Value Capture in the Face of Known and Unknown Unknowns

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VALUE CAPTURE IN THE FACE OF KNOWN AND UNKNOWN UNKNOWNS*

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Abstract

A large theoretical literature on value capture following [Brandenburger and Stuart Jr. (1996)] uses cooperative games under complete information to study how and why firms earn supernormal profits. However, firms often have different information, beliefs, or creative foresight. We extend value capture theory to incomplete information (“known unknowns”) or unawareness (“unknown unknowns”), and illustrate some conceptual issues with that extension. Using the case study of Cirque du Soleil, we show how an entrepreneurial firm can profit even when it does not contribute materially to value creation.

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“Reports that say that something hasn’t happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns, the ones we don’t know we don’t know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.”

Donald Rumsfeld, United States Secretary of Defense, February 12, 2002

1 Introduction

In their pioneering paper, Brandenburger and Stuart Jr. (1996) clarify the link between a firm’s added value (potential to create economic value) and its economic performance (its actual profitability). Their development of this link provides a concrete instance of their more general claim, that cooperative game theory is a promising formalism for strategy research. That paper, along with the contemporaneous practitioner-oriented book Co-opetition (Brandenburger and Nalebuff, 1996), spawned a substantial body of work now known as value capture theory.¹

Central to value capture theory is the core solution concept applied to cooperative games with complete information. In these games, potential for value capture is summarized by a characteristic function. This function states the economic surplus each subset of market agents can produce among themselves. The core provides bounds on profit given these characteristic function values. In particular, no group can collectively earn more than their added value in the broader market. Further, every group must earn at least as much as they could earn transacting only with each other.²

Because these definitions feature complete information, it implies that agents share expectations of the surplus the market will create. Further, they also share expectations about what any group could produce among themselves.³

The starting point for this paper is the observation that market participants generally do not agree about the economic surplus various groups can produce. There is often considerable uncertainty about other agents’ willingness to pay for a given product, their costs, the products they could produce if they formed a joint venture, and so on, and some of this uncertainty may be asymmetric among agents. Therefore, market participants may neither share expectations about the consequences of the actual deals they intend, nor about alternative deals that determine their ranges of value capture possibilities. Thus, even in a world in which everyone can conceive of all events relevant to value-creation and can evaluate them using well-formed probabilities (i.e.,

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¹For a recent survey, see Gans and Ryall (2017).

²See MacDonald and Ryall (2004) for a static example of complete information, cooperative game setup used in a strategy setting. The concept of “biform” games, introduced in Brandenburger and Stuart Jr. (2007), provides a dynamic framework for complete information games and includes the possibility of noncooperative, strategic interaction between firms.

³It is worth pointing out that although the approach features complete information, it does not necessarily imply that it could not be applied to settings with imperfect information. Rather, the assumption is that, all agents have symmetric information about the value and it is commonly known that all agents have symmetric information about the value.
“known unknowns” in Rumsfeld’s terminology), differences of opinion regarding the economic value of a particular deal are likely to be rule rather than the exception.

The problem for real-world markets may go beyond incomplete information about events relevant to value-creation and capture. Economic agents may even be unable to conceive of some of those events, which we refer to as unawareness (or “unknown unknowns” in Rumsfeld’s terminology). That agents may be unaware of the options available to them and others is, if not explicitly then implicitly, central to those streams of strategy scholarship in which the focus is upon knowledge creation and diffusion (Durand et al., 2017) describe the breadth of such inquiry in their review). Beyond strategy, the early literature on entrepreneurship portrays actors with unique imaginative foresight, superior awareness, and alertness to market opportunities (Schumpeter (1911), Knight (1921), Kirzner (1973)). In many contexts of strategy, a firm’s rivals may not realize the implications of an innovative business model based upon a novel synthesis of well-understood constituents. Even informal, practitioner-popular theories of value capture like “Blue ocean strategy” (Mauborgne and Kim (2005)) rely on exactly this possibility. In short, disagreement about the value that a group could create, or even disagreement about the very possibility of creating value, is at the heart of much work in strategy.

We take these qualitative contributions seriously. Our primary contribution is to extend the formal theory of value capture to cases with incomplete information and unawareness. The challenge is to overcome the impossibility of modeling unawareness with standard state-spaces as shown by Dekel et al. (1998). To wit, we marry value capture theory to recent developments on mathematical modeling of incomplete information and unawareness. In particular we devise cooperative games with incomplete information and unawareness by applying unawareness structures developed in Heifetz et al. (2006), Heifetz et al. (2008), and Heifetz et al. (2013a), and extend the coarse core (Wilson (1978)) to those games. In Section 2 we illustrate our approach informally using the case of Cirque du Soleil (Casadesus-Masanell and Aucoin, 2009).

In Sections 3 and 4 we show how to formally extend cooperative games to settings with incomplete information and unawareness. We allow the characteristic function to be state-dependent. Thus, value-creation is now allowed to depend on a wide variety of events, including: agents’ willingness to pay for a given product, suppliers’ costs, the products rivals could produce if they formed a venture, and so on. States also describe beliefs and awareness of those events for each agent, including any higher-order beliefs about beliefs and awareness. This allows agents to have asymmetric expectations about the consequences of the actual deals they intend as well as about the alternative deals that are not intended but which, nevertheless, could be implemented.

Once we establish the setup, we extend the solution concept that is predominantly used in value-capture theory, the core, to incomplete information and unawareness. In value-capture theory, the core specifies the distribution of value agents can capture as constrained by competition. The captured values must be feasible in the sense that a distribution in the core cannot distribute more value than created among all agents. And the captured values must be such that

\[4\] This is expressed quite eloquently by Drucker (1985, p. 135), who writes “Indeed, the greatest praise an innovation can receive is for people to say: ‘This is obvious. Why didn’t I think of it?’”.

\[5\] In the current paper we focus both on incomplete information, formalized with probabilistic beliefs, and unawareness. It is possible to extend our framework to allow for ambiguity and Knightian uncertainty, in which groups may also lack confidence in forming probabilistic judgments. While we find ambiguity uncertainty highly relevant in the context of strategy (see for example Ryall 2009, Ryall and Sampson 2017), we reserve that extension for a later paper.
no group of agents would want to deviate from them and pursue alternative side-deals instead (i.e., competitive consistency). We extend the core to cooperative games with incomplete information and unawareness by using the coarse core due to Wilson (1978), who proposed a special version of it in the context of exchange economies with incomplete information. We allow distributions of value among agents to be state-dependent. That is, the profit of an agent may depend on events about which some agents are uncertain or unaware. In our framework, agents can always form expectations about their own value capture and the value capture by other agents. Similar to the core, the coarse core requires feasibility and competitive consistency. Feasibility now means that the allocation of values must be feasible in every state. That is, in every state no more value can be allocated in aggregate than the aggregate created. Competitive consistency is now more involved. It requires that no group of agents can agree among each other to deviate from the proposed distribution of value. Here agreement means that there is explicit common knowledge among group members, which is very much in the spirit of Aumann (1976) famous no-agreeing-to-disagree theorem.” While perfect information models feature common knowledge in an informal way, the fact that we now allow agents to disagree about value capture means we must make this requirement explicit.

Finally, in Section 5 we discuss limitations of our current approach and avenues to address them in future research. Some further details of our formal framework are relegated to an appendix.

2 Illustrative Example: Cirque du Soleil

As an example of a successful strategy of the “blue ocean” kind that they admonish managers to pursue, Mauborgne and Kim (2005) present the case of Cirque du Soleil. Cirque implemented a fresh take on the traditional circus – an ingenious synthesis of existing categories of live performance into a novel category of entertainment. Important features of this synthesis included: keeping the cachet of a circus “big-top” tent; dropping live animal acts; using street performers rather than star acrobats; and imbuing the performance with a sophisticated, dramatic narrative played out to an original soundtrack. This insight led Cirque’s revenue to increase twenty-fold between 1995 and 2007.

Since Schumpeter (1911), innovation has been characterized a process of recombination of known components. In some inventive activities, agents may be aware of all possible recombinations. They are simply uncertain which ones yield the desired success. The well-defined quest of research & development is then to find out the right recombination. This is a situation of incomplete information. There are other settings in which some agents are unaware of the relevant consequences of various recombinations. It appears that, prior to Cirque, no one was aware of its inventive business model. All of its novel components – elements of a traditional circus, street performance, drama, and a concert – existed in plain sight for any market participant. From a strategy perspective, unawareness explains the central question of why no one come up with this winning synthesis earlier. The providers of traditional circus acts were unaware of the opportunity Cirque envisioned and then implemented, right under their noses. To illustrate our approach to analyzing this important class of business strategy phenomena, let us consider a highly stylized version of the Cirque case.

6 Casadesus-Masanell and Aucoin (2009) provide a detailed business case study.
Consider three firms: Cirque (C), Street Performance Inc. (P), and Big Top Circus (B). Firms P and B have extensive experience operating in the street performance and big-top circus market segments, respectively. P and B are not aware of any opportunity to work together in some innovative fashion and C is inactive. To stick with the popular metaphor, we call this the “red ocean” state and denote it \( r \). In state \( r \), we assume P captures value of $10 and B captures value of $30.

Now, suppose that C has the idea of combining P and B into an altogether novel form of entertainment that is much more valuable than what P and B create independently. Assume that if the market is receptive to it, the value of such a novel form of entertainment is $100. Otherwise, if it flops, no more value is produced than under the traditional way of operating ($40 in aggregate). P and B remain unaware of this possibility – even though the idea is simple, P and B do not have it. Importantly, we also assume that C is certain that P and B do not have it. Finally, although C is aware of the idea to recombine P and B in a novel way, it is uncertain as to the receptivity of such novel form of entertainment by the market. Specifically, assume it assigns probability \( \frac{1}{2} \) to the recombination of P and B being a hit and \( \frac{1}{2} \) to being a flop. We may call these the “blue ocean” states \( h \) and \( f \), respectively.

Note that C brings nothing to the table (in terms of value production): P and B could create the $100 without C if only they thought of it. Moreover, C does not know whether or \( h \) or \( f \) is true. In the technical sense that an agent “knows” the event (subset of outcomes) to which it assigns probability equal to one, C knows nothing. Indeed, C is maximally uncertain as to whether the true state of the world is that its idea is a hit or a flop. The only thing C has going for it is awareness of the possibility of combining P and B into a novel form of entertainment which, at this point, is just an idea.

![Unawareness Structure in the Cirque Example](image)

Figure 1: Unawareness Structure in the Cirque Example

How can these possibilities be modeled? As we will discuss in more depth in the following section, standard state-space models in decision theory do not allow agents to be unaware of the possibility of some states. We therefore use the unawareness structures introduced by Heifetz et al. (2013a). The approach is illustrated for the Cirque example in Figure 1. There are two
state spaces, the Blue Ocean Space and the Red Ocean Space. The Red Ocean Space (lower space) contains just one state, \( r \), which represents P and B conducting independent operations in their traditional markets and C being inactive. Firms P and B can only reason about \( r \), as shown by the dashed circle around it in the diagram. Circles and ovals indicate supports of probability distributions: P and B are certain of \( r \).

Moreover, because they are unaware of any other possibility, they must also assume that everyone else also reasons only in the Red Ocean Space. Thus, P and B imagine the state of mind of C is indicated by the solid-lined circle in the Red Ocean Space. In reality, C is aware of its novel idea. This is represented by the Blue Ocean Space, where C’s true state of mind is given by the solid-lined oval that contains both states \( h \) and \( f \) (with its beliefs shown above each state). Since at both \( h \) and \( f \) the beliefs of P and B are concentrated on \( r \) in the Red Ocean Space (as indicated by the dashed arrows emanating from both \( h \) and \( f \) and pointing to \( r \)), C is certain that both P and B are unaware of its novel idea.

Simple as it is, Figure 1 illustrates how unawareness structures go beyond standard state spaces. First, there are several state spaces, one for each awareness level. These spaces have a natural order by expressiveness. The Blue Ocean Space is more expressive than the Red Ocean Space: its states describe the success or failure of the innovative form of entertainment, which is not possible in Red Ocean Space. Second, an agent’s belief at a state in some space may be concentrated on states in a less expressive space. For instance, at any state in the Blue Ocean Space, firms B and P’s beliefs are on the Red Ocean Space. They have no beliefs about states in the Blue Ocean Space at all, which is how B and P’s unawareness of the Blue Ocean is formalized in this structure.

