of a solution.

**Theorem 9:** If $M \geq Q(r) > 0$ for $a \leq r < \infty$, then the distance between successive zeros of a solution of Eq. (5) is at least $(\pi/\sqrt{M})$, and if $Q(r) \geq m > 0$ for $a \leq r < \infty$, then the distance is at most $(\pi/\sqrt{m})$. (5)

Example 7: Considering scattering processes in which $\pi = 2$, $\kappa < 0$, and the power potential $V(r) = -a/r, a > 0$, Eq. (30) has the form

$$\psi'' + \left(|\psi| + \kappa + (n - 1)/2\right) - \left((n - 1)/2\right)/r^2 + a/r\psi = 0. \quad (34)$$

[Naturally, $V(r)$ does not satisfy inequality (8).] If, for example, $a = 2m^2/h^2$, then (34) describes the proton-electron scatterings. Let $(-b) = \kappa + \left((n - 1)/2\right)/r^2 - (n - 1)/2$, where $b > 0$, then $Q(r) = \eta - (b/r^2) + a/r$. The graph of $Q(r)$ can be seen in Fig. 2. Because of Theorem 9, the distance between two successive zeros of a solution of (34) is at least $[\pi/\eta] + (2\pi/4b^2)$ and at most $[\pi/\eta]$ in $2b/a \leq r < \infty$.

**Theorem 10** [Liapunov’s inequality]: Let $Q(r)$ be real valued and continuous on $a \leq r \leq b$. A necessary condition for (5) to have a solution $\psi(r) \neq 0$ possessing two zeros is that

$$\int_a^b Q(r)dr > 4/(b - a). \quad (35)$$

Theorem 10 is developed in the next statement.

**Theorem 11** (Galbraith): Let $r_1$ and $r_2$ be two zeros of $Q(r)$ and $Q(r) \geq 0$ if $r_1 < r_2$. Further, let $r_1 \leq a < b \leq r_2$ and $Q(r)$ concave on $[a, b]$. If there exist $T_1$ and $T_2$ such that $a \leq T_1 < T_2 \leq b$ and

$$(T_2 - T_1) \int_{T_1}^{T_2} Q(r)dr > \pi^2, \quad (36)$$

then any solutions $\psi$ of Eq. (5) have at least two zeros in $T_1 \leq r \leq T_2$. (17)

Example 8: In the case of attractive power potentials $V(r) = -a/r, a > 0$ and for fixed (eigen) states $n \neq 2, \kappa < 0$, the state Eq. (5) has the following form:

$$\psi'' + \left(|\psi| + \kappa + \left((n - 1)/2\right)/r^2 - 1/4r^2\right) + a/r\psi = 0. \quad (37)$$

Example 7: Considering scattering processes in which $\pi = 2, \kappa < 0$ and the power potential $V(r) = -a/r, a > 0$, Eq. (30) has the form

$$\psi'' + \left(|\psi| + \kappa + (n - 1)/2\right) - \left((n - 1)/2\right)/r^2 + a/r\psi = 0. \quad (34)$$

[Naturally, $V(r)$ does not satisfy inequality (8).] If, for example, $a = 2m^2/h^2$, then (34) describes the proton-electron scatterings. Let $(-b) = \kappa + \left((n - 1)/2\right)/r^2 - (n - 1)/2$, where $b > 0$, then $Q(r) = \eta - (b/r^2) + a/r$. The graph of $Q(r)$ can be seen in Fig. 2. Because of Theorem 9, the distance between two successive zeros of a solution of (34) is at least $[\pi/\eta] + (2\pi/4b^2)$ and at most $[\pi/\eta]$ in $2b/a \leq r < \infty$.

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$$(T_2 - T_1) \int_{T_1}^{T_2} Q(r)dr > \pi^2, \quad (36)$$

then any solutions $\psi$ of Eq. (5) have at least two zeros in $T_1 \leq r \leq T_2$. (17)

Example 8: In the case of attractive power potentials $V(r) = -a/r, a > 0$ and for fixed (eigen) states $n \neq 2, \kappa < 0$, the state Eq. (5) has the following form:

$$\psi'' + \left(|\psi| + \kappa + \left((n - 1)/2\right)/r^2 - 1/4r^2\right) + a/r\psi = 0. \quad (37)$$
(For example, in the hydrogen atom $a = 2me^2/k^2$.) With the notation 
$\langle -b; = n(1-2) - [(n-1)/2]^2, b > 0, Q(r) = -|b| - b/r^2 + a/r$. A graph of $Q(r)$ can be seen in Fig. 3. The graph $Q(r)$ is concave if 
$|b| < \sqrt{\lambda}/a$. For $\lambda = 1$, there exists $T_1 < T_2 < \sqrt{\lambda}/a$, such that (36) is true, so 
any $\psi$ solution of (37) has at least two roots in $T_1 \leq r \leq T_2$.

