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PEIRCE'S "DIAGRAMMATIC REASONING" AS A SOLUTION OF THE LEARNING PARADOX*

Michael H. G. Hoffmann

The interest of philosophy for the *process of gaining knowledge* goes back to Plato's statement of the so-called "learning paradox" in his *Meno* dialogue. "To put it most simply, the paradox is that if one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced or complex as the one to be acquired" (Bereiter 1985, 202). One goal of this paper is to show that the two paradigmatic attempts at solving this problem which were offered by, respectively, Plato and Aristotle are inadequate. Plato's apriorism and Aristotle's inductivism may be seen as two horns of a dilemma: While the inductivist cannot justify any representation of data without assuming *a priori* given hypotheses, the apriorist cannot justify why a certain application of given ideas is correct without being caught in an infinite regress. The other objective of my paper is to explore how Peirce's concept of *diagrammatic reasoning*—that has to be interpreted within the context of his *evolutionary realism*—can avoid this dilemma, and thus makes it possible to explain the knowledge acquisition process.

1. The Paradox of Learning

Plato's major thesis is that knowledge cannot be instilled into souls from without. The "power of learning," as shown in his *Republic*, must be given from the outset in the learner's soul. The "art" of education is "the art of leading around" (τέχνη τῆς περιαγωγῆς) or of redirecting what is already given in the soul: the "eyes" of the soul must be turned away from what is visible and be directed to the "eternal forms" (Plato *Rep.* VII, 518b ff.). In the *Meno*, he argues that all human beings have seen before birth the "eternal form" of each thing, so that during their lifetime they come to perceive visible particulars for what they are. Thus for Plato, all learning is recollecting what was already known but subsequently forgotten. Before discussing Plato's recollection thesis, I will describe *the problem* which this thesis tried to solve. This problem seems to be even more interesting than the solution itself. The starting point is the following question raised by Meno: Is it possible to search for what we do not know at all? For Socrates, this question suggests a fundamental *paradox*: On the one hand, it is superfluous to search for what we already know, on the other hand, we cannot search for what we do not know, for we do not even know what to search for (Plato *Men.*, 80d,e).

As Moravcsik (1994) has shown, the paradox is not applicable to all conceivable forms of learning, but only to those in which "the projected learning must be given direction by the learner himself." Thus, the point of the discussion is not "the learning of non-intellectual skills such as riding, or learning by being told,

or by imitation," but it is "learning taking the form of inquiry" (p. 113). Another restriction which Moravcsik formulates with regard to the applicability of the paradox, is that the learning in question does not pertain to the verification of what can be empirically established. If we take, for example, the question "Is it cold in Escanaba?" we clearly know what we are looking for. The question merely aims at establishing the confirmation or denial of the claim expressed in the proposition "It is cold in Escanaba." However, the paradox would apply to the same question, if the learner were not to understand the sentence. Thus, as Moravcsik points out, the paradox arises only in an "*a priori* inquiry, since the grasping of meanings of sentences entails knowledge of certain *a priori* propositions" (p. 114).

Moravcsik's thesis that Plato concentrates his efforts on what he calls "*a priori* inquiry" can be supported by a further observation. For Plato, the dichotomy of knowing and not knowing depends essentially on the *entities* to be known: Real knowledge means knowing the "eternal forms," the true "paradigms" of all visible things. Without knowledge of these eternal forms, and without the assumption that these forms exist, it would be impossible to explain how we can *define*, and even how we can *perceive* what everything is. In the *Republic* he emphasizes that in any speaking about beautiful or good things we have to presuppose "the beautiful and the good in itself," that is we have to define the many particular instances "according to one form for every particular," in order to call each thing "what it is" (ὃ ἔστιν, Plato *Rep.* VI, 507b; see also Hoffmann 1996, 81-84, 116-118). In a more recent formulation, Plato's thesis says nothing but "*that no knowledge is possible unless concepts and hypotheses are already 'in the mind' before anything is observed at all*" (Piattelli-Palmarini 1980, p. 258).

Within this framework, we can reconstruct Plato's paradox of learning in the following form:

(1) In order to grasp something as "what it is," it is necessary to know its eternal form *a priori*.
(2) If we have such knowledge of an eternal form, it is superfluous to search for it.
(3) If we have no such knowledge of an eternal form, it makes no sense to search for it, because we could not even perceive its particular instances in the visible world.