Next, we explain how value creation works by introducing the state-dependent characteristic function shown in Table 1. The table shows, for each state (and space) and each group of firms, the quantity of value created by the associated group in the associated space. There are two items worth noting. First, in every state, C adds zero value to all groups. Second, the value created in state \( f \) is the same as that in state \( r \). Thus, C does not know whether the innovative idea will lead to a hit or business as usual. In fact, since C holds a uniform belief over \( h \) and \( f \), it is maximally uncertain about hit versus failure. Given that C is unable to contribute any productive capacity or information, it is intuitive to conjecture that C should also be unable to capture any value. Nevertheless, our intuition is that C should receive some payoff because, after all, the idea really is rare and valuable. In fact, the insight of the example will be to demonstrate that an agent can capture value by just contributing awareness (instead of contributing any productive capacity or information).

What do values represent here? We will discuss this point in greater depth, but it is important to see both that the characteristic function is state-dependent, and that values in these state-dependent characteristic function represent the actual value which will be created in those states given agents' beliefs and unawareness. That is, when we say that the group \( \{ P, B, C \} \) has a value of 100 in state \( h \), we mean that, from the perspective of the modeler, 100 would actually be created by that group if that state obtains. Recall that, in cooperative game theory, characteristic functions represent value created in free-form interaction between groups or subgroups. So implicitly \( h \) represents all the events and unmodelled actions of the agents that would lead to the creation of 100 worth of value were this state to occur.

This concludes the specification of value creation under asymmetric known and unknown unknowns for this example. Next we consider what this implies about value capture. To this
Table 1: Values of the State-Dependent Characteristic Function in the Cirque Example

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The diagram shows all ways of distributing the aggregate value associated with that state.

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A simplex shows all ways of distributing the aggregate value associated with that state.
capture in state $h$ is $100$. In state $f$ in the Blue Ocean Space and state $r$ in the Red Ocean Space, that maximum is only $40$. We denote by $\pi_B(h)$ the value captured by B in state $h$, by $\pi_P(f)$ the value captured by P in state $f$, and so on. The subscript refers to the firm. The argument refers to the state. With this notation we can discuss the coarse core.

Begin with the Red Ocean Space. At state $r$, all firms are unaware of the Blue Ocean (according to the unawareness structure in Figure 1). B can generate a value of $30$ on her own, P $10$ on her own, and C nothing (as shown in Table 1). In this state, aggregate value capture cannot exceed $40$. At the same time, given their independent alternatives in this state, it must be that $\pi_B(r) \geq 30$ and $\pi_P(r) \geq 10$. These latter, linear constraints are shown in the diagram as lines in the simplex corresponding to state $r$. The only distribution of value satisfying these conditions is $\pi_B(r) = 30$, $\pi_P(r) = 10$, and $\pi_C(r) = 0$ (as indicated by the dot in the simplex corresponding to state $r$). In the Red Ocean world of state $r$, these are values that would be captured by the three firms.

Next, consider the Blue Ocean states. According to the unawareness structure in Figure 1, B and P are unaware of the Blue Ocean at any state in the Blue Ocean Space. C is aware of the Blue Ocean at either $h$ or $f$. C is also certain that B and P are not aware of the Blue Ocean.
Thus, when C considers putting together a deal involving B and P, it realizes that the relevant alternatives, from the point-of-view of B and P, are the ones associated with the Red Ocean Space. Specifically, at any state of the Blue Ocean Space, B must capture at least $30 and P at least $10 because these are the amounts of value they know they can obtain independent of C. Again, these are linear constraints, easily depicted as lines shown in the simplices of the Blue Ocean Space.

Summing up, any state-contingent distribution of value in the coarse core must respect all of these constraints. Specifically, any distribution $(\pi_B, \pi_P, \pi_C)$ satisfying

\[
100 \geq \pi_B(h), \pi_B(f), \pi_B(r) \geq 30 \\
100 \geq \pi_P(h), \pi_P(f), \pi_P(r) \geq 10 \\
100 \geq \pi_C(h), \pi_C(f), \pi_C(r) \geq 0 \\
\pi_B(h) + \pi_P(h) + \pi_C(h) \leq 100 \\
\pi_B(f) + \pi_P(f) + \pi_C(f) \leq 40 \\
\pi_B(r) + \pi_P(r) + \pi_C(r) \leq 40
\]

is in the coarse core. At state $h$ of the Blue Ocean Space, this corresponds to any distribution in the grey triangle. At states $f$ of the Blue Ocean and $r$ of the Red Ocean, the distribution of payoffs in the coarse core corresponds to the black dot in which firm B captures $30, firm P $10, and firm C nothing.

Consistent with our earlier intuition, in state $h$ there is nothing preventing C from capturing a substantial amount of value – even though it does not contribute anything to the value created and has no knowledge of the true state. What C does contribute is imaginative foresight, her awareness of a novel business idea which creates the Blue Ocean. For instance, the distribution $(\pi_B(h), \pi_P(h), \pi_C(h)) = (30, 10, 60)$ is in the coarse core, in which both B and P’s constraints are binding and C captures the entire residual. Even so, C is not certain whether state $h$ or $f$ obtains, assigning probability $\frac{1}{2}$ to both states. Thus, if C captures all the residual value when the outcome is a hit, her expected value capture is $30, which is still substantial. We call this an awareness rent or payoff due to entrepreneurial foresight. Without C’s awareness, no one would have thought to innovate in the maximally productive way.

How would such distributions be implemented? Under these conditions, we imagine that C could simply propose bilateral employment contracts to B and P, guaranteeing them wages equal to $30 and $10, respectively. In fact, the emergence of bilateral employment contracts is what essentially happened in the real-life Cirque du Soleil case. Of course, bilateral contracts with non-compete clauses, etc., may also help to sustain firm C’s supernormal value capture by creating legal barriers to B and P cutting C out of the picture once B and P have the “why didn’t we think of this,” response. This illustrates that, in a world of unawareness, other legal institutions besides those that protect intellectual property may facilitate value capture. Alternatively, C may develop crucial knowledge with respect to producing and marketing such shows, such that it continues to enjoy positive value capture into the future.\[8\]

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\[8\] Since $(\pi_B(h), \pi_P(h), \pi_C(h)) = (30, 10, 60)$ and $(\pi_B(f), \pi_P(f), \pi_C(f)) = (30, 10, 0)$.

\[9\] See, for example, the recent analysis by Gans and Stern (2017) that analyzes entrepreneurial value capture through formal intellectual property protection (which they refer to as a “control” approach) versus through learning advantages (which they refer to as an “execution” approach).

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As mentioned in the Introduction, this discussion immediately points toward those lines of strategy research that focus on knowledge-based advantage. Sustainable value capture advantage may well begin with the strategic insight, which then establishes first-mover advantages due to learning. As Drucker (1985) remarks, market incongruities that yield new business opportunities are often overlooked by insiders, “(t)ypically, these incongruities are macro-phenomena, which occur within a whole industry or a whole service sector. The major opportunities for innovation exist, however, normally for the small and highly focused new enterprise, new process, or new service. And usually the innovator who exploits this incongruity can count on being left alone for a long time before the existing businesses or suppliers wake up to the fact that they have new and dangerous competition.” Being left alone for a while may be enough to build know-how and exploit the learning curve in ways that assure ongoing value capture. These dynamic considerations are beyond the scope of the present paper. Cooperative games with unawareness provide strategy theorists with a rich analytical framework, one that generalizes the approach used in the extant value capture stream, and can be used to conduct rigorous examination of issues of this kind. Our hope is that the model presented here motivates future research that goes beyond our static setting (see our discussion in Section 5).

3 Value Capture under Known Unknowns

In this section we focus on extending value capture theory to incomplete information, in which agents are aware of all the states, but may have asymmetric beliefs about them. The extension to unawareness is left to the next section. Focusing on incomplete information first allows us to introduce the modelling complexities step-by-step.

3.1 Incomplete Information

Incomplete information refers to situations with known unknowns. Begin with an indexed set of market participants, denoted $N \equiv \{1, \ldots, n\}$, where $n$ is finite. A generic participant, or agent, is denoted $i \in N$. To represent events of relevance to the market participants, we introduce a finite state space, denoted $S$, with typical element $\omega \in S$. Each element in $S$ is a description of events that affect the economic values that agents can create by transacting with one another. An event is a subset of states, $E \subseteq S$. For instance, $E$ could collect all states in which agent $i$ has available some particular production technology. Let $\Sigma \equiv 2^S$ denote the set of all events; i.e., all subsets of $S$. We assume that agents are able to form probabilistic beliefs. To this end, let $\Delta(S)$ denote the set of all probability distributions over states in $S$, with typical element $\mu \in \Delta(S)$.

Unfortunately, just specifying a probability distribution over states is not enough for modeling beliefs of agents in strategic settings. The reason is that agents need to form also beliefs about other agents’ beliefs, beliefs about that etc. The standard approach for this in game theory is the use of type spaces (Harsanyi 1967, Mertens and Zamir 1985), in which a state encodes not only the “brute facts” that affect value creation, but also the beliefs of all the agents about those facts and about each others’ beliefs. Specifically, for every agent $i \in N$, a type map-

\[^{10}\text{To avoid substantial mathematical technicalities that add little additional insight, we take } S \text{ to be finite. Our approach extends to infinite state spaces under appropriate measurability assumptions.}\]
*ping* is a function from states to probability distributions over states, \( t_i : S \rightarrow \Delta(S) \), where \( t_i(\omega) \) represents agent \( i \)'s beliefs over states when the true state is \( \omega \). In this way, a state implicitly describes the beliefs of all the agents. The type mapping “pulls beliefs out of the state” to make them explicit. We impose the condition that, while agents can be uncertain about others’ beliefs, they are always certain of their own.\(^{11}\) This condition is called *Introspection*.\(^{12}\)

### 3.2 Value Creation under Incomplete Information

The next step is to model value creation. As is standard in value capture theory, agents in a market are assumed to face myriad opportunities for the joint production of economic value, via simple arm’s length transactions or more complex activities organized under elaborate contracts. As is the case in cooperative games with complete information, the value creation potential of arbitrary groups of agents is summarized by a characteristic function. However, in order to allow for incomplete information, we assume the characteristic function is state-dependent. Since states describe not only the direct determinants of value creation but also the beliefs of market participants, agents may now disagree about the quantity of value producible by the various groups.

Let \( \mathcal{G} \equiv \{ G \subseteq N \mid G \neq \emptyset \} \) denote the set of all nonempty groups of agents. The characteristic function \( v : S \times \mathcal{G} \rightarrow \mathbb{R}_+ \) assigns to each state \( \omega \in S \) and each group of agents \( G \in \mathcal{G} \) a nonnegative value \( v(\omega, G) \geq 0 \). We interpret \( v(\omega, G) \) as the ex post value that is created by group \( G \) when the true state is \( \omega \). By implication, \( v(\omega, N) \) is the aggregate value that is produced at \( \omega \) by the entire market.\(^ {13}\) Since, as we have seen above, states also pin down interim beliefs of agents, implicitly ex post values may even depend on interim beliefs of agents. This is intuitive, as interim beliefs of agents may affect deal-making, production decisions etc. of agents that albeit not explicitly modelled in cooperative games are at least implicitly captured by the characteristic function.

We restrict the values of the characteristic function to non-negative real numbers. Under complete information, this is fairly innocuous since agents can and, presumably, would choose to avoid transactions known to result in losses. However, under incomplete information, a venture may result in substantial profits in some states and substantial losses in others. Naturally, groups may still decide to take calculated risks and join such a venture when its expected value is sufficiently positive. Nevertheless, we rule out such situations to sidestep the issues of bankruptcy and default. We return to this point when discussing our solution concept in the next subsection and when we discuss limitations of our current model in Section 5.

The characteristic function approach has both its advantages and disadvantages. Rather

\(^{11}\)This condition rules out some particular mistakes in information processing (e.g., Geanakoplos (1989), Samet (1990)). Formally, for each agent \( i \in N \) and state \( \omega \in S \), the agent’s belief at \( \omega \), \( t_i(\omega) \), assigns probability 1 to the set of states in which \( i \) has such beliefs: \( t_i(\omega)(\{ \omega' \in S \mid t_i(\omega') = t_i(\omega) \}) = 1 \).