In Examples 2 and 3 we gave examples of oscillatory equations with an 
unbounded solution. The next few theorems give conditions that ensure 
that various types of oscillatory equations have only bounded solutions 
with 
$$|\psi(r)| < K < \infty \quad (a < r < \infty). \quad (38)$$

[The more precise bounds of the form $f(r) \leq \psi(r) \leq g(r) \quad (a < r < b)$ 
can be found in Refs. 5 and 18.] Because of the nature of the Liouville 
transformation, the boundedness is not influenced by multiplication (or 
division) by the function $r^{1-n/2}$ in the interval $0 < r < \infty$ and thus is 
the same for Eq. (5) and Eq. (4).

**Theorem 12:** If 
$$0 < m < Q(r) < M < \infty \quad (a < r < \infty) \quad (39)$$
and $Q$ has a continuous second derivative with $\int_0^\infty |Q''(r)|dr < \infty$, then 
all solutions of Eq. (5), together with their first derivatives, are bounded 
for the interval $0 < r < \infty$.\(^\text{(5)}\)

Example 9: See (26) in $\exp(1/4) \leq r < \infty$ and (34) in $2b/a < r < \infty$, etc.

Results have been obtained for equations with unbounded functions $Q$. 
We give one such theorem.

**Theorem 13:** If $Q(r)$ is nondecreasing and if $Q(r)$ tends to infinity with $r$, 
then every solution of Eq. (5) is bounded, and at least one solution tends to 
zero as $r$ tends to infinity.\(^\text{(5)}\)

Example 10: See the first equation of (23) in Example 4.

Many other equations can be dealt with by using the following 
comparison theorem, which says that a small $V_p(r)$ perturbation of an 
equation does not destroy the boundedness of solutions.

**Theorem 14:** If every solution of Eq. (5) is bounded, and if $\int_0^\infty |V_p(r)|dr < \infty$, 
then every solution of 
$$\psi'' + [Q(r) + V_p(r)]\psi = 0 \quad (a < r < \infty) \quad (40)$$
is bounded.\(^\text{(5)}\)

Example 11: In the case $a, b \geq 1$, the singular potentials given in (12) 
satisfy the above theorem. [Naturally, they cannot satisfy inequality (8).]

The next theorem applies when $Q(r)$ is an approximate positive 
constant for large values of $r$.

**Theorem 15** (Kakutani): Let
$$\psi'' + [m + Q(r)]\psi = 0 \quad (a \leq r < \infty), \quad (41)$$
where $m > 0$. If 
$$\int_0^\infty |Q(r)|dr < \infty, \quad (42)$$
then the general solution of (41) is
$$\psi = c_1 \sin(\sqrt{mr} + c(r)), \quad (43)$$
where $\lim_{r \to \infty} c(r) = 0$.\(^\text{(19)}\)

Example 12: In the case $a > 0$, Eq. (26) satisfies the conditions of 
Kakutani’s theorem, because $\int_0^\infty \ln r|f(r)|dr < \infty$, so choosing $m = |\eta|$ 
the general solution of (26) is precisely (43).

The next theorem generalizes Theorem 15.

**Theorem 16:** Let
$$Q(r) = \lambda + R(r) + S(r), \quad (44)$$
where $\lambda > 0$, the integrals $\int_0^\infty |R(r)|dr < \infty$ and $\int_0^\infty |S(r)|dr < \infty$ are 
both finite and $S(r) \to 0$ as $r \to \infty$. Then every solution of Eq. (5) is bounded.\(^\text{(5)}\)

Example 13: See Eqs. (30) or (31) in Example 6.

Using Theorems 14 and 16, a strong and general statement can be 
formulated.

**Theorem 17:** Let $Q(r) = \hat{Q}(r) + m/r^4$ and $m > 0$. If $\hat{Q} \in C^1(\mathbb{R}^+ - \{0\})$ and $\hat{Q}(r)$ satisfies the differential equation $r^4\hat{Q}(r) + 4r^3\hat{Q}'(r) = f(r)$, $\int_0^\infty |f(r)|dr < \infty$ and $r^4\hat{Q}(r) \to 0$ as $r \to 0$, then Eq. (5) is oscillatory 
in $0 < r < \infty$ and the successive relative maximums of $|\psi|$ form a 
strictly monotone increasing function of $r$, where $\lim_{r \to 0} \psi(r) = 0$. Furthermore, 
if every solution of the equation $\psi'' + \hat{Q}(r)\psi = 0$ is bounded 
in $0 < r < \infty$, then every solution of (5) is bounded in $0 < r < \infty$.

Proof of Theorem 17: Let
$$\psi(r) = ry(1/r) \quad (45)$$
Then
$$y(t) = t\psi(1/t), \quad \dot{y}(t) = \psi(1/t) - (1/t)\dot{\psi}(1/t),$$
$$\ddot{y}(t) = -(1/t^2)\psi(1/t) + (1/t^3)\dot{\psi}(1/t) - (1/t^2)\ddot{\psi}(1/t); \quad (46)$$
that is,
$$\psi(r) = (1/t)\psi(t) \quad (47)$$
Substituting (47) into Eq. (5), we get
$$t^3\ddot{y}(t) + (1/t)\dot{\psi}(1/t)y(t) = 0. \quad (48)$$
Substituting the relation $\psi(1/t) = \hat{Q}(1/t) + mt^4$ into (48):
$$\ddot{y} + \left[\frac{\hat{Q}(1/t)}{t^4} + m\right]y = 0 \quad (1/b \leq t < \infty). \quad (49)$$
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If \( R(r) = 0 \) in Theorem 16 then we get the following lemma.