Jerry Fodor (1980) has reformulated—in a controversy with Piaget—this paradox with regard to cognitive means in general: How can a stronger cognitive apparatus be generated by means of a given cognitive apparatus, if the stronger means are neither *deductively* derivable from the given cognitive means nor available solely by *induction* from experience.

2. Plato's, Chomsky's, and Fodor's Solution: Apriorism

Fodor's conclusion is "not only that there is no learning theory but that in certain senses there certainly *couldn't* be" one (p. 143). As a possible alternative he hints at Chomsky's nativism. For Chomsky, the thesis of a genetically predetermined "generative grammar" is a possible explanation of the astonishing fact that children acquire in a rather short time a highly developed and complex language that is

"hopelessly underdetermined by the fragmentary evidence available" (Chomsky 1976, 10; see also Chomsky 1980, 187f.). The problem is the *poverty of the stimulus* and thus the question: "How do we know so much from so little? What makes the *insides* of our mind so full and systematic when the *outsides* are so motley?" as William Frawley puts the question (1997, 35). Chomsky's solution is in principle the same as Plato's: while Chomsky assumes the existence of a "universal grammar" that has been developed together with the human genome and that every human being actualizes depending on the different contexts in different empirical languages (Chomsky 1980, 187f.), Plato assumes that the immortal human soul has "seen" the eternal forms before birth and can "recollect" them in adequate situations. I do not want to describe in detail the more or less mythological conception of the soul's prenatal visions which play a great role not only in Plato's *Meno*, but also in his *Phaedo* and *Phaedrus* (see in detail Klein 1965). It should be mentioned, however, that the Platonic approach—albeit without its metaphysical trappings—continues to play a role, even in recent computational theories of cognitive science, as William Frawley has shown (p. 36).

In the debate between Piaget and Chomsky, the latter held "the whole question of nativism" to be "beside the point, in that the general thesis is so obviously true that it is even not worth discussing" (Piattelli-Palmarini 1980, 261; see also Otte 1998, 434). His argument rests mainly on the famous problem of *induction*, as we shall see after we have learned something about the Aristotelian solution, which indeed attempts to solve the paradox of learning in terms of inductive reasoning.

3. Aristotle's Solution: Acquiring Knowledge by Induction

One of Aristotle's greatest merits was his introduction of the concept of ἐπαγωγή into the philosophic discussion of scientific method, a concept that has come to be known as "induction" ever since Cicero translated the term in this way. In spite of the familiarity of this concept, however, it is not very easy to see what it exactly means for Aristotle (see von Fritz 1971, Hintikka 1980, McKirahan 1983, Niiniluoto 1994/95, and Knuuttila 1993).

In his *Posterior Analytics,* Aristotle emphasizes that it would be "impossible to gain a view of universals except through induction" (ἀδύνατον δὲ τὰ καθόλου θεωρῆσαι μὴ δι' ἐπαγωγῆς, Aristotle *An. Post.-a* I 18, 81b2f.), and goes on to define the process described by this term as starting from particulars, and thus as depending on "sense-perception, because it is sense-perception that apprehends particulars" (82b6f.). He continues this line of thought in the last chapter of the *Posterior Analytics*, where we can find the fullest account in Aristotle's writings of "how we obtain knowledge of first principles" (Aristotle *An. Post.-a* II, 19, 99b17f.). In opposition to Plato's assumption of inborn knowledge of forms, Aristotle says here that the only sort of capacity we need to presume is a "discriminatory capacity, which is called perception," and which we share with all animals. But whereas in some animals "the act of perception leaves no lasting impression, in others the impression persists and gives rise to memory; and... repeated memories produce experience," which is the first step in the development of science (Tredennick in Aristotle *An. Post.-a*, 17). As Aristotle puts it, with every perception "which has come to rest in the soul...there is a primitive universal

(πρῶτον...καθόλου) in the soul; for although you perceive particulars, perception is of universals,—e.g. of man, not of Callias the man....Thus it is plain that we must get to know the primitives (τὰ πρῶτα) by induction; for this is the way in which perception instills universals" (Aristotle *An. Post.-b* II 19).