\(^ {12}\)Note that this formalism allows the modeling of beliefs about another agent’s beliefs, beliefs about those beliefs about beliefs, etc. Harsanyi’s suggestion that such a representation can model arbitrarily higher order beliefs of agents was subsequently shown to be true by Mertens and Zamir (1985) and others under various assumptions about \( S \).

\(^ {13}\)If this aggregate value is produced by a value network of agents in \( N \) (e.g., vertical supply chains or, more generally, “value partitions” as described by Montez et al. (2018), then these are also encoded by \( \omega \). See Section 3.3 for a simple example.
than elaborating a framework to explicate how exogenous events and strategic actions by agents with various beliefs give rise to the actual deals that groups would strike to create value, we skip directly to the last step. That is, we assume all the details relevant to value creation are encoded in an abstract “state” and then take the state-contingent characteristic function as given. This allows for a detail-free approach to modeling the productive potentials of groups and, as such, greater generalization than that possible under a more special structure, such as a contract-theoretic (e.g., Bolton and Dewatripont (2005)) or industrial organization approach. Even so, this detail-freeness does introduce ambiguities about how to build a characteristic function from lower-level primitives. For example, the timing of actions in any deal-making process is an important determinant of the deals that get struck. We discuss this further in Section 5. For now we proceed under the assumption that, in each application, the interpretation of the characteristic function is fixed in a sensible way.

Taking stock of the framework we have introduced so far yields the notion of a characteristic function game with incomplete information. We summarize this formally in the following definition.

**Definition 1** A finite cooperative game with incomplete information \( \langle N, v, S, t_1, \ldots, t_n \rangle \) consists of

- a finite set of agents, \( N \);
- a finite state space \( S \);
- for each agent \( i \in N \), a type mapping \( t_i : S \rightarrow \Delta(S) \) satisfying Introspection; and
- a state-contingent characteristic function \( v : S \times 2^N \rightarrow \mathbb{R}_+ \), where \( 2^N = \{ G \subseteq N \mid G \neq \emptyset \} \).

In a cooperative game with incomplete information, the value creation by groups of agents is state-dependent. Each state determines also a belief for each agent about states (and thus beliefs about the values created by various groups, beliefs about other agents’ beliefs etc.). Agents may have different opinions about values created. Note that, if \( S \) is a singleton (i.e., there exists only one state, which eliminates all uncertainty), then the preceding definition collapses to a standard cooperative game with complete information; i.e., in which each group is certain of the values created by various groups and this is common knowledge among all agents.

### 3.3 Value Capture under Incomplete Information

While the cooperative game with incomplete information constitutes a model of value creation in the face of known unknowns, it does not tell us who captures how much of the value created. This is the question to which we now turn.

Let \( \pi : S \rightarrow \mathbb{R}_+^N \) be a function that assigns to each state \( \omega \in S \) a distribution \( \pi(\omega) \) of value. This distribution \( \pi(\omega) = (\pi_1(\omega), \ldots, \pi_n(\omega)) \) consists of a vector indicating quantities of value captured, one for each agent \( i \in N \), where \( \pi_i(\omega) \) is the value captured by agent \( i \) in the state \( \omega \). We now require a solution concept that ties the state-contingent value creation possibilities, as described by \( v \), and the interim beliefs of the agents, as described by their types, to “sensible” distributions of value, as described by \( \pi \).
The core has become the central solution concept in value capture theory because it has the nice interpretation of identifying the effects of competition on an agent’s ability to capture value. Similar to the cooperative game itself, this concept is relatively detail-free, essentially capturing the outcomes of “free-form” bargaining among the agents. A distribution of value among agents is in the core if two properties are satisfied: First, the distribution of value among agents must be feasible: it cannot specify an aggregate level of value capture that is greater than the aggregate amount of value created in the market as a whole. Second, the distribution must be consistent with competition among agents: No group of agents should have an incentive to block the distribution. That is, there does not exist alternative distribution of value that is feasible among agents in the group and that allows them to capture an amount of value larger than the amount specified in the original distribution.

It is not straightforward to define an analogous concept for cooperative games with incomplete information. Agents can have asymmetric beliefs about their joint opportunities. In such cases, how do we resolve when a distribution of value is consistent with these beliefs? Should the focus be on particular types of agents or all types? Is a distribution problematic when all agents of a group believe they have a strictly better alternative, or is it enough that some believe the alternative is strictly better and the rest are indifferent? Beyond this, is any information revealed — and, if so, by whom and to whom — during the implicit deal-making stage? Are some sort of updated, interim beliefs required? Can distributions of value depend on the private information of some agents and, if so, should incentives to reveal such information be checked? Different answers to these questions potentially lead to different solution concepts.

We now make several assumptions designed to mitigate these subtleties and specify a solution concept that satisfies them.

- First, we assume that any distribution of payoffs that constitutes a solution to the game must be feasible by state. That is, at any state, more value cannot be distributed than is created. Otherwise, since state-contingent characteristic functions represent ex-post values, it is not clear how such an infeasible distribution could be implemented.

- Second, we assume no losses. While we have already assumed that the characteristic function is non-negative, we also assume that individual levels of value capture are never negative. This is an uncontroversial assumption under complete information. No agent would agree to a venture in which she is sure to lose. However, it is not without loss of generality under incomplete information. For example, an agent might be willing to trade off losses in one state with substantial profits in another. By making this assumption, we avoid concerns about how to enforce losses ex post, especially when it may potentially bankrupt some groups and yield to default. We will discuss this problem further in Section 5.

- Third, we assume that groups take their (interim) beliefs into account when comparing the value they capture against the productive opportunities facing them. Each agent uses the beliefs specified by her type when forming expectations about value created by various groups and the value captured in a particular distribution. This means that no information is shared among agents at the interim stage. This means that an agent’s implicit willingness or unwillingness to enter into alternative options does not convey additional information to other agents. Again, this assumption is not without loss of generality.
One could consider settings in which the incentives of agents to share information at the interim stage is important. We avoid these complications here.

Formally, we extend the coarse core due to Wilson (1978), who defined a core for the special setting of an exchange economy with asymmetric information and showed that, in this setting, it is nonempty (see also Kobayashi (1980)). We generalize the coarse core to cooperative games with incomplete information. This solution concept satisfies all the assumptions mention above, beginning with feasibility which we formalize as follows:

**Definition 2 (Feasibility)** Given a cooperative game with incomplete information \( \langle N, v, S, t_1, \ldots, t_n \rangle \), a distribution of value \( \pi \) is feasible for group \( G \) if

\[
\sum_{i \in G} \pi_i(\omega) \leq v(\omega, G) \text{ for all } \omega \in S.
\]

A distribution of value is feasible if it is feasible for \( N \).

That is, a distribution of value \( \pi \) is feasible for group \( G \) if, at any state the agents in \( G \) do not capture more value than they can create at that state. A special case is feasibility for all groups in \( N \). Feasibility is the first ingredient of the coarse core.

Next, we need to formalize a notion of competitive consistency. That is, we need to specify how competition constrains the set of feasible distributions of values captured. While under complete information, the values of the groups are commonly known, under incomplete information, agents may disagree about those values. Yet, agents form (possibly different) expectations about the values created and values captured by various groups. These expectations form the basis for agents to agree or object to various distributions of value. Given a distribution of value \( \pi \) and a state \( \omega \in S \), any agent \( i \in N \) can form expectations about her captured value. To form these expectations, agent \( i \) at state \( \omega \in S \) uses her belief \( t_i(\omega) \).

\[
E[\pi_i | t_i(\omega)] := \sum_{\omega' \in S} \pi_i(\omega') \cdot t_i(\omega)(\{\omega'\}),
\]

is agent \( i \)'s interim expected value capture at state \( \omega \in S \). For example, if \( \omega \) is the true state, then agent \( i \) assigns probability \( t_i(\omega)(\{\omega'\}) \) to the state \( \omega' \).

Intuitively, a distribution of value is consistent with the expectations of a group of agents if there are no feasible alternative distributions that imply better expected levels of value capture for the group. Thus, for any two distributions of value, \( \pi \) and \( \pi' \), we define the event in which \( \pi' \) dominates \( \pi \) for agent \( i \) in expectation:

\[
\{ \omega \in S : E[\pi'_i | t_i(\omega)] > E[\pi_i | t_i(\omega)] \}.
\]

From here, define the event in which \( \pi' \) dominates \( \pi \) for a group \( G \in \mathcal{G} \). This is simply the event that \( \pi' \) dominates \( \pi \) for \( i \in G \) and for \( j \in G \) and ... (for all agents in \( G \)):

\[
\bigcap_{i \in G} \{ \omega \in S : E[\pi'_i | t_i(\omega)] > E[\pi_i | t_i(\omega)] \}.
\]

When would a group of agents \( G \) disagree with a proposed distribution of value? Suppose that agent \( i \) believes that he and agent \( j \) have a joint venture opportunity to create value
and distribute it to themselves according to a distribution $\pi'$ which, consistent with $i$'s beliefs, dominates another distribution $\pi$ for both $i$ and $j$. Can $\pi$, therefore, be ruled out? Not necessarily. For instance, $j$ might be uncertain about $i$’s assessment of the joint opportunity. The fact that $i$ proposes $\pi'$ to $j$ becomes now valuable information to $j$. Consequently $j$ should update her beliefs based on the fact that $i$ proposed $\pi'$ to her. Yet, based on such update, maybe $\pi'$ does not look that promising to $j$ after all. Or perhaps there is now another distribution of value $\pi''$ that looks even more promising to $j$. Similarly, $j$ accepting $i$’s proposal for $\pi'$ may be valuable information to $i$. All of this could result in a very complicated game of signaling and countersignaling, with alternative deals being proposed at each iteration.

This is an interesting issue, but one that is more properly addressed in a dynamic framework rather than in the static setting of this paper (see Section 5 for further discussions). Thus, we rule these sorts of machinations out by assuming that a distribution $\pi$ is blocked by a group of agents $G$ if it is common certainty among agents in $G$ that some feasible $\pi'$ dominates $\pi$ for the group $G$. Common certainty in $G$ here means that every agent in $G$ assigns probability 1 to the event that $\pi'$ dominates $\pi$ for group $G$, every agent in $G$ assigns probability 1 to the event that everybody in $G$ assigns probability 1 to the event that $\pi'$ dominates $\pi$ for group $G$, etc. (In the appendix, we provide a definition of common certainty.)

**Definition 3 ( Blocking Group) ** Given a cooperative game with incomplete information $\langle N, v, S, t_1, ..., t_n \rangle$, we say that a group of agents $G \in G$ blocks a feasible distribution $\pi$ at state $\omega \in S$ if there exists another distribution $\pi'$ that is feasible for group $G$ and it is common certainty among agents in $G$ that $\pi'$ dominates $\pi$ for group $G$.

We now have all the ingredients for our solution concept, the coarse core of Wilson (1978) extended to characteristic function games with incomplete information:

**Definition 4 (Coarse Core) ** A distribution of value $\pi$ is in the coarse core of the cooperative game with incomplete information $\langle N, v, S, t_1, ..., t_n \rangle$ if and only if

(i) $\pi$ is feasible, and

(ii) for all $\omega \in S$ no group blocks $\pi$.

A distribution of payoffs is in the coarse core if, at every state, it is feasible and there is no common certainty among any group of agents that some other distribution feasible for them results in them capturing strictly greater expected value.

For the coarse core, we formalized the notion of competitive consistency with the absence of blocking groups. This somewhat different from the notion of competitive consistency used in value capture theory under complete information. There, a distribution of values satisfies competitive consistency if for every group, the total value captured by this group is at least the value created by the group. We refer to this as group rationality. Under complete information, competitive consistency means that for any group $G \in G$, $\sum_{i \in G} \pi_i \geq v(G)$. That is, the sum of values captured by a group is not less than the value produced by the group. This has been called “coalitional rationality” elsewhere (e.g., [Maschler et al. (2013)]).
it is well-known that both notions of consistency are equivalent. That is, there are equivalent
definitions of the core using either the absence of blocking groups or group rationality (Owen,
1995, Section X.3-4). This begs the question whether or not the coarse core as defined here can
also be equivalently recast using the more familiar notion of group rationality.