**Lemma:** Let

\[
y' + [m + S(t)]y = 0 \quad (a \leq t < \infty)
\]

and \( m > 0 \). If

\[
\int_a^\infty |S(t)|dt < \infty
\]

and \( S(t) \to 0 \) as \( t \to \infty \), then every solution of (50) is bounded in \( a \leq t < \infty \) and if \( t \to \infty \), then (50) has the solution

\[
y = c_1 \sin(\sqrt{mt} + c).
\]

Applying the Lemma:

\[
S(t) = \frac{\hat{Q}(1/t)}{t^4} \quad \text{and} \quad \dot{S}(t) = \frac{-4 \hat{Q}(1/t)}{t^5} - 4 \frac{\hat{Q}(1/t)}{t^5}.
\]

Since

\[
t = 1/r, \quad dt/dr = -1/r^2 \quad \text{and} \quad dt = -(1/r^3)dr,
\]

so condition (51) is equivalent with

\[
\int_a^\infty |\dot{S}(t)|dt = \int_a^\infty \left| \frac{\hat{Q}(1/t)}{t^6} - 4 \frac{\hat{Q}(1/t)}{t^5} \right| dt
\]

\[
= \int_{r_0}^\infty \left| \frac{\hat{Q}(r)}{r^7} - 4 \frac{\hat{Q}(r)}{r^5} \right| dr
\]

\[
= \int_{r_0}^\infty |r^{-5}\hat{Q}'(r) - 4r^{-3}\hat{Q}'(r)|dr \Rightarrow r^5\hat{Q}'(r) + 4r^3\hat{Q}'(r)
\]

\[
= f(r) \quad \text{and} \quad \int_{r_0}^\infty |f(r)|dr < \infty,
\]

and it is finite because of the condition of the theorem. So the Lemma can be used and the solution (52) has the form

\[
y(1/r) = c_1 \sin(\sqrt{m(1/r) + c}).
\]

Substituting (56) into (45):\n
\[
\psi(r) = c_1 r \sin(\sqrt{m(1/r) + c}).
\]

Because of the condition of the Lemma and the theorem \( r^4\hat{Q}(r) = S(1/r) \to 0 \) as \( r \to 0 \), Eq. (5) is oscillatory in \( 0 < r \leq b \) and

\[
\lim_{r \to 0} \psi(r) = \lim_{r \to 0} c_1 r \sin(\sqrt{m(1/r) + c}) = 0.
\]

It follows from Theorem 14 that if every solution of \( \psi'' + \hat{Q}(r)\psi = 0 \) is bounded in \( 0 < r < \infty \), then, because of \( \int_a^\infty |m/r^4|dr < \infty \), every solution of

\[
\psi'' + Q(r)\psi = 0, \quad Q(r) = \hat{Q}(r) + m/r^4
\]

is bounded in \( 0 < r < \infty \) thus proving the theorem.

**Remark 2:** If \( \hat{Q}(r) = 0 \) in Theorem 17, then \( Q(r) = m/r^4 \), and so Eq. (5) has the form

\[
\psi'' + (m/r^4)\psi = 0 \quad (m > 0).
\]

The general solution of (60) is

\[
\psi = Ar \sin(\sqrt{m(1/r)} + C),
\]

which in the case of \( C \neq 0 \) is unbounded and in the case of \( C = 0 \) \( \psi(r) \leq A\sqrt{m} \) (See Fig. 4.)

Oscillatory behavior and boundedness of solutions of Eq. (4) can be considered on the basis of the Liouville transformation, since \( \varphi(r) = \psi(r)[(1-n)/2] \) and

\[
\varphi(r) = c_1 r^{(3-n)/2} \sin[\sqrt{m(1/r) + c}]
\]

on the basis of (57). We can see from (62) that 1) in the case \( n < 3 \), \( \varphi(r) \) has infinitely many zeros in \( 0 < r \leq b \) and the amplitude of \( \varphi \) is a strictly monotone increasing function of \( r \), so that \( \lim_{r \to 0} \varphi(r) = 0; 2) \) in the case \( n = 3 \), \( \varphi(r) \) oscillates in \( 0 < r \leq b \) with amplitude \( c_1 \), but in the case \( r \to 0 \) \( \varphi(r) \) has no limit at \( r = 0 \), the function is not continuous; 3) in the case of \( n > 3 \), \( \varphi(r) \) oscillates in \( 0 < r \leq b \) and the amplitude of \( \varphi \) is a strictly monotone decreasing function of \( r \), and \( \varphi(r) \) has no limit at \( r = 0 \), the function is not continuous.