As Jonathan Barnes emphasizes, "the ultimate source of knowledge is, in Aristotle's view, perception" (Barnes 1982, 58). Perception, memory, and experience are the steps leading to knowledge. But it remains unclear what exactly the term "induction" refers to in the above quote. Kurt von Fritz has argued that according to the statement quoted above, *all* perception for Aristotle is perception of universals, so that already the first "coming to rest" of an "undifferentiated" perception instills a "primitive" or "first universal" in our soul (προτον...καθόλου). This means, as von Fritz summarizes, that "according to (Aristotle's) approach, the universal as such is really *contained* in the things, and does not only subjectively arise in our mind by seeing together mere similarities" (pp. 653f., my italics). Thus, *any perception* of something as such-and-such there *involves an induction*. This induction, however, is not an "abstraction that lets arise universal concepts out of hundreds or thousands of similar particular sense-perceptions" (p. 653), but it is something that could best be classified as "*intuitive* induction."

Just this kind of "intuitive induction" is also relevant for gaining knowledge of universal propositions according to von Fritz's interpretation: ἐπαγωγή should be understood as "an insight in a general relationship, a relationship which will become immediately evident as universal by leading up (a student) to a single particular instance" (von Fritz 1971, 657; indeed, "leading up to..." is one of the original meanings of επ-αγω). Thus, von Fritz finds a common and essential characteristic of all the different forms of Aristotelian induction in the consideration that "by leading up (a student) to a particular case or particular cases a (real or only fictitious) insight in a general and necessary relationship is elicited. The several kinds of ἐπαγωγή have to be differentiated, then, according to their reliability and exactness" (pp. 663, 667; see also Ross 1971, 41, 55).

According to Aristotle, *perception* is the starting point of gaining knowledge. The objects of immediate intuition are not Platonic forms but universals which, for Aristotle, are contained in the perceivable particulars (see for example Aristotle, *Met.* XIII 4, 1078b28). In this way, all that is necessary for knowledge is given in the objects out there.

4. Two Horns of One Dilemma

A. Problems with Inductivism

Understanding Aristotelian induction as "intuitive induction" which is already the basis for any single act of perception, immediately presents a problem. There are only two possibilities: either the student or scientist *grasps* the universal contained in a particular, or he does not. *Tertium non datur*. This view, however, fails to yield a solution of the learning paradox, for the situation remains exactly the same: If you "get it," you need not search for it, and if you don't, there is not even a chance to find it, for you don't know at all what it is you are looking for. It is "all or nothing."

If we understand, on the other hand, induction in the modern sense of generalization from particular data, the situation isn't any better. Hempel (1965) has

described the core of the problem with utter clarity. I choose to present his view as formulated by Fodor because the latter pointed out that the paradox of induction is exactly the same as the paradox of learning:

> Hempel says: suppose you have a set of points in a two-dimensional Cartesian coordinate system. Once plotted, the points can be fit by a curve that represents an induction about the hypothetical relationship between the two variables. The problem of induction, in this situation, is the problem of deciding which curve to choose. This choice is underdetermined by the data. For any finite set of data points, there will be an infinite set of curves, all of which fit the data equally well. Imagine, for example, a set of points arrayed in a perfectly straight line. In this case, one would surely want to choose a linear curve to represent the relationship. But the data do not force this choice. For in addition to the linear solution, there is an infinite set of sinusoidal curves, each of which fits the points as well as the straight line. These sine waves vary freely in amplitude but have a fixed period, chosen so that every one travels through each of the observation points. Given this set of alternatives, we are obviously in need of some principle that will select the straight line and block the sine waves.
>
> What does it mean to say that we have to block them? It means that *before* we look at the data we have to have the possible curves ordered in terms of preference, so that given just these data we choose the straight line, given slightly anomalous data we choose the closest approximation to a straight line, and so on. You can call that simplicity, or an a priori ordering of the functions, or nativism. This leads to the same point as before: *you can't carry out an induction, it is a logical impossibility to make a nondemonstrative inference without having an a priori ordering of hypotheses*. This general point about nativism is so self-evident that it is superfluous to discuss it; the only question is, how specific are the innate constraints? (Fodor in Piattelli-Palmarini 1980, 259f.; see also the illuminating example of statistic representations in Hoffmann 1999, part I).

For Fodor and Chomsky, the argument against the possibility of induction is an argument in favor of apriorism, thus leading us from inductivism back to Plato's approach. The problems which forced us in this direction are clear. Observation is "theory-laden," sets of data can be described by very different functions or laws, and any mathematical function can be continued in infinitely many ways. The choice of a certain interpretation depends in itself on given ideas, hypotheses, and systems of classification that *mediate* between facts and representation. Thus, the problems with inductivism that forces us back into Plato's arms come down to what may be called the problem of *interpretation* or *representation*, for the main problem is interpreting or representing particular facts.