The problem is, of course, that under incomplete information, agents may disagree with
respect to their expectations of the value created and value captured by a group of agents.
There may even be disagreement within a group. Nevertheless, we can define consistency à la
group rationality under incomplete information by requiring that every agent in a group believes
that the group captures more under a focal distribution than it could create independently. More
formally:

**Definition 5 (Group rationality)** Given a cooperative game with incomplete information
\langle N, v, S, t_1, ..., t_n \rangle, a distribution of value \( \pi \) is group rational for \( G \in \mathcal{G} \) at \( \omega \in S \) if for every
agent \( i \in G \),

\[
\mathbb{E} \left[ \sum_{j \in G} \pi_j \mid t_i(\omega) \right] \geq \mathbb{E} [v(G, \cdot) \mid t_i(\omega)].
\]

A distribution of value \( \pi \) satisfies group rationality at \( \omega \in S \) if it is group rational for every
group \( G \in \mathcal{G} \).

It seems intuitive that if a distribution of value is group rational, then there should not
exist a blocking group. Yet, our definition of a blocking group under incomplete information
required common certainty that everybody in the blocking group gains. But it may be case
that everybody in the blocking group thinks she gains but also thinks that someone else loses.
It turns out that this does not occur when: (i) each agent’s type mapping is consistent with
a common prior; and (ii) no types are ruled out by that prior. Therefore, under those two
assumptions, group rationality at every state, together with feasibility, implies the coarse core.

Typically, a prior is interpreted as a probability distribution from which the agent’s beliefs
are derived. However, assuming a prior stage appears to be artificial in a static setting like ours.
To wit, we can simply interpret a prior as a consistency condition on agent beliefs instead of
explicitly introducing a prior stage. In particular, a prior for agent \( i \) is a convex combination
of agent \( i \)’s beliefs – a kind of “average belief”. A common prior among group \( G \in \mathcal{G} \) is a prior
for every \( i \in G \). In principle, the prior may assign zero probability to some types of an agent.
It is convenient to rule this out and assume a positive common prior. See the appendix for a
rigorous definition of a positive common prior. The relation between group rationality and the
coarse core can now be stated:

**Proposition 1** Given a cooperative game with incomplete information \( \langle N, v, S, t_1, ..., t_n \rangle \) with
a common prior, if a feasible distribution of value \( \pi \) satisfies group rationality at every state
\( \omega \in S \), then \( \pi \) is in the coarse core.

A more general version of this result is proved in the Appendix. We were not able to prove
the direct converse but only weaker versions of the converse, which we omit.

At a conceptual level, Proposition shows that when competitive consistency as usually un-
derstood in value capture theory under complete information is suitably extended to incomplete
information, then it is consistent with the coarse core. Note that when a distribution of value satisfies group rationality at every state, it is automatically common certainty among all groups. Thus, an immediate corollary is that common certainty of group rationality implies absence of blocking groups under the common prior assumption.

Note further that Proposition 1 provides a sufficient condition for non-emptiness of the coarse core. Currently it is an open question whether general conditions exist with respect to the primitives of the model that are sufficient for non-emptiness of the coarse core. It is well known that the coarse core is nonempty for classes of cooperative games with incomplete information that are important in economics. [Wilson (1978)] proved non-emptiness of the coarse core for exchange economies with asymmetric information.

We refer the interested reader to Appendix A in which we provide a more rigorous exposition of this section with further details.

3.4 Example

We close this section on incomplete information with an example that illustrates the formalism introduced so far. The example will also illustrate how uncertainty with respect to which value chains actually form in a market can be modeled using our framework. In real markets, value is generated by complex webs of consummated transactions – some in the form of simple, arm’s-length purchases in a spot market (buying paper clips at the local retailer), some in the form of more complex negotiated agreements (such as employment contracts, formal supply deals, or multilateral technology alliances involving both formal and relational contracts). Thus, behind the aggregate value produced in the market, say \( v(\omega, N) \), is some collection of disjoint networks of actual, consummated transactions that connect agents down the supply chain and through to the end-user.

To see how all of this maps into the formalism, let us return to a simplified version of the Cirque example. Suppose P and B are aware of the opportunity to create value by forming an alliance to offer a novel form of live entertainment that represents a synthesis of their traditional acts. (We leave C out of the picture in this example.) Let’s assume that the basic brute facts are that P and B can each operate in their traditional segments, yielding them payoffs of 30 each. In all circumstances, P and B know this with certainty. Alternatively, the two firms can form the alliance. The uncertain possibilities are with respect to whether the alliance is a hit (\( h \)), resulting in aggregate value creation of 100, or a flop (\( f \)), resulting in aggregate value creation of 40. The firms cannot simultaneously form an alliance and continue to operate in their traditional markets. Regarding types, suppose there are two types for each agent. One type, \( \theta^1 \), believes \( h \) and \( f \) each occur with a probability \( \frac{1}{2} \). The other type, \( \theta^2 \) believes \( h \) and \( f \) occur with probabilities \( \frac{1}{4} \) and \( \frac{3}{4} \), respectively. Both agents know that they are of the same type.

These details can be summarized by a state space \( S \), with typical element \( \omega = (x, \theta) \) with \( x \in \{ h, f \} \) and \( \theta \in \{ \theta^1, \theta^2 \} \). The type mapping (for both agents) is \( t(\omega)((h, \theta^1)) = t(\omega)((f, \theta^1)) = \frac{1}{2} \) for \( \omega \in \{(h, \theta^1), (f, \theta^1)\} \) and zero otherwise, \( t(\omega)((h, \theta^2)) = \frac{1}{4} \) for \( \omega \in \{(h, \theta^2), (f, \theta^2)\} \) and zero otherwise, and \( t(\omega)((f, \theta^2)) = \frac{3}{4} \) for \( \omega \in \{(h, \theta^2), (f, \theta^2)\} \) and zero otherwise. The state space and type mapping is illustrated in Figure 3.

The state-contingent characteristic function indicates both the actual value produced in
that state as well as the true values of the value-creating alternatives available to each group. It makes sense to assume that in states with $\theta = \theta^1$, the firms form the joint venture. Yet, in those with $\theta = \theta^2$, each firm independently pursues its traditional offering. In states in which the joint venture forms, $v((x, \theta^1), N) = 100$ if $x = h$ and $v((x, \theta^1), N) = 40$ if $x = f$. In states in which the venture does not form, $v((x, \theta^2), N) = 60$, regardless of $x$, which is the sum of the values produced by each firm sticking with its traditional market segment. Suppose that, if P and B agree to a joint venture, then they agree to an even split of the aggregate value actually produced. Refer to this distribution of value as $\pi$. These details, along with the state-contingent, interim expected values for each agent, are summarized in Table 2.

<table>
<thead>
<tr>
<th>States ($\omega$)</th>
<th>$(h, \theta^1)$</th>
<th>$(f, \theta^1)$</th>
<th>$(h, \theta^2)$</th>
<th>$(f, \theta^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(\omega, N)$</td>
<td>100</td>
<td>40</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>$v(\omega, P)$</td>
<td>30</td>
<td>30</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$v(\omega, B)$</td>
<td>30</td>
<td>30</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$\pi_i(\omega)$, $i \in {P, B}$</td>
<td>50</td>
<td>20</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$\pi'_B(\omega)$</td>
<td>30</td>
<td>12</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$E[\pi_i \mid t(\omega)]$, $i \in {P, B}$</td>
<td>35</td>
<td>35</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$E[\pi'_B \mid t(\omega)]$</td>
<td>21</td>
<td>21</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 2: State-contingent value creation and capture

Finally, we check whether the state-contingent distribution of value, $\pi$, indicated in Table 2 is in the coarse core. The two conditions are feasibility and no-blocking-groups. Feasibility requires the distribution to be feasible for $N$ (only), which is clearly the case in each state. In order for a focal distribution $\pi$ to be blocked, all the agents in a group must agree that there is some other distribution that is feasible for that group such that the latter dominates the focal distribution. In this example, the only relevant “groups” are the ones composed of the individual agents themselves. The only alternative distributions for these groups are the ones that pay the respective agent 30 with certainty. Checking against the second-from-bottom row of Table 2 we see that neither agent blocks $\pi$ at any state because in every state their expected value of $\pi$ is weakly larger than 30.

Suppose instead that, in states where the agents believe the joint venture creates the maximum aggregate value, P proposes a 70/30 split in its favor. Label this distribution $\pi'$. The characteristic function, ex-post value capture for B and interim expected value capture for B under $\pi'$ are also shown in Table 2. Here, $\pi'$ is not in the coarse core. To see this, consider...
again the distribution that yields agent B 30 in every state. As we observed already, such a distribution is feasible for group \{B\} since, at every \(\omega\), \(v(\omega, B) = 30\). Moreover, 30 dominates \(\pi'\) at \((h, \theta^2)\) because \(\mathbb{E}[\pi'_B \mid t(h, \theta^1)] = 21\), which is strictly less than 30. Therefore, \{B\} blocks \(\pi'\) at state \((h, \theta^1)\), which violates the coarse core conditions.

By analogous reasoning, \{B\} also blocks \(\pi'\) at \((f, \theta^2)\), but it is already enough that \(\pi'\) is blocked at one state in order to conclude that \(\pi'\) is not in the coarse core. “Group” \{B\} does not block \(\pi'\) at states \((h, \theta^2)\) or \((f, \theta^2)\), but this is now irrelevant for whether or not \(\pi'\) is in the coarse core. Note that in those states, the agents’ beliefs cause them to pursue their traditional offerings, consistent with which \(\pi'\) resulting in an expected payoff of 30 (as shown in Table 2). No alternative distribution that is feasible for \{B\} can do strictly better than that.

 Compared to the original Cirque example in Section 2, this example illustrates the framework confined to incomplete information only. Here, it is worth emphasizing one aspect of this illustration, namely the possibility to use our framework to model uncertainty about value chain formation. While in states \((h, \theta^1)\) and \((f, \theta^2)\) both P and B join together, they remain separate in the market in states \((h, \theta^2)\) and \((f, \theta^2)\).

4 Value Capture under Unknown Unknowns

Cooperative games of incomplete information allow us to model asymmetric uncertainty of agents with respect to events that are pertinent to their value creation opportunities. Yet, as illustrated by our introductory example, business strategies based upon superior creativity, foresight, imagination, ideas, innovation, and alertness can lead to positive economic profits. Such strategies exploit the unknown unknowns of one’s rivals, aspects of the world of which those rivals are simply unaware. This seems to be the type of situation Penrose (1959, Chapter III) had in mind when she emphasized the importance of the “subjective productive opportunity of the firm” for explaining systematic performance heterogeneity among firms. In this section, we introduce general tools for rigorous analysis of unawareness in market settings.

4.1 Unawareness Structures

In general, the modeling of unknown unknowns has proven to be a challenge. Importantly, Dekel et al. (1998) demonstrate that any decision theory featuring a standard state space (e.g., subjective expected utility theory and non-expected utility theory, like maximin expected utility or Choquet expected utility used for ambiguity analyses) is inadequate to the task.\footnote{To understand this impossibility result at an intuitive level, let’s consider what states really mean. We use states to let ‘something’ depend on them (e.g., like the value created by firms). So whatever effects this ‘something’ is implicitly described by states even though states are often just taken as some abstract set of indices or points in some space. Taking this thought more seriously, we can think of a state as a comprehensive description of all relevant events like “drug A cures X, X is caused by Y, ...”. Now given that a decision maker can reason like in this one state, she could also use her language to reason about alternative states like the state “drug A cures X, X is not caused by Y, ...”. More generally, the language to express the events that occur in one state, can be used to reason about occurrence and non-occurrence of any events in the language, and thus describe any state referring to those events. Consequently, a decision maker whose reasoning is modelled implicitly with a state space can not be unaware of any event expressible in this state space because implicitly she has the language to reason about all events modelled in this state space.}

To model...
unawareness one must go beyond a state space. There are several approaches to doing so, developed in artificial intelligence, logic, and game theory (see Schipper (2015) for a survey). Some of these approaches require the user to know epistemic logic, which, unfortunately, is not in the repertoire of most social scientists.

Fortunately, there is one approach that does not require the user to know epistemic logic (even though although it can be cast in those terms) – the unawareness structures developed by Heifetz et al. (2006), Heifetz et al. (2008), and Heifetz et al. (2013a). This is the approach we adapt to cooperative games. It is illustrated in our earlier Cirque du Soleil example. The solution is to use more than one state space. Intuitively, by allowing more than one state space we can allow for differently rich perceptions of the world’s possibilities. When an agent forms beliefs over events in one space (like B and P in the Red Ocean Space), she is blind to some events expressed in the richer space (like the Blue Ocean Space). This simple idea is generalized by unawareness structures.