**Corollary 3:** If \( Q(r) = m/r^4 \quad (m > 0) \) in Eq. (5), then \( \hat{Q}(r) = \eta + \xi + [(n - 1)/2] - [(n - 1)/2] \) \, \( - V(r) - m/r^4 \) and \( \hat{Q}(r) = -2\xi + [(n - 1)/2] - [(n - 1)/2] \, \( r^2 - V(r) + 4(m/r^2) \). Because of the latter relation, the condition \( \int_{r_0}^\infty |r^4\hat{Q}'(r) - 4r^3V'(r)|dr < \infty \) reduces to

\[
\int_{r_0}^\infty |r^4V''(r) + 4r^3V(r)|dr < \infty
\]

and \( V \in C^1(R^+ - \{0\}) \). A \( V(r) \) potential satisfies (63) if and only if \( V(r) \) is a solution of the differential equation

\[
r^4V'' + 4r^3V = g(r), \quad \int_{r_0}^\infty |g(r)|dr < \infty.
\]

[The function \( g(r) \) can be, for example, a polynomial too.] In the case \( r \to 0 \), the condition \( r^4\hat{Q}'(r) \to 0 \) reduces to the condition

\[
\lim_{r \to 0} (r^4V(r) + m) = 0.
\]

Thus Theorem 17 is valid for every potential that satisfies (64) and (65).
Example 14:
(1) Let \( g(r) = 0 \) in (64). Then the solution is

\[
V(r) = \text{const}/r^4. 
\]

(66)

Because of (65) and (66), the const = -m (since \( m > 0 \), we have that \( V(r) < 0 \)). Therefore, (66) is an attractive singular potential. The Regge formalism (the theory of Regge poles and Regge trajectories) and the S-matrix theory of scattering processes also use spherically symmetric singular potentials \( V(r) \sim r^{-\nu}, \nu \geq 4 \). Such potentials are, for example,

\[
V(r) = -V_0/(\mu^2 r^4) 
\]

discussed by Spector et al.\(^{20}\) or in the case \( \nu \geq 4 \)

\[
V(r) = -g^2/r^4. 
\]

(67)

discussed in Ref. 21. If in (67) \( V_0 > 0 \) and \( V_0/\mu^2 = m \) as well as in (68) \( g \in \mathbb{R}, g^2 = \mu \) and \( \nu = 4 \), then the attractive potentials (67) and (68) satisfy conditions (64) and (65) in \( 0 < r \leq b \). With potentials (67) and (68)

\[
\mathcal{Q}(r) = \eta + \frac{k_j + [(n - 1)/2] - [(n - 1)/2]^2}{r^2}. 
\]

(69)

If \( \eta > 0 \), then because of the Kakutani theorem, every solution of the equation \( \psi'' + \mathcal{Q}(r)\psi = 0 \) is bounded in \( 0 < r < \infty \). Because of Theorem 14, the boundedness is transmitted to Eq. (59) too.

(2) In (64), let \( g(r) = a + \beta r + \gamma r^2 + \delta r^3 + \ldots + \omega r^j \) be a polynomial, where \( \alpha, \beta, \ldots, \omega \in \mathbb{R} \) and \( j \in \mathbb{N} \). Then solving (64) with the method of constant varying, the general solution is

\[
V(r) = \frac{m}{r^3} + \frac{\alpha}{r^3} + \frac{\beta}{r^2} + \frac{\gamma}{3 r} + \frac{\delta}{4} + \ldots + \frac{\omega}{j + 1} r^{-j-3}. 
\]

(70)

Naturally, (70) satisfies the conditions (64) and (65) in \( 0 < r \leq b \). With potential (70)

\[
\mathcal{Q}(r) = \eta + \frac{k_j + [(n - 1)/2] - [(n - 1)/2]^2}{r^2} - \frac{\alpha}{r^3} - \frac{\beta}{2 r^2} - \frac{\gamma}{3 r} - \frac{\delta}{4} - \ldots - \frac{\omega}{j + 1} r^{-j-3}. 
\]

(71)

If \( j \leq 3, 0 < (\eta - \delta/4) = \lambda \) and

\[
\mathcal{Q}(r) = \lambda - \frac{\alpha}{r^3} + \frac{\kappa_j + [(n - 1)/2] - [(n - 1)/2]^2 - \beta/2 - \gamma/3}{r^2} \]

then because of Theorem 16, every solution of the equation \( \psi'' + \mathcal{Q}(r)\psi = 0 \) is bounded in \( 0 < r < \infty \) and because of Theorem 14, this boundedness is transmitted to Eq. (59) too.

In the case \( g(r) = \gamma r^2 \), the particular solution of (64) is \( V_{\text{Coul}}(r) = (\gamma/3)1/r \), which is exactly the Coulomb potential multiplied by \( 2m/\hbar^2 \) with \( \gamma = -2 \times 3m^2/\hbar^2 \), but it does not satisfy the relation (65). More generally, the relations (64) and (65) can be satisfied by the potentials

\[
V_{\text{Coul}}(r) = V_{\text{Coul}}(r) + V_{\text{Coul}}(r), \quad \text{where} \quad V_{\text{Coul}}(r) = -m/r^4 \quad (m > 0)
\]

is the solution of Eq. (64).