B. Problems with Apriorism

But a return to the approach of Plato, Chomsky, and Fodor, at once confronts us with a similar problem. Here, however, it must be formulated the other way around:

while for inductivism the problem is gaining universals from particulars which are given in the world, the problem with the complementary position that bases knowledge in *a priori* given structures is how to *apply* an inborn or otherwise given system of universal conditions. The question is how to get the correct facts for a given general form. For, if a certain set of applications of a general form is supposed to be an interpretation or representation of this form, we are again forced to say that this interpretation must be *mediated* by some third element without which the relation between forms and facts remains utterly arbitrary.

The problem of mediating between given universals in the mind and particulars to which the universals must be applied was already discussed by Plato in a later dialogue, the *Theaetetus*. Though the work does not contain any explicit reference to his theory of recollection, the problems discussed in fact amount to what may be called Plato's *implicit* self-critique of this doctrine (see also McDowell 1973, 218f., 222f.).

The problem discussed in the *Theaetetus* is the problem of the definition of ἐπιστήμη, "knowledge." Several definitions are presented and criticized, and the dialogue finally ends in an *aporia* (see Hoffmann 1996, 33–110). Within the context of our problem, the most interesting hypothesis is that knowledge is nothing but "true opinion" (Plato *Tht.*, 187b). The main problem with this definition is how to distinguish *true* from false opinions. But in the end, the nature of the question is seen to differ according to whether it is seen from the perspective of the problem of *recognizing* a certain particular as such-and-such or from the perspective of the problem of *thinking* without any perception. Plato illustrates the first perspective by using the image of a "wax block" in our souls, into which is "imprinted" what we have in our mind or memory. In that case, recognizing something as such-and-such simply means that, whenever something is perceived, a certain idea or imprint is "activated" or "triggered." Accordingly, false opinion may be seen as "the *mismatching* of perceptions with imprints" (McDowell 1973, 212).

To illustrate the problem of false opinions in acts of sense-independent *thinking*, Plato offers another fantastic image in which the mind is compared to an *aviary*, a large cage in which birds are kept. The problem which this image is supposed to solve is described by the example of calculating 7 + 5, an operation which is of course independent of sense-perception. Here too, false opinions are possible, for someone may think that adding 7 to 5 yields 11 instead of 12. To explain this possibility, Plato suggests a distinction between *possessing* and *having* pieces of knowledge. In the former case, "learning" is nothing but "capturing" pieces of knowledge or "birds" and putting them in our minds as into a kind of aviary, so that we *possess* them. Thus, we "know" the numbers 11 and 12 because they are "flying" in our minds like birds in an aviary. The problem in calculating the sum of 7 and 5, is to make sure we "catch" the right bird. When we catch the false one, we are "having" knowledge or opinion, albeit a false one.

At first glance, the whole story seems to be a witty joke, but we should read it—at least partially—as a model for a serious problem. As McDowell points out (p. 222f.), both the problem and its solution, which consists in distinguishing between *possessing* and *having* knowledge, have a "structural or formal" parallel in the paradox of learning of the *Meno*. According to the theory of recollection, what the "learner" experiences is nothing but *actualizing* or *having in hand* what he already possesses as prenatally envisioned forms. In the *Theaetetus*, however, Plato takes a completely new point of view with regard to the problem and its solution: Here the problem of false opinion reveals an implication of the theory of recollection, which transcends the horizon of the *Meno*. For, if it is possible to have false opinions, the major question is obviously *how to distinguish false opinions from true ones*. My thesis is that it is precisely this question which provides the background for an objection formulated in a rather obscure passage (pp. 199c7–200c7). The objection is formulated by an hypothetical "expert in refutation" against Theaetetus's suggestion that there are also "pieces of *ignorance*" flying about in our mind, so that "one makes a false judgment because of the piece of unknowing, and a true one because of the piece of knowledge" (p. 199e4–6). As a rhetorical question the same "destructive critic" asks him:

> Or are you going to start all over again and tell me that there are yet more pieces of knowledge of those pieces of knowledge and unknowing, which their possessor has shut up in yet more ridiculous aviaries or moulded lumps of wax, and which he knows as long as he possesses them, even if he doesn't have them readily available in his mind? Are you going to let yourselves be forced, in that way, to keep coming round, time after time, to the same point, without making any progress? (Plato *Tht.*, 200b5–c4).