Begin with a finite set of state spaces $\mathcal{S} = \{S, S', S'', \ldots\}$. We assume that each state space is nonempty and finite. Moreover, we assume that any two state spaces are disjoint. Importantly, they are ordered according to the relative richness of their descriptions of the world’s possibilities. This order is not necessarily complete because one event may be described in space $S$ but not in space $S'$ while a different event may be described in space $S'$ but not in $S$. However, for any two spaces we can consider a (weakly) richer space that describes all events that can be described in either space and a (weakly) poorer space that describes all events that can be described in both spaces. In mathematical terms, we require a lattice of spaces $(\mathcal{S}, \succeq)$, where $\succeq$ is a partial order on the spaces in $\mathcal{S}$. It is also useful to have notation for all states, no matter in which space: let $\Omega = S \cup S' \cup S'' \cup \ldots$ denote the union of spaces in $\mathcal{S}$.

State spaces are not independent. In our Cirque example, the states “novel synthesis a hit” and “novel synthesis a flop” in the Blue Ocean Space are refinements of the state “red ocean” in the Red Ocean Space – the possibilities pertaining to the novel synthesis are not discernible in the simpler space, where the novel synthesis is not even imagined. To model such relations, we introduce projections from more expressive to less expressive spaces. Formally, for any two spaces $S, S' \in \mathcal{S}$ with $S' \succeq S$, we define a function $r^{S'}_S : S' \rightarrow S$ that maps states in the more expressive space $S'$ to states in the less expressive space $S$. Interpret projections as “erasing” some details from the description of a state in the richer space to yield a somewhat impoverished state description in the poorer space.

Projections are assumed to be onto: every state in the poorer space has a state in the richer space that is related to it. In the general setting there can be many spaces that are ordered, for instance $S'' \succeq S' \succeq S$. This raises the need to impose some consistency on projections. Specifically, we require that the projection from $S''$ to $S$ yield the same as the projection from $S''$ to $S'$ followed by the projection from $S'$ to $S$: $r^S_{S''} = r^S_{S'} \circ r^{S''}_{S'}$.

Projections map states in richer state spaces to those in poorer state spaces. Based upon these, it is also possible to identify all the states in richer spaces that are related to a state in a poorer space. For any two spaces $S, S' \in \mathcal{S}$ with $S' \succeq S$ and state $\omega \in S$, let $\rho^S_{S'}(\omega)$ denote the subset of states in $S'$ that project to $\omega$: $\rho^S_{S'}(\omega) \equiv (r^{S'}_S)^{-1}(\omega)$, the inverse image of $\omega$ under $r^{S'}_S$. We call $\rho^S_{S'}(\omega)$ the ramifications of $\omega$ in $S'$.

At this point, it may be helpful to illustrate the formalism introduced so far. Consider Figure 4. There are four spaces. In the upmost space, $S_{pq}$, there are states that describe
whether or not event $p$ or event $q$ happen. For instance, at state $pq$ both event $p$ and event $q$ happen. At state $p\neg q$ event $p$ happens but $q$ does not happen, and so on. The leftmost space $S_p$ is poorer than space $S_{pq}$ because nothing can be expressed with regard to event $q$. Similarly, the rightmost space $S_q$ is poorer than $S_{pq}$ because nothing can be expressed with regard to event $p$. Spaces $S_p$ and $S_q$ are incomparable to each other with respect to their expressiveness: $S_p$ can express whether or not $p$ happens but $S_q$ cannot; and $S_q$ can express whether or not $q$ happens but $S_p$ cannot. Finally, there is a least expressive space, $S_\emptyset$, in which none of these details can be elaborated. This is a simple example of a nontrivial lattice of spaces.

16 The "¬" symbol stands for "not".

17 For illustration purposes, we simply index spaces by primitive events that can be expressed in them.

In a standard state space, an event corresponds to subset of states. For instance, in the example of Section 3.4 the event that the combined offering of $P$ and $B$ is a hit is the subset of states $\{(h, \theta^1), (h, \theta^2)\}$. In our setting with multiple state-spaces, if there is a space in which an event obtains in some states, then also at elaborations of those states in more expressive spaces this event obtains (and may be others as well). This follows from the lattice order of spaces and the projections. For instance, in Figure 4 let $p$ stand for "penicillin has antibiotic properties". Then also at $pq$ we have "penicillin has antibiotic properties" (and some $q$ like "penicillin can be synthesized for mass production"). Formally, take any subset of states $D \subseteq S$ in some space $S \in S$. Denote by $D^\uparrow = \bigcup_{S' \succeq S} \rho_S^{S'}(D)$ the union of elaborations of $D$ in more expressive spaces. This is the union of elaborations of $D$ over all spaces (weakly) more expressive than $S$ (i.e., all states in spaces where what is implicitly described in $D \subseteq S$ can also be described). An event $E$ in the lattice structure has the form $E = D^\uparrow$ for some set of states $D$ in some space $S$. We call $D$ the base of the event $E$ and $S$ the base-space of the event. The base space of an event $E$
is denoted by $S(E)$.

![Figure 5: Event $[p]$](image)

We illustrate the notion of event in Figure 5. It depicts the same lattice of spaces as in Figure 4. Consider now all states in space $S_p$ in which $p$ obtains. This is only the state $p$ in space $S_p$. We indicate it with the dotted area in space $S_p$ of Figure 5. Now consider the elaboration of this set in the more expressive space $S_{pq}$ using the projections. This collects states $pq$ and $p\neg q$, the dotted area in space $S_{pq}$ of Figure 5. The union of these two dotted areas in Figure 5 represents the event $[p]$ (we may write it as $[p]$ in order to distinguish it from the state or the description). It comprises of all the states in all spaces in which $p$ obtains. For this event, $\{p\} \subset S_p$ is the base and $S_p$ is the base space.

Note well that, according to the preceding definition, not every subset of states is an event. For instance, the (singleton) set of states $\{p\}$ is not an event because it misses the elaborations in more expressive spaces. Alternatively, the union of $\{p\}$ and $\{q\}$ is not an event because there is no unique base space that contains $p$ and $q$. Let $\Sigma$ denote the set of events.

Like in the case of incomplete information, we proceed by introducing probability distributions on state-spaces. For any state space $S \in \mathcal{S}$, let $\Delta(S)$ be the set of probability distributions on $S$. Even though we consider probability distributions on each spaces $S \in \mathcal{S}$, we can talk about probability of events that as we just have seen are defined across spaces. To extend probabilities to events of our lattice structure, let $S_\mu$ denote the space on which $\mu$ is a probability measure. Whenever for some event $E \in \Sigma$ we have $S_\mu \succeq S(E)$ (i.e., the event $E$ can be expressed in space $S_\mu$) then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \nsubseteq S_\mu$ (i.e, the event $E$ is not expressible in the space $S_\mu$ because either $S_\mu$ is strictly poorer than $S(E)$ or $S_\mu$ and $S(E)$ are incomparable), then we leave $\mu(E)$ undefined.

To model an agent’s awareness of events and beliefs over events and awareness and beliefs of other groups, we introduce type mappings. Given the preceding paragraph, we see how the
belief of an agent at state $\omega \in S$ may be described by a probability distribution over states in a less expressive space $S'$ (i.e., $S \succeq S'$). This would represent an agent who is unaware of the events that can be expressed in $S$ but not in $S'$. These events are “out of mind” for him in the sense that he does not even form beliefs about them at $\omega$: his beliefs are restricted to a space that cannot express these events.

More formally, for every agent $i \in N$ there is a type mapping $t_i : \Omega \rightarrow \bigcup_{S \in S} \Delta(S)$. That is, the type mapping of agent $i \in N$ assigns to each state $\omega \in \Omega$ of the lattice a probability distribution over some space. Now a state does not only specify which events effecting value creation may obtain, and which beliefs agents hold over those events, but also which events agents are aware of. Recall that $S_\mu$ is the space on which $\mu$ is a probability distribution. Since $t_i(\omega)$ now refers to agent $i$’s probabilistic belief in state $\omega$, we can write $S_{t_i(\omega)}$ as the space on which $t_i(\omega)$ is a probability distribution. $S_{t_i(\omega)}$ represents the awareness level of agent $i$ at state $\omega$. This terminology is intuitive because at $\omega$ agent $i$ forms beliefs about all events in $S_{t_i(\omega)}$.

For a type mapping to make sense, certain properties must be satisfied. The most immediate one is Confinement: if $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$. That is, the space over which agent $i$ has beliefs in $\omega$ is weakly less expressive than the space contains that $\omega$. Obviously, a state in a less expressive space cannot describe beliefs over events that can only be expressed in a richer space. We also impose Introspection, which played a role in our prior discussion of incomplete information: every agent at every state is certain of her beliefs at that state. Appendix A discusses additional properties that guarantee the consistent fit of beliefs and awareness across different state-spaces and rule out mistakes in information processing.

![Figure 6: Unawareness structure](image-url)

It might be helpful to illustrate type mappings with an example. Figure 6 depicts the same lattice of spaces as in Figures 4 and 5. In addition, we depict the type mappings for three different groups. At any state in the upmost space $S_{pq}$, the blue agent is aware of $p$ but unaware of $q$. Moreover, she is certain whether or not $p$ depending on whether or not $p$ obtains. This is modeled by her type mapping that assigns probability 1 to state $p$ in every state where $p$ obtains
and probability 1 to state \( \neg p \) in every state where \( \neg p \) obtains. (The blue circles represent the support of her probability distribution that must assign probability 1 to the unique state in the support.) An analogous interpretation applies to the red agent except that she is an expert in \( q \).

In contrast, the green agent is aware of both \( p \) and \( q \) but knows nothing with certainty, modeled by her probabilistic beliefs in the upmost space that assigns equal probability to each state in it.\(^{18}\)

Unawareness structures allow us to model an agent’s awareness and beliefs about another agent’s awareness and beliefs, beliefs about that etc. This is because, as in the incomplete information case, beliefs are over states and states also describe the awareness and beliefs of groups. Return to Figure 6. At state \( pq \) the green agent assigns probability 1 that the blue group is aware of \( p \) but unaware of \( q \). Moreover, he assigns probability 1 to the blue agent believing with probability 1 that the red group is unaware of \( p \).\(^{19}\)

### 4.2 Value Creation under Incomplete Information and Unawareness

The preceding structure provide a foundation by which to model the reasoning of agents about known and unknown unknowns. To model value creation and capture under known and unknown unknowns, we must augment the formalism with cooperative game theory. The good news is that the extension of cooperative games with incomplete information to those with unawareness is now surprisingly straightforward. We simply replace the state space and type mappings in the definition of cooperative game with incomplete information (Definition 1 in Section 3) by an unawareness structure. The definitions follow almost verbatim!

As before, the characteristic function is state-dependent. However, instead of depending on a state in a single state space, the characteristic function can depend on any state in any state space in the lattice of spaces. Formally, \( v : \Omega \times \mathcal{G} \rightarrow \mathbb{R}_+ \) assigns to each state \( \omega \in \Omega \) (i.e., in the union of spaces) and each group \( G \in \mathcal{G} \) a nonnegative value \( v(\omega, G) \geq 0 \). As before, \( v(\omega, G) \) is the “ex post value created by \( G \) if the events described in \( \omega \) obtain.” The new wrinkle is that \( v(\omega, G) \) is the actual value created ex-post as perceived in the mind of someone whose awareness level is given by the state space that contains \( \omega \). Because \( \omega \) may now leave out important features of the world that affect the amount of value created, this perception may not be fully accurate.

One may be tempted to impose conditions on how values are related across spaces. For instance, in the Cirque du Soleil example, there was a state in the Blue Ocean Space, \( f \), that obviously projected to the state \( r \) in the Red Ocean Space. Moreover, the values for various agents at \( f \) coincided with the values of various agents at \( r \). Alternatively, in some situations the value at a state in a poorer space may be the average (with respect to some prior) of the values in some more expressive space (i.e., agents miss some events, but are correct on average). Such alternative assumptions on how values are related across spaces may differ with the applications and do not seem to be perfectly general. Hence, we refrain from imposing

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\(^{18}\) The example is taken from Schipper (2016) who shows how a generalist (i.e., the green agent) emerges as an entrepreneur and forms a firm made of specialists (i.e., the blue or red agents) in a knowledge and awareness-based theory of the firm using strategic network formations games under incomplete information and unawareness.