**Corollary 4:** Let

\[
Q_1(r) = \eta + \frac{k_j + [(n - 1)/2] - [(n - 1)/2]^2}{r^2} - V_1(r), 
\]

(72)

where \( V_1(r) = V_{\text{Coul}}(r) \) and

\[
Q_2(r) = \eta + \frac{k_j + [(n - 1)/2] - [(n - 1)/2]^2}{r^2} - V_2(r), 
\]

(73)

in which \( V_2 \in C^1(\mathbb{R}^n - \{0\}) \). Because of (64), (65), and \( V(r) = V_{\text{Coul}}(r) \)

\[
|V(r)| > \eta + \frac{k_j + [(n - 1)/2] - [(n - 1)/2]^2}{r^2} \quad (0 < r \leq b), 
\]

(74)

so if

\[
|V_2(r)| = \eta + \frac{k_j + [(n - 1)/2] - [(n - 1)/2]^2}{r^2} \quad (0 < r \leq b), 
\]

(75)

in the interval \( 0 < r \leq b \), and if

\[
V_2(r) \leq V_1(r) < 0 \quad (0 < r \leq b), 
\]

(76)

then inequality (29) is valid in \( 0 < r \leq b \). It follows from Sturm’s comparison theorem, Theorem 17, and Corollary 3 that over Sturm’s minorant equation \( \psi'' + Q_1(r)\psi = 0 \) the Sturm’s majorant equation

\[
\psi'' + Q_2(r)\psi = 0 
\]

(77)

is also oscillatory in the interval \( 0 < r \leq b \) and \( \psi_2 \) has at least one zero between any two successive zeros of \( \psi_1 \) in \( 0 < r \leq b \). Because of (77), \( Q_2 \in C^1(\mathbb{R}^n - \{0\}) \); thus if \( Q_2(r) < 0 \) in the interval \( 0 < r \leq b \), then from the Sonin-Pólya theorem it follows that the successive relative maxima of \( |\psi_2| \) form a strictly monotone increasing function of \( r \) in \( 0 < r \leq b \). [Because of Theorem 17 and Corollary 3, \( \lim_{r \to 0} \psi_2(r) = 0 \), but this property is not shared by \( \psi_2(r) \) which satisfies (77).]

Example 15: Let \( V(r) = V_{\text{Coul}}(r) \) be a potential given by (64) and

\[
V_2(r) = \frac{a}{r^j} + \frac{b}{r^{j-1}} + \ldots + \frac{z}{r} + \alpha \]

\[
+ \beta + \gamma r^{2} + \ldots + \omega r^{j}, 
\]

(78)

in which \( a, b, \ldots, z \in \mathbb{R}, a > 0 \) and \( \alpha, \beta, \ldots, \omega \in \mathbb{R} \), as well as \( i, j \in \mathbb{N} \cup \{0\} \) and \( i > 4 \). It can be seen on the basis of Corollary 4 and (78) that over Sturm’s minorant equation \( \psi'' + Q_1(r)\psi = 0 \), Sturm’s majorant equation (77) is also oscillatory in \( 0 < r \leq b \) and the successive relative maxima of \( |\psi_2| \) form a strictly monotone increasing function of \( r \). Substituting (78) into (77) and supposing that \( i = 4 \), we get for \( Q_2(r) \)

\[
\mathcal{Q}_2(r) = \frac{a}{r^j} + \frac{b}{r^{j-1}} + \ldots + \frac{z}{r} + \alpha \]

\[
+ \beta + \gamma r^{2} + \ldots + \omega r^{j}, \quad \text{where} \quad \mathcal{Q}_2(r) = V_{\text{Coul}}(r) + V_2(r), 
\]

(79)

In the case \( g(r) = \gamma r^2 \), the particular solution of (64) is \( V_{\text{Coul}}(r) = (\gamma/3)1/r \), which is exactly the Coulomb potential multiplied by \( 2m/\hbar^2 \) with \( \gamma = -2 \times 3m^2/\hbar^2 \), but it does not satisfy the relation (65). More generally, the relations (64) and (65) can be satisfied by the potentials \( V_{\text{Coul}}(r) = V_{\text{Coul}}(r) + V_2(r) \) only, where \( V_{\text{Coul}}(r) = -m/r^4 \quad (m > 0) \) is the solution of Eq. (64).
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So, because of Theorem 16, every solution of the equation \( \dot{\psi}_r^2 + \ddot{\psi}(r)\psi_r = 0 \) is bounded in \( 0 < r < \infty \), and because of Theorem 14, the boundedness is transmitted to Eq. (77). Equation (78) can demonstrate, for example, the potential of intermolecular forces and dispersion forces\(^{22}\) or liquid metals\(^{23}\).

Remark 3: Let \( a, b, \ldots, z \in R \) and \( a, z < 0 \); \( i, j \in N \cup \{ 0 \} \) and \( i > 4 \), as well as \( \int_0^\infty r^4P(r)dr < \infty \), \( \lim_{r \to \infty} r^4P(r) = \alpha = 0 \) and \( \alpha > 0 \). Then it can be seen from Theorems 7 and 17, just as from their corollaries, that the wave functions \( \psi(r) \) of Eq. (5) — independently from the dimension number as well as from values of \( \eta \) and \( \kappa_r \) — have infinitely many zeros in the interval \((0, \beta)\) in the case of potentials such as \( V(r) = a/r + b/r^{1-1} + \ldots + c/r^4 + P(r) \). Naturally, \( V(r) \)'s do not satisfy inequality (8).] That's why in cases of this kind of potentials the connection \( \eta = (2m/h^2)E = -m^2c^2/h^2N^2 \) of \( \eta \) and \( N \) is not valid in the three-dimensional eigenstates either. Furthermore, the well-known connections of the principal quantum number \( N \), the secondary quantum number \( l \), and the magnetic quantum number \( m \) are also not valid.