\(^{19}\) We note, it has been shown that under appropriate assumptions on spaces \( S \in \mathcal{S} \) and the type mapping, unawareness structures are rich enough to model any higher order beliefs of agents (see the working paper version of Heifetz et al., 2013a).
such further assumptions on the characteristic function in our general set-up and leave it to the researcher to fill-in the appropriate characteristic function values consistent with the application being modeled.

At the interim stage, when agents form their beliefs given their awareness, they can consider expected value creation. Of course, the expectations about the potentials of groups to create value may differ across agents, depending on their awareness and beliefs. Moreover, since unawareness structures allow for reasoning by agents about other agents’ beliefs and awareness, an agent may realize that another agent expects different value creation because she knows that this agent is unaware of some events that affect it. This is the case for C in the Cirque du Soleil example, who is certain that B and P are both unaware of the Blue Ocean and, consequently, do not perceive the opportunity of the novel form of entertainment.

We summarize our discussion in the definition of cooperative game with incomplete information and unawareness:

Definition 6 A finite cooperative game with incomplete information and unawareness
\[ \langle N, v, \langle S, \succeq \rangle, (r^S_{S'}), t_1, ..., t_n \rangle \] consists of

- a finite set of agents \( N \),
- a finite unawareness structure \( \langle S, (r^S_{S'}), t_1, ..., t_n \rangle \), where \( S \) lattice of finite state spaces with projections \( r^S_{S'} : S' \rightarrow S \) for any \( S' \succeq S \), \( S, S' \in S \), \( \Omega = \bigcup_{S \in S} S \), and for each agent \( i \in N \), \( t_i : \Omega \rightarrow \bigcup_{S \in S} \Delta(S) \) is a type mapping satisfying properties outlined in Appendix A,
- a state-contingent characteristic function \( v : \Omega \times G \rightarrow \mathbb{R}_+ \) with \( G = \{ G \subseteq N \mid G \neq \emptyset \} \).

It should be clear that cooperative games with incomplete information are a special case of cooperative games with incomplete information and unawareness in which the lattice structure contains just a single state space.

4.3 Value Capture under Incomplete Information and Unawareness

Given cooperative games with incomplete information and unawareness, the theory of value capture under incomplete information and unawareness is now also a straightforward extension of value capture under incomplete information. Distributions of value are now defined for each state in the entire lattice structure. That is, \( \pi : \Omega \rightarrow \mathbb{R}_+^N \) is a function that assigns to each state \( \omega \in \Omega \) a distribution value of \( \pi(\omega) \). As before, \( \pi(\omega) = (\pi_1(\omega), ..., \pi_n(\omega)) \) is a vector of state-contingent quantities of value captured, one for each agent \( i \in N \). Since \( \omega \) may be just a partial description of events that actually happen (because it is in a poorer space), \( \pi_i(\omega) \) may not be the precise amount of value captured by agent \( i \) but, instead, the capture of agent \( i \) as perceived by someone whose awareness level corresponds to the space that contains \( \omega \).

To identify a set of reasonable value distributions from among the enormous number of possibilities, we extend the solution concept of the coarse core to this setting. The definitions developed in Section 3 for cooperative games with incomplete information apply verbatim to cooperative games with incomplete information and unawareness once we replace the state space.
with the union of spaces in the unawareness structure and use the type mappings as defined for the unawareness structure. This extension brings about additional conceptual features. In particular, when agents contemplate whether a group blocks a distribution of value, not only is no information revealed among agents, but neither is any awareness.

Agents may be surprised in the coarse core – especially when they realize ex-post that there were events relevant to value creation of which they were unaware of and, consequently, did not take into proper account at the interim stage. By the time they realize this, the static game ended. To be sure, an interesting question is what happens in future interactions after some agent becomes aware of some relevant events. However, this requires a dynamic framework, which we leave to future research. (See our discussion in the next section.)

The Cirque du Soleil example in Section 2 illustrates the framework just introduced here. In the Appendix, we provide further details on cooperative games under incomplete information and unawareness. We write out all definitions for cooperative games with incomplete information and unawareness and prove for this class of games a more general version of Proposition stated earlier for the special case of cooperative games with incomplete information.

5 Discussion

In this section, we discuss some important conceptual issues with our framework, its limitations, and opportunities for extending our setting.

5.1 Ex Ante Versus Interim Versus Ex Post Perspectives

We model agents using the interim perspective, in which the true state specifies each agent’s awareness and beliefs prior to the revelation of the state itself. To see that this is the relevant perspective for the extension of cooperative games to incomplete information and unawareness, consider the alternatives. Under the ex-post perspective, the game starts with the state fully resolved. At this point, there is no incomplete information and the approach collapses to cooperative games under complete information.

At the other extreme, consider an ex-ante perspective. Usually ex-ante refers to a situation in which agents have not yet received their private information. Such a situation is often considered under a common prior. That is, all agents hold the identical beliefs. This perspective can make no sense of unawareness because each agent’s awareness is determined at the interim stage. It would be absurd to assume that everyone is aware of everything ex-ante then, at the interim stage, some agent suffers amnesia and becomes unaware of some events. Thus, consider the ex ante perspective when every agent has full awareness instead. Then with a common prior all agents have identical ex ante expectations of all relevant variables. We can now apply the tools of cooperative game theory under complete information except that we use commonly expected values of the characteristic function. Again, this approach essentially boils down to cooperative games under complete information and the traditional approach to value capture use in extant strategy research. If, instead, no common prior is assumed at the ex ante stage, the situation is no different from our interim setting. Hence, the interim approach is the relevant one when considering extensions of cooperative games to incomplete information and unawareness.
5.2 Losses

Our definitions rule out losses both in terms of the value created and value captured. We did this for pragmatic reason of avoiding having to specify how losses are enforced and whether agents can face bankruptcy or default.20 This is a non-issue in cooperative game theory under complete information since no agent would join a venture that yields losses for certain. Yet, under incomplete information, agents may take calculated risks of losses at some states if they are offset by sufficient profits in other states (see the following subsection). Moreover, under unawareness, agents may not even realize that some venture may lead to losses. We hope that this issue will be tackled in future contributions to the theory.

5.3 Implicit Timing and Renegotiation

Although our present framework is static, the characteristic function may implicitly assume some particular timing of events relevant to value creation. This highlights the pros and cons of the characteristic function approach. On the one hand, its detail-freeness requires us to just specify the characteristic function without the need of modeling the minute specifics of non-cooperative interaction. On the other hand, the implicit timing assumed of events affecting value creation need to be taken carefully into account when interpreting characteristic functions.

An example may help to make this more transparent. There are two suppliers, F1 and F2, as well as a buyer B. The buyer wants to buy at most one good from a supplier. Table 3 illustrates how characteristic functions are derived from more basic features like a state-dependent function representing the buyer’s willingness to pay for the good of F1 and F2, and costs of production under different assumptions on the implicit timing of contracting and production within the value chain. The good of F1 is either very valuable (at state $\omega_1$) or useless (at state $\omega_2$) to the buyer. In contrast, the good of supplier F2 is always somewhat useful. F1 has higher cost of production than F2. Assume that all groups have common beliefs, assigning equal probability to either state.

Case I The buyer must select the supplier before the state is resolved and the supplier’s production decision is done before the state is resolved. The commonly expected value from the buyer contracting with F1 and F2 being inactive yields 45 while contracting with F2 instead yields 40. Thus, it makes sense for the buyer to contract with F1. Yet, when state $\omega_2$ occurs this yields a loss because production costs are sunk at this time.21 This illustrates the issue raised in the earlier subsection. In particular, we observe that it might be natural to consider losses in value creation under incomplete information. But such losses could also affect the entire value network when for instance (in a more complex version of the example) the supplier also has contractual obligations to supply to another buyer but at state $\omega_2$ is now unable to do so because of bankruptcy.

Case II The characteristic function features the same willingness to pay and costs of production, but a different timing of production. The buyer still selects the supplier before the state is resolved. Yet, now, the firm postpones its decision to produce until after the state is resolved.

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20This problem is analogous to bankruptcy in general equilibrium under incomplete information. See, for instance, [Dubey et al. (2005)].

21We purposefully depart from our framework in order to illustrate a case with losses.
Table 3: Implicit Timing Modelled in the Characteristic Function

### Case I: Supplier choice and production before the state resolves

<table>
<thead>
<tr>
<th>Value of group</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${F_1, F_2, B}$</td>
<td>120</td>
<td>-30</td>
</tr>
<tr>
<td>${F_1, F_1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${F_1, B}$</td>
<td>120</td>
<td>-30</td>
</tr>
<tr>
<td>${F_2, B}$</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

### Case II: Supplier choice before but production after the state resolves

<table>
<thead>
<tr>
<th>Value of group</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${F_1, F_2, B}$</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td>${F_1, F_1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${F_1, B}$</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td>${F_2, B}$</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

### Case III: Supplier choice and production after the state resolves

<table>
<thead>
<tr>
<th>Value of group</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${F_1, F_2, B}$</td>
<td>120</td>
<td>40</td>
</tr>
<tr>
<td>${F_1, F_1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${F_1, B}$</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td>${F_2, B}$</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

(perhaps due to interim market research). Thus, the firm may cancel production when it becomes clear that the buyer is not willing to take the good after all. This saves production costs and eliminates losses.

**Case III** Retain the willingness to pay and costs of production, but change the timing of contracting and production. Assume the buyer can postpone final selection of the supplier until after the state resolved, at which point production also occurs. Even though the buyer may intend to go forward with $F_1$ initially, she may decide to contract with $F_2$ if state $\omega_2$ transpires.

Case III may also arise when considering renegotiation in Case II. After it becomes clear in Case II, that the buyer is not interested in $F_1$’s good, the market configuration may change when the buyer approaches $F_2$ (in case $F_2$ is still available). So if in Case II commitment to a supplier at the interim stage is “soft” and renegotiation may occur, it is actually more appropriate to consider Case III to begin with.

Clearly, different assumptions about the implicit timing of the market relationships affect modeling the characteristic function. To avoid ambiguities, the modeler needs to consider these issues when specifying the state-contingent characteristic function.

### 5.4 Information and Awareness Sharing at the Interim Stage

We assume that no information or awareness is shared at the interim stage when agents assess their deals with respect to their alternative opportunities. However, the distributions of value
that may arise in the coarse core could give agents reasons to question their beliefs. The coarse core does not require agents to “rationalize” distributions of value. It just requires no agreement by any group to block those distributions. It is a relatively weak solution concept that has its origin in general equilibrium of exchange economies with asymmetric information (Wilson, 1978) where allocations in the coarse are best interpreted as proposed by some neutral market maker without any own information. Realistically, though, if deals are proposed by market participants with private information and awareness, such proposals may be informative. Cooperative game theory does not provide a framework for modeling “proposal actions” or the resulting signals they provide to other agents. Nevertheless, our framework may already offer a glimpse toward how such dynamic updating of information and awareness may play out. For instance, a sequence of unawareness structures could be used to model updates of awareness based on strategic communication among market participants.

We illustrate this with a simple example. Again, let there be two suppliers, F1 and F2, and a buyer B. B wants to buy at most one unit. Initially B is unaware of some events affecting value creation. Both suppliers are fully aware. Suppose there are three awareness levels given by spaces \( S_1 \succ S_2 \succ S_3 \), all of which are singletons for simplicity (i.e., we are assuming complete information given the awareness level). Initially, B’s awareness level is represented by the least expressive space, \( S_3 \). This is indicated in the unawareness structure at time \( t = 1 \) in Figure 7 by the dashed red arrows and information set/support of B’s beliefs. The suppliers’ information and awareness is given by the blue circles.

At state \( \omega_3 \), the only state that the buyer considers possible, F1 has an advantage over F2 (see the characteristic function tabulated at the very left of the Figure 7). Thus, B wants to buy from F1. Yet, if B’s awareness level were \( S_2 \), then he would realize that F2 has an advantage over F1 (see the corresponding characteristic function). It is only natural to assume that F2 would want to raise B’s awareness to \( S_2 \). Consequently, the “updated” awareness of B is depicted
in the unawareness structure at $t = 2$ (at the middle of Figure 7). At this point, if F1 raises B’s awareness to $S_1$, then F1 would have an advantage over F2 once again. (See the unawareness structure at the very right of Figure 7). In this sense, suppliers compete “in awareness” for the buyer.