In the case of the (spherically symmetric) potentials \( V(r) \), which satisfy the conditions (64) and (65), and \( V_2(r) \), which satisfy (73), (75), and (76), because of (2) and (3) the solutions of (1) and (5) naturally cannot be WKB approximated if \( r \to 0 \).

Returning to the investigation of Eq. (1) and considering the notations of Eq. (5), we get for Eq. (1)

\[
\frac{d^2\psi}{dr^2} - \frac{\psi}{r} + \left( \frac{\eta + \kappa_r/r^2}{\eta + \kappa_r/r^2} - V(r) \right)\psi = 0. \tag{80}
\]

The first-order differential equation system of the second-order differential Eq. (80) can be written based on the transfer principle as follows:

\[
d_{\psi}/dr = \varphi, \quad dv/dr = \left[ \left( 1 - n \right)/n \right] \varphi
+ \left[ V(r) - \eta - \kappa_r/r^2 \right] \psi \quad (r \neq 0). \tag{81}
\]

**Theorem 18:** Let \( V(r) = \eta + \kappa_r/r^2, \ n > 2 \) and domain \( T = \{ r > 0, \ \varphi \in R, \ w \in R \cap \{ 0 \} \} \). Then the equilibrium solution \( (\varphi, \psi) = (\text{const}, \ 0) \) of the system (81) is **globally, uniformly, asymptotically stable** on \( T \).

In the case of \( n \leq 2 \) the equilibrium solution \( \varphi(r) = \text{const} \) is **unstable**.

Proof of Theorem 18: We can see, based on the qualitative theory\(^{24}\) that (81) is a nonautonomous system; that's why the construction of Liapunov's function is more difficult than in the case of an autonomous system. [See, for example, Ref. 25.] In the case of the potential \( V(r) = \eta + \kappa_r/r^2 \) and \( n > 2 \) the solution of (81) [and (80)] is

\[
\varphi(r) = c_1([1/(2 - n)]r^{2-n} + c_2, \tag{82}
\]

where \( c_1 \) and \( c_2 \) are arbitrary integration constants (see Fig. 5). We can see from (82) that the value (or change) of \( c_2 \) does not exercise an influence on stability, because it only shifts the function \( \varphi(r) \) by a constant. So it is enough to consider the effect of values \( c_1 \) on stability, because an alteration of the initial value (or the perturbation) is accomplished via the change of \( c_1 \). If \( \varphi(c_1) \to \varphi(c_2) \), then \( \varphi(r) \to \varphi(r) \) for every \( 0 < r < \infty \) and \( r \to r \geq r \); this is why the equilibrium solution \( \varphi(c_1) \) is **uniformly stable** in \( 0 < r < \infty \).\(^{25}\) In the case \( n > 2 \) and \( c_1 \neq 0 \), it can be seen from the form \( \varphi(r) \) that if

According to definition\(^{25}\) Eq. (83) means that the equilibrium solution \( \varphi_0 \) is globally asymptotically stable for the value \( (\varphi_0, \ w_0) = (R, R - \{ 0 \}) \). So it has been proved that the equilibrium solution \( \varphi_0 = (c_2, 0) \) is globally, uniformly, asymptotically stable on \( T \) (see Fig. 5).

The solutions of (81) and (80) in one and two dimensions are as follows:

\[
\varphi(r) = c_1r + c_2 \quad \text{and} \quad \varphi(r) = c_1 \ln r + c_2. \tag{84}
\]

On the basis of the Proof of Theorem 18 and (84), it can be seen that in the case \( n < 2 \), the equilibrium solution \( \varphi_0 = \text{const} \) is unstable. [Naturally, the equilibrium solution \( \varphi(\eta) = \text{const} \neq 0 \) does not satisfy the condition \( \varphi_0 \in \mathcal{C}^2(\mathcal{R}_+) \).]

Example 16: In the case of noncentral collisions \( (\kappa_r 
eq 0) \) and \( \eta = 0 \) as well as \( n > 2 \) in scattering processes of attractive potential field

\[
V(r) = \kappa_r/r^2, \quad \kappa_r < 0, \tag{85}
\]

the equilibrium solution \( (\varphi_0, \ w_0) = (c_2, 0) \) of the equation system (81) is globally uniformly asymptotically stable.

Remark 4: Einstein's gravitation equations \( R_{sk} - i_\delta_{sk}R = (8\pi G/c^4)T_{sk} \) are reduced to

\[
R_0^0 - (1/2)R = (8\pi G/c^4)T^0_0 \tag{86}
\]

in the case of time-symmetric\(^{26}\). Equation (86) can be written in the form

\[
R_0^0 - (1/2)R = i_\delta_{sk} - (R_0^0 - R_0^0) - R_{sk} = (8\pi G/c^4)T^0_0, \tag{87}
\]

too. It leads to

\[
(0)R = + (16\pi G/c^4)T^0_0 = 0 \tag{88}
\]

in the case of three-space dimension.\(^{13}\) In Eq. (87) \( R_{sk} = R_{sk} \delta_{sk} \delta_{sk} = R_{sk} \delta_{sk} \delta_{sk} \) is the three-dimensional invariant curvature \( (a, b, \gamma, \delta = 1, 2, 3) \).

In a vacuum, the initial value condition of the time-symmetric pure
gravitational waves\textsuperscript{(13)} is

\[ R = 0. \]  
(88)

It is well known\textsuperscript{(13)} that given any three-dimensional metric \( g_{\alpha \beta} \) the new metric \( \tilde{g}_{\alpha \beta} = \Phi^4 g_{\alpha \beta} \) (or \( d\tilde{s}^2 = \Phi^4 d^2 \)) is a solution of (88) if and only if \( \Phi \) satisfies the following equation:

\[ (\Delta + (1/8)R)\Phi = 0. \]  
(89)

Here, \( \Delta \) and \( R \) are the three-dimensional Laplace operator and the invariant curvature in the original metric \( g_{\alpha \beta} \). Equation (89) is the three-dimensional Schrödinger equation of (weak) time-symmetric gravitational waves\textsuperscript{(27)} given by (6), with effective attractive potential \( V_{\text{eff}}(x, y, z) = - (1/8)R(x, y, z) \) and extreme scattering states in the case of zero energy \( \eta = 0 \). Let \( (1/8)R \) be a spherically symmetric potential. (See the Schwarzschild solution, the de Sitter solution, the Einstein solution, etc.) Then, \( \Psi(x, y, z) = \Phi(r, \theta, \omega) \). We are looking for the solution in the form \( \Psi(r, \theta, \omega) = \psi(r)Y(\theta, \omega) \). Then the form of (89) is

\[
\left[ \Delta + \left( \frac{1}{8} R \right) \right] \psi(r) = \left( \frac{d^2}{dr^2} + \frac{2 \frac{d}{dr}}{r} + \frac{k^2}{r^2} + \frac{1}{8} R \right) \psi(r) = 0,
\]  
(90)

where \( B \) means the two-dimensional Laplace-Beltrami operator. In the case of the eigenvalue equation \( BY = \kappa \psi \), (90) becomes

\[
\left[ \Delta + \left( \frac{1}{8} R \right) \right] \psi(r) = \left( \frac{d^2}{dr^2} + \frac{2 \frac{d}{dr}}{r} + \frac{k^2}{r^2} + \frac{1}{8} R \right) \psi(r) = 0,
\]  
(91)

where \( k_i = -l(l + 1) \) and \( l = 0, 1, 2, \ldots \). The right side of (91) with the \( \psi(r) = \psi(r)^{-1} \) Liouville transformation leads to

\[ \psi'' + Q(r)\psi = 0, \]
\[ Q(r) := \kappa r^2 + (1/8)R(r). \]  
(92)

Equation (5) is equivalent to (92) in the case of \( n = 3 \) and \( V(r) := - (1/8)R(r) \). That is why the above-mentioned results may be applicable for the radial Schrödinger equations (91) and (92) of (weak) time-symmetric gravitational waves. We mention only two instances: 1) Brill and Weber have investigated\textsuperscript{(27)} those cases when \( (1/8)R \) is \( = \kappa r^2 \), where \( \kappa > 0 \) and \( i = 0 \) as well as \( \psi(r) > 0 \) is positive definite in the interval \( a < r < \infty \) and asymptotically \( \psi(r) \to 1 \); furthermore, the space has axial symmetry. In the case of axial symmetry, \( \Psi \) is independent of \( \omega \) and so \( Y(\theta, \omega) = Y(\theta) \). This case corresponds to the elastic scattering of short-range forces from the viewpoint of quantum mechanics, which is generally investigated by the method of partial waves.

1a) Let \( i = 4 \) for the development of "s waves" \((k_i = i = 0)\) during the "central collision." Then \( Q(r) = \kappa r^4 \), \( \kappa > 0 \). It follows from Theorem 8 that (92) is not oscillatory in \( 2\sqrt{\kappa} < r < \infty \). It follows from Theorem 8 that (92) is not oscillatory in \( 2\sqrt{\kappa} < r < \infty \). But (92) is oscillatory in \( 0 < r < 2\sqrt{\kappa} \) according to Theorem 17. Then the general solution has the form of (61). Since \( \psi_i(r) = \psi_i(r)^{-1} \), then \( \lim_{r \to \infty} \psi_i(r) = 1 \) may be obtained with the adequate choice of the integration constants \( A \) and \( C \). It can be seen from Fig. 4 that \( \psi_i(r) > 0 \) in \( 2\sqrt{\kappa} < r < \infty \).

1b) Let \( i > 4 \). Then \( Q(r) = \kappa r^i \) and \( i > 4 \). It follows from Theorem 8 that (92) is not oscillatory in the interval \( (4\kappa)^{1/(i-2)} < r < \infty \). According to Theorems 7 and 17, \( \psi \) has at least one zero between any two successive zeros of \( \psi \) in \( 0 < r < (4\kappa)^{1/(i-2)} \). But according to Theorem 9, distances between successive zeros of \( \psi \) increase in the interval \( 0 < r < (4\kappa)^{1/(i-2)} \), as \( r \) increases from 0 to \( (4\kappa)^{1/(i-2)} \). If \( r \to \infty \), then (92) has the form \( \psi = 0 \); the solution is \( \psi = \alpha + b \). Since \( \psi(r) = \psi(r)^{-1} \), and if \( a = 1 \) and \( b = 0 \), then \( \lim_{r \to \infty} \psi(r) = 1 \). Thus \( \psi(r) > 0 \) may be still true in the interval \( (4\kappa)^{1/(i-2)} < r < \infty \).