What is the effect of all of this on value capture according to the coarse core? Initially, B is happy with 13. F1, who knows that the value is actually 22, may propose a contract that promises only 13 to B and captures the residual value. Similarly, upon becoming aware of $S_2$, F2 may offer B 18, who would happily accept that deal, and take the residual for himself (knowing that the actual value created is 20). Finally, at time $t = 3$, B and F1 could agree to split F1’s added value equally. Note that profits go down as suppliers compete in awareness. Of course, F1 does not have any incentive to preempt F2 by raising B’s immediately to $S_1$. Thus, competition in awareness may drive down awareness rents, but this may also be a gradual process.

A more sophisticated extension of our framework would add a non-cooperative stage before the cooperative stage, in spirit of bi-form games by Brandenburger and Stuart Jr. (2007). In bi-form games with complete information and unawareness, the persuasive interaction among firms could be modeled explicitly before the cooperative stage takes place. This is left to future research.

5.5 Information Sharing Ex Post, Incentive Compatibility, Commitment, and Verifiability

We allowed distributions of value in the coarse core to be state-dependent. This may be problematic when the value an agent captures relies upon private information of another agent because the latter may have no incentive to reveal such information to the former. In the interim, agents may anticipate the need to incentivize information revelation. This could rule out some distributions of value in the coarse core. Such considerations led to an active literature on incentive compatible versions of the core for exchange economies with asymmetric information, (see Forges and Serrano (2013) and Forges et al. (2002), for surveys). Essentially, it amounts to molding non-cooperative considerations of incentive compatibility into the cooperative solution concept. This approach does not extend immediately to characteristic function games under incomplete information and unawareness since we take as a primitive the characteristic function rather than individual utility functions with which incentives to reveal information and awareness could be measured. The incentive-compatible core and related solution concepts also require commitment to an incentive compatible mechanism and thus an institution, such as a court of law, that ultimately enforces such commitments. We believe that, rather than overburdening, a cooperative solution concept like the coarse core with non-cooperative features, it would be more fruitful to model revelation of information and awareness explicitly in a prior non-cooperative stage via an extension of our work to bi-form games with incomplete information and unawareness. This is left to future research.

For now, concerns with incentive compatibility can be alleviated by assuming that states are verifiable ex post. While this is not without loss of generality, we argue that there are many situations of interest in which states do eventually become public. For instance, it did eventually become obvious that Cirque’s reinvention of the circus was a hit. Assuming that the state becomes verifiable ex post does beg the question “In which state space?” when unawareness is present. The natural answer is: the space that results when awareness of all agents is pooled.
This is not necessarily the state in the most refined space (which would imply that all agents become aware of everything ex post).

5.6 Awareness of Unawareness

In many situations, agents have some sense that there is “something” out there of which they are unaware. We may refer to this as “awareness of unawareness”. Our notion of (un)awareness is propositionally determined in the sense that awareness is about basic events like “penicillin has antibiotic properties”. In unawareness structures, if an agent is unaware of an event, then she is unaware that she is unaware of the event. It does not rule out a situation where the agent has some sense that there is “something” of which she is unaware. While there have been models of awareness of unawareness in epistemic logic (see Schipper (2015) for a survey), all those approaches are not syntax-free and, hence, are difficult to adapt directly to game theoretic settings.

Our approach is essentially neutral to a general awareness of unawareness. Agents take everything of which they are aware into account. They may have a vague sense of unawareness. Yet, in order for such a sense of unawareness to have any effect, agents should either have some actions available to investigate their unawareness (e.g., asking a lawyer, a doctor, or some other kind of expert who could be aware of it), or they should have an attitude towards such feelings that affects how they rank their available actions (since different actions may expose them to different unawareness levels). In any case, in order to tackle this type of analysis, a non-cooperative stage in which agents can take actions to address their vague sense of unawareness is required. Again, this motivates an extension of our approach to bi-form games with unawareness, where a non-cooperative stage would use the formalism of extensive-form games with unawareness developed by Heifetz et al. (2013b). The latter allows for awareness of unawareness as well as surprises and changes of awareness.

5.7 Unawareness versus Zero Probability

One may wonder how a model with unawareness differs from a model with a single state space where the events of which an agent is unaware of are simply assigned zero probability. At a conceptual level the difference is immediate. Being aware of a state versus being aware of it but assigning zero probability to it are two distinct epistemic states of mind. In particular, in latter case the agent may be willing to bet her entire wealth against a zero probability state, while in former case such bet is not even meaningful to him. This is because assigning zero probability to a state means that the agent assigns probability 1 to its non-occurrence.

The idea of betting can be used to identify events of which an agent is unaware (rather than assigning probability zero to it). By varying the payoff consequences of a contract in a more expressive space while keeping it constant in less expressive spaces and, then, observing an agent’s choices indicates whether or not she is aware of events in the more expressive space. Schipper (2013) Section 4) suggested this as a method to reveal unawareness and a test for unawareness versus zero probability.

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22Stuart Jr. (2017) studies the role of gambles in value creation. This is the only other paper of which we are aware that studies value capture under incomplete information.
5.8 Closing Thoughts

Where do we stand? Private information, awareness of opportunities, entrepreneurial foresight: all are surely of first-order importance in explaining the differential profitability of firms. These concepts have not been easy to incorporate into existing formal theories of firm performance. We show how to extend one particularly well-known model of firm performance - the value capture model - to incorporate information and awareness differences. Our extension uses tools developed in logic, decision theory, and game theory.

Nonetheless, the model here is only a first step. While we have developed in this paper a theory of value capture under known and unknown unknowns in the static case, the preceding discussion makes it clear that extending the model to dynamic settings appears to be the most promising direction. Such an extension could address implicit timing issues, the exploration of strategic revelation of information and awareness, and awareness of unawareness.

We suggest that the most promising path for incorporating these issues will require explicitly modeling strategic information and awareness transmission. For the reasons described above, a model based solely on cooperative game solutions with characteristic functions is limited in its ability to incorporate important aspects of that communication. We therefore argue that theorists should focus on extending the bi-form game approach of [Brandenburger and Stuart Jr. (2007)] by introducing a non-cooperative dynamic stage in spirit of [Heifetz et al. (2013b)] before the cooperative stage. This is left for future research.

A Further Details

In this section we provide a brief but rigorous and self-contained exposition of our general static theory of value capture under known and unknown unknowns. This, of course, contains also the static theory of value capture under known unknowns as a special case.

Denote by \((\mathcal{S}, \succeq)\) the nonempty finite lattice of nonempty finite disjoint state-spaces and let \(\Omega := \bigcup_{S \in \mathcal{S}} S\) the union of all state spaces. For any \(S, S' \in \mathcal{S}\) with \(S' \succeq S\), there is a surjective projection \(r^{s'}_S : S' \rightarrow S\) for which \(r^S_S\) is the identity for any \(S \in \mathcal{S}\). Moreover, projections commute, i.e., for any \(S, S', S'' \in \mathcal{S}\) with \(S'' \succeq S' \succeq S\), we have \(r^{S''}_S = r^{S'}_S \circ r^{S''}_{S'}\). For any \(\omega \in S\) and \(S' \succeq S\), the inverse image \((r^{S'}_S)^{-1}(\omega)\) denotes the ramifications in \(S'\) of state \(\omega\).

For a subset of states \(D \subseteq S\), for some space \(S \in \mathcal{S}\), denote by \(D^S = \bigcup_{S' \succeq S} \left( (r^{S'}_S)^{-1}(D) \right)\). An event has now the form \(E = D^S\) with \(D \subseteq S\), for some \(S \in \mathcal{S}\). \(D\) is called the base of the event \(E\) and \(S\) the base-space of the event \(E\) denoted by \(S(E)\). If \(E \neq \emptyset\), then \(S\) is uniquely determined by \(E\). Otherwise, we write \(\emptyset^S\) for the vacuous event that is based in space \(S\). To understand this, note that the empty set is a subset of any state space. When we take the empty subset of a state space, we can consider also the union of its inverse images in more expressive spaces, which of course is empty as well. This is a vacuous event. But all these vacuous events are different because they have different base spaces. While this may look strange at first, it makes perfect sense. A vacuous event corresponds to a contradiction, a description that is contradictory like “the sun is shining and the sun is not shining”. There is no state of the world where this is true. However, contradictions can be more or less rich depending on how rich is the language with which they are described. This is essentially specified with the base space, and that’s why
we must have different vacuous events. We denote by $\Sigma$ the set of events. We mention that although $\Sigma$ is not an algebra (because it can have more than one vacuous event) it essentially “works” like an algebra, which comes in handy when we define beliefs.

For any state space $S \in S$, let $\Delta(S)$ be the set of probability measures $S$. We consider this set itself as a measurable space endowed with the $\sigma$-field $\mathcal{F}_{\Delta(S)}$ generated by the sets $\{\mu \in \Delta(S) : \mu(D) \geq p\}$, where $D \in 2^S$ and $p \in [0,1]$. In order model beliefs at different levels of awareness, we need to relate probability measures on a richer space to probability measures on poorer spaces. Formally, for a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of $\mu$ on $S \preceq S'$ is defined by

$$
\mu_{|S}(D) := \mu \left( \left( r_{S'}^S \right)^{-1}(D) \right), \quad D \in 2^S.
$$

To extend probability measures to events of our lattice structure, let $S_\mu$ denote the space on which $\mu$ is a probability measure. Whenever for some event $E \in \Sigma$ we have $S_\mu \succeq S(E)$ (i.e., the event $E$ can be expressed in space $S_\mu$) then we abuse notation slightly and write

$$
\mu(E) = \mu(E \cap S_\mu).
$$

If $S(E) \not\in S_\mu$ (i.e., the event $E$ is not expressible in the space $S_\mu$ because either $S_\mu$ is strictly poorer than $S(E)$ or $S_\mu$ and $S(E)$ are incomparable), then we say that $\mu(E)$ is undefined.

For each agent $i \in N$ there is a type mapping $t_i : \Omega \rightarrow \bigcup_{S \in S} \Delta(S)$. We impose the following properties:

(i) **Confinement:** If $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$. This property has already been discussed in the main text.

(ii) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \Delta(S')$ then $t_i(\omega_S) = t_i(\omega)|S$. Property (ii) compares the types of an agent in a state $\omega \in S'$ and its projection to $\omega_S$, for some less expressive space $S \preceq S'$. Suppose an agent’s awareness level at $\omega$ is $S'$, which means that agent $i$’s belief at state $\omega$ is over states in $S'$. What should the agent’s beliefs be at a poorer description of $\omega$ at an awareness level $S$ below $S'$? Property (ii) says that the agent should hold the same belief over an event $E$ as he does at $\omega$ provided that he is still aware of the event $E$. In this sense, the types at $\omega$ and $\omega_S$ just differ in their awareness.

(iii) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \Delta(S)$ then $S_{t_i}(\omega) \succeq S$. Property (iii) also compares the types of an agent in a state $\omega \in S'$ and its projection to $\omega_S$, for some less expressive space $S \preceq S'$. Property (iii) means that at $\omega$ an agent cannot be unaware of an event that she is aware of at the projected state $\omega_{S'}$.

(iv) **Introspection:** $t_i(\omega) \left( \left\{ \omega' \in \Omega : t_i(\omega')|S_{t_i}(\omega) = t_i(\omega) \right\} \right) = 1$. Property (iv) means that at every state, agent $i$ is certain about her own beliefs. More precisely, for every state $\omega$, the type of agent $i$ at $\omega$ is certain of the set of states at which agent $i$’s type or the marginal thereof coincides with her type at $\omega$. This property implies what is called introspection (i.e., Property (va) in Proposition 4 in Heifetz et al. 2013a). It rules out mistakes in information processing.

We denote by $\langle \langle S, \succeq \rangle, (r_{S_\beta}^{S_\alpha})_{S_\beta \preceq S_\alpha}, (t_i)_{i \in N} \rangle$ a finite interactive unawareness structure. This completes the model of beliefs and awareness of groups.
Next, we define characteristic function games with incomplete information and unawareness. The characteristic function \( v : \Omega \times \mathcal{G} \rightarrow \mathbb{R}_+ \) assigns to each state \( \omega \in \Omega \) and each subset of agents \( G \in \mathcal{G} \) a value \( v(\omega, G) \) which is some non-negative real number. As before, we let \( \mathcal{G} \) denote the set of all nonempty subsets of \( N \). The definition of cooperative game with incomplete information and unawareness \( \left( N, v, (\mathcal{S}, \succeq), (r_{S_\beta}^{S_{\alpha}})_{S_\beta \subseteq S_{\alpha}}, (t_i)_{i \in N} \right) \) is now given in Definition 6 in the main text.