2a) Let \( k_i \neq 0 \), \( i = 4 \) for the "running out of sphere waves" in cases of "noncentral collisions." Then \( Q(r) = \kappa r^4 + \alpha r^4 = -l(l + 1) r^2 + \alpha r^4 \), \( \alpha > 0 \). It follows from Theorem 8 that (92) is not oscillatory in \( (4\kappa/[(l + 1) + 1])^{1/2} < r < \infty \). But according to Theorem 17, (92) is oscillatory in the interval \( 0 < r \leq (4\kappa/[(l + 1) + 1])^{1/2} \). In the case of \( r \to 0 \) the solution has the form of (61). In the case of \( r \to \infty \) the solution has the form of \( \psi = \gamma r^2 + \gamma r^{-2} \), where \( \omega := \{k_i - 1/4\}^{1/2} \). It follows from Theorem 8 that (92) is not oscillatory in \( (4\kappa/[(l + 1) + 1])^{1/2} < r < \infty \). It is possible that \( \gamma_i < 0 \). Furthermore, the boundedness and the condition \( \lim_{r \to \infty} \psi(r) = 1 \) cannot be guaranteed.

2b) Let \( k_i \neq 0 \), \( i > 4 \). Then \( Q(r) = \kappa r^i + \alpha r^i = -l(l + 1) r^2 + \alpha r^i \), \( \alpha > 0 \). It follows from Theorem 8 that (92) is not oscillatory in the interval \( (4\kappa/[(l + 1) + 1])^{1/2} < r < \infty \). According to Theorems 7 and 17, \( \psi \) has at least one zero between any two successive zeros of \( \psi \) in \( 0 < r < (4\kappa/[(l + 1) + 1])^{1/2} \). It follows from Theorem 6 that the successive relative maxima of \( |\psi|^2 \) decrease in \( 0 < r < (4\kappa/[(l + 1) + 1])^{1/2} \). But according to Theorem 9, distances between successive zeros of \( \psi \) increase in the interval \( 0 < r < (4\kappa/[(l + 1) + 1])^{1/2} \), as \( r \) increases from 0 to \( (4\kappa/[(l + 1) + 1])^{1/2} \). Consequently, the same can be said about asymptotic behavior as in point 2a). Furthermore, the boundedness and the condition \( \psi(r) > 0 \) cannot be guaranteed.

2) In the case of \( (1/8)R \) is \( = -\kappa r^2 \), it follows from Theorem 18 that the equilibrium solution \( (r_0, \omega) = (\text{const}, 0) \) of (91) is globally uniformly asymptotically stable on \( r > 0 \). It follows from Theorem 18 that the equilibrium solution \( (r_0, \omega) = (\text{const}, 0) \) of (91) is globally uniformly asymptotically stable on \( r > 0 \). It follows from the above-mentioned statements that if \( 0 < r \leq 2\sqrt{\kappa} \), \( i = 4 \) [in the case of 1a)] and \( 0 < r \leq (4\kappa)^{1/(i-2)} \), \( k_i := 0 \), \( i > 4 \) [in the case of 1b)] as well as \( \alpha = -\kappa r^2 \), \( k_i = 0 \), \( i = 4 \) [in the case of 2a)], Furthermore, if \( 0 < r < \infty, k_i \neq 0, i > 4 \) [in the case of 2b)], then the Schwarzschildian mass\textsuperscript{(27)} given by

\[ M = \frac{c^2}{2\pi G} \int \sqrt{\frac{\Phi}{\Phi'}} \, \sqrt{g} \, d^3x, \quad \Phi(x, y, z) = \psi(r)^{-1}Y(\theta, \omega) \]  
(93)

is not well defined (is not a positive definite quantity), because \( \Phi \) is not positive definite. Since the general solution of (92) should be continuous
among other things in the interval $0 < r < \infty$, then the Schwarzschildian mass of (weak) time-symmetric gravitational waves calls for another explanation.

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Résumé

La présente étude porte sur les états diffus et stationnaires de la théorie quantique non-relativiste. Entre autres énoncés, nous montrons que si $V(r) = -\alpha/r + b/r^{i-1} + \ldots z/r + a + \beta r + \gamma r^2 + \ldots + \omega r^i$ est un potentiel sphériquement symétrique à $n$ dimensions, tel que $a, b, \ldots, z \in R, a > 0$ et $a, \beta, \ldots, \omega \in R$, ainsi que $i, j \in N \cup \{0\}$, et $i \geq 4$, une fonction d'onde radiale $\psi(r)$ possède alors un nombre fini de zéros dans l'intervalle $0 < r < \infty$, plus l'amplitude de $\psi$ est une fonction monotone strictement croissante de $r$.


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