A state-dependent distribution of payoffs is \( \pi = (\pi_i)_{i \in N} \) with \( \pi_i : \Omega \rightarrow \mathbb{R}_+ \) for all \( i \in N \). \( \pi_i(\omega) \) represents agent \( i \)'s payoff in state \( \omega \).

Given a cooperative game with incomplete information and unawareness, 
\[
\left( N, v, (\mathcal{S}, \succeq), (r_{S_\beta}^{S_{\alpha}})_{S_\beta \subseteq S_{\alpha}}, (t_i)_{i \in N} \right),
\]
we say that a profile of payoff functions \( \pi \) is \emph{feasible} for group \( G \) if
\[
\sum_{i \in G} \pi_i(\omega) \leq v(\omega, G) \text{ for all } \omega \in \Omega.
\]
We say that \( \pi \) is feasible if it is feasible for all groups \( G \in \mathcal{G} \).

Next we want to define blocking groups. To this extend, we need to formalize what is common belief among a group. For each \( i \in N \), define the certainty operator \( B^1_i : \Sigma \rightarrow \Sigma \) on events by for all \( E \in \Sigma \)
\[
B^1_i(E) = \{ \omega \in \Omega : t_i(\omega)(E) = 1 \}
\]
if there is a state such that \( t_i(\omega)(E) = 1 \), and by \( B^1_i(E) = \emptyset^{S(E)} \) otherwise. For each event \( E \in \Sigma \), \( B^1_i(E) \) is the set of states in which agent \( i \) is certain (i.e., assigns probability 1) to the event \( E \). By Heifetz et al. (2013a Prop. 2), for any event \( E \in \Sigma \), \( B^1_i(E) \) is an \( S(E) \)-based event in \( \Sigma \). That is, the certainty operation is well-behaved and easy to work with.

For any group \( G \subseteq N \) define the mutual certainty operator by \( B^1_G(E) = \bigcap_{i \in G} B^1_i(E) \). That is, in every state of \( B^1_G(E) \) every agent in group \( G \) is certain of the event \( E \). Again, the mutual certainty operator is tractable because for every event \( E \in \Sigma \) and group \( G \), the set of states \( B^1_G(E) \) is an \( S(E) \)-based event in \( \Sigma \).

We can now iterated the mutual certainty operator to formalize that not just everybody in \( G \) is certain of the event \( E \) but also everybody in \( G \) is certain of that fact, and everybody in \( G \) is certain of this etc. For any group \( G \subseteq N \) define the common certainty operator by \( CB^1_G(E) = \bigcap_{n=1}^{\infty} (B^1_G)^n(E) \). Again, for any event \( E \in \Sigma \) and group \( G \), the set is \( CB^1_G(E) \) an \( S(E) \)-based event. For properties of these operators, see Heifetz et al. (2013a Propositions 4-7) for unawareness-belief structures and see Monderer and Samet (1989) for standard type spaces.

Given a state-dependent distribution of payoffs \( \pi \) and a state \( \omega \in \Omega \), any agent \( i \in N \) can form expectations about the sum of payoffs that a group \( G \in \mathcal{G} \) receives. We denote by
\[
\mathbb{E} [x_i \mid t_i(\omega)] := \sum_{\omega' \in \Omega} \pi_i(\omega') \cdot t_i(\omega)(\{\omega'\})
\]
agent \( i \)'s (conditional/interim) expectation of the sum payments received by group \( G \) conditional on state \( \omega \in \Omega \). By Confinement we could write the right-hand side by summing over \( \omega' \in S_{t_i(\omega)} \) instead of the entire \( \Omega \).
For any group $G \in \mathcal{G}$, define the set of states

$$[\pi' G \text{-dominates } \pi] := \bigcap_{i \in G} \left\{ \omega \in \Omega : \mathbb{E}[\pi_i | t_i(\omega)] > \mathbb{E}[\pi_i | t_i(\omega)] \right\}$$

**Definition 7 (Blocking Group)** For a profile of payoff functions $\pi$ that is feasible, a group $G$ blocks $\pi$ at state $\omega \in \Omega$ if there exists a profile of payoff functions $\pi'$ that is feasible for group $G$ such that $\omega \in CB^1_G([\pi' G \text{-dominates } \pi])$.

A profile of payoff functions $\pi$ that is feasible for the entire set of agents $N$ is blocked by group $G$ at state $\omega \in \Omega$ if there exists a profile of payoff functions $\pi'$ that is feasible for group $G$ and at $\omega$ it is common certainty among group $G$ that $\pi'$ yields each member of $G$ a strictly higher expected payment than $\pi$.

**Definition 8 (Coarse Core)** The profile of payoff functions $\pi$ is in the coarse core if and only if

(i) $\pi$ is feasible for the entire set of agents $N$, and

(ii) for all $\omega \in \Omega$ no group blocks $\pi$.

Next, we investigate the relationship between group rationality and no blocking group.

**Definition 9 (Group Rationality)** A profile of payoff functions $\pi$ is $G$-group rational at $\omega \in \Omega$ if for all $i \in G$,

$$\mathbb{E} \left[ \sum_{j \in G} \pi_j | t_i(\omega) \right] \geq \mathbb{E} \left[ v(G, \cdot) | t_i(\omega) \right].$$

A profile of feasible payoff functions $\pi$ is group rational at $\omega \in \Omega$ if it is $G$-group rational for every group $G \in \mathcal{G}$.

It seems intuitive that if a profile of payoff functions is group rational then there should not be a blocking group. Yet, for a blocking group there is common certainty that everybody in the blocking group gains. But it can be case that everybody in the blocking group thinks she gains but also thinks that someone else loses. Such cases are ruled out when the agents’ beliefs are somehow consistent (even though agents may have different awareness) as implied by the common prior assumption.

To define a common prior on unawareness-belief structures we require some notation. Typically, a prior is interpreted as a probability measure from which the agents’ beliefs are derived. Yet, agents may be unaware of some events. That’s why we need a more explicit formalism that allows us to model what events agents are aware of. For each agent $i \in N$, define the awareness operator on events $E \in \Sigma$ by

$$A_i(E) := \{ \omega \in \Omega : t_i(\omega) \in \Delta(S), S \succeq S(E) \}$$

Note that these sets of states may not be events in the unawareness belief structure. Yet, the belief operators can be extended in a straightforward way to those sets. See Heifetz et al. (2013a).
if there is a state \( \omega \in \Omega \) such that \( S_{t_i}(\omega) \supseteq S(E) \), and by \( \psi^S(E) \) otherwise. That is, agent \( i \) is aware of event \( E \) in all states in which her belief is defined on a space that can express the event \( E \). [Heifetz et al. (2013a, Proposition 1)] show that for any agent \( i \in \mathbb{N} \) and event \( E \in \Sigma \), \( A_i(E) \) is an \( S(E) \)-based event in \( \Sigma \). Moreover, [Heifetz et al. (2013a, Proposition 5 and 7)] show that indeed it models non-trivial awareness in unawareness structures and captures properties of awareness introduced in the prior literature.

In a standard state space, a prior is a probability measure on the space. In unawareness-beliefs structures there is a collection of spaces with an order structure. That why in our context of awareness introduced in the prior literature.

**Proof.** If a profile of payoff functions \( \pi \) is feasible for group \( G \) and group rational at every state \( \omega \in \Omega \), then for all \( \omega \in \Omega \) and all groups \( G \in \mathcal{G} \),

\[
\mathbb{E} \left[ \sum_{j \in G} \pi_j \mid t_i(\omega) \right] \geq \mathbb{E} \left[ v(G, \cdot) \mid t_i(\omega) \right] \text{ for all } i \in G.
\]

By Jensen’s Inequality, for all \( \omega \in \Omega \) and groups \( G \in \mathcal{G} \),

\[
\sum_{j \in G} \mathbb{E} \left[ \pi_j \mid t_i(\omega) \right] \geq \mathbb{E} \left[ v(G, \cdot) \mid t_i(\omega) \right] \text{ for all } i \in G. \tag{1}
\]

In principle the prior may assign zero probability to some types of agents. It is convenient to rule this out. Denote for any \( i \in \mathbb{N} \) and \( \omega \in \Omega \), \( \{t_i(\omega)\} := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\} \). This is the set of states in which agent \( i \) has the type \( t_i(\omega) \). A common prior \( P = (P^S)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S) \) is positive if and only if for all \( i \in \mathbb{N} \) and \( \omega \in \Omega \): If \( t_i(\omega) \in \Delta(S') \), then \( P^S((\{t_i(\omega)\} \cap S') \uparrow \cap S) > 0 \) for all \( S \supseteq S' \).

The relation between group rationality and the coarse core can now be stated. The result generalized Proposition 1 to cooperative games with incomplete information and unawareness. The proof applies the No-speculative-betting theorem under unawareness [Heifetz et al. (2013a)].

**Proposition 2** Assume a positive common prior. If a profile of feasible payoff functions \( \pi \) is group rational at every state \( \omega \in \Omega \), then \( \pi \) is in the coarse core.
Suppose by contradiction that \( x \) is not in the coarse core. Then there exists \( \omega \in \Omega \), a group \( G \in \mathcal{G} \), and a profile of payoff functions \( \pi' \) feasible for group \( G \) such that \( \omega \in CB^{1}_G([\pi' G - dominates \pi]) \). \( \omega' \in [\pi' G - dominates \pi] \) if and only if

\[
\mathbb{E}[\pi'_i | t_i(\omega')] > \mathbb{E}[\pi_i | t_i(\omega')] \quad \text{for all } i \in G.
\]

Suppose that there exists \( \omega \in CB^{1}_G([\pi' G - dominates \pi]) \) and \( i \in G \) such that

\[
\mathbb{E}[\pi_j | t_i(\omega)] < \mathbb{E}[\pi_j | t_i(\omega)] \quad \text{for some } j \neq i.
\]

Then since the unawareness-beliefs structure satisfies a positive common prior, this contradicts the “No-speculative-betting” theorem of Heifetz et al. (2013a, Theorem 1). Hence, for all \( \omega \in CB^{1}_G([\pi' G - dominates \pi]) \) and \( i \in G \)

\[
\mathbb{E}[\pi'_j | t_i(\omega)] \geq \mathbb{E}[\pi_j | t_i(\omega)] \quad \text{for all } j \in G.
\]

Summing for all \( j \in G \), we obtain for all \( \omega \in CB^{1}_G([\pi' G - dominates \pi]) \) and \( i \in G \)

\[
\sum_{j \in G} \mathbb{E}[\pi'_j | t_i(\omega)] \geq \sum_{j \in G} \mathbb{E}[\pi_j | t_i(\omega)].
\]

In fact, this inequality holds strictly because for all \( \omega \in CB^{1}_G([\pi' G - dominates \pi]) \) and \( i \in G \),

\[
\mathbb{E}[\pi'_i | t_i(\omega)] > \mathbb{E}[\pi_i | t_i(\omega)].
\]

We conclude, for all \( \omega \in CB^{1}_G([\pi' G - dominates \pi]) \) and \( i \in G \)

\[
\sum_{j \in G} \mathbb{E}[\pi'_j | t_i(\omega)] = \sum_{j \in G} \mathbb{E}[\pi_j | t_i(\omega)]. \tag{2}
\]

Since \( \pi' \) is feasible for group \( G \), we have for all \( \omega \in \Omega \),

\[
\sum_{i \in G} \pi'_i(\omega) \leq v(G, \omega).
\]

This implies (by linearity of the expectations operator)

\[
\sum_{j \in G} \mathbb{E}[\pi'_j | t_i(\omega)] \leq \mathbb{E}[v(G, \cdot) | t_i(\omega)] \quad \text{for all } i \in N, \omega \in \Omega.
\]

Together with Inequality \( \boxed{1} \) it implies for all \( \omega \in \Omega \),

\[
\sum_{j \in G} \mathbb{E}[\pi_j | t_i(\omega)] \geq \sum_{j \in G} \mathbb{E}[\pi'_j | t_i(\omega)] \quad \text{for all } i \in G. \tag{3}
\]

This yields a contradiction to Inequality \( \boxed{2} \). \( \square \)
References


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