According to the aviary model of the mind, the only way to determine whether 11 or 12 is the correct result of calculating 7 + 5 would indeed be to presuppose an infinite series of "meta-aviaries," each of them containing pieces of knowledge of a higher level to determine what are correct combinations of lower level pieces of knowledge. Thus, each *application* of "possessed" knowledge needs further possessed knowledge of a higher level, etc. *ad infinitum*. It would be inherently impossible for any knowledge we possess in the form of "pieces of knowledge" to by itself determine its own application.

The same is true, of course, with regard to recognizing something as such-and-such by perception: If we want to distinguish between true opinions and false ones as based on a mismatching of perception and imprint, we are forced to presuppose again a higher level of "imprints" in order to judge the correctness of applying a certain imprint to a certain perception, and so on *ad infinitum*.

Thus we might conclude: The main problem of apriorism is to distinguish between *adequate* and *inadequate* applications of internal given universal structures to particular facts or events. The question is, how to avoid arbitrariness in the application of universals. Within an approach that is exclusively based on *a priori* forms, any solution to this problem necessitates the assumption of higher level universals in the mind, which govern the application of lower level universals. Such solution, however, necessarily leads to an infinite regress to ever higher levels of universals.

Taking together the problems of *inductivism* and those of *apriorism*, we now can summarize our considerations as follows: While *Inductivism* claims that universals can be obtained by perception and by abstraction from perceivable facts, it cannot explain the process of gaining correct universal representations of what is given in the world without. And, while *apriorism* claims that all perception of

something as "what it is" necessarily depends on *a priori* given universals, it fails to provide the means for judging the correctness of the application of such universals without getting caught in an infinite regress. These are the two horns of a dilemma. Neither apriorism nor inductivism can by themselves alone explain the process of acquiring knowledge.

5. Peirce's Solution of the Learning Paradox through Diagrammatic Reasoning

One of Peirce's greatest merits from the point of view of the philosophy of science was his introduction of the distinction between "abduction" and "induction" as two modes of synthetic or ampliative reasoning. For Peirce, any step in scientific inquiry involves three inferential operations in a fixed order: *abduction* is the first step, *deduction* the second, while *induction* is the last. These three "are the only elementary modes of reasoning there are," as Peirce says in 1905. *Abduction* "consists in examining a mass of facts and in allowing these facts to suggest a theory. In this way we gain new ideas; but there is no force in the reasoning. The second kind of reasoning is *deduction*, or necessary reasoning." After abduction has "suggested a theory, we employ *de*duction to deduce from that ideal theory a promiscuous variety of consequences to the effect that if we perform certain acts, we shall find ourselves confronted with certain experiences. We then proceed to try these experiments, and if the predictions of the theory are verified, we have a proportionate confidence that the experiments that remain to be tried will confirm the theory" (Peirce CP 8.209). Induction, thus, is restricted to increasingly *confirming* or *disconfirming* given universal propositions on the basis of particular data, while abduction is the process of *forming* such universal propositions as explanatory hypotheses (see also Peirce CP 2.755). In Hans Reichenbach's terminology, we might say that induction hints at the problem of justification of general assertions while abduction is concerned with the problem of discovery (see also my discussion in Hoffmann 1999).

Thus, what we have discussed above as the well-known problem of *induction* may now be seen as a conglomerate of two different problems. On the one hand, there is the problem of confirmation as discussed, for example, in Goodman's famous theory of projectibility. For Goodman, the major problem of induction is to *exclude* certain predicates such as his famous "grue" in the proposition "all emeralds are grue." The latter predicate as it would appear in the proposition "All emeralds are grue" is supported by data available at a certain time *t*, just as much as the predicate "green" in the assertion "all emeralds are green" (something is called "grue" if it has the property "green" before *t*, and the property "blue" after *t*; see Goodman 1983; see also the discussion in Stalker 1994). This "new riddle of induction," as Goodman calls it, is exactly the same as the problem to which we have referred above, of determining a lawlike representation of points in a coordinate system, in as much as the common underlying question is how to *block* representations that are intuitively more or less implausible but equally well supported by available data. Such a question, however, can arise only if we are already confronted with certain predicates or general representations of data, and we must decide what is the better representation and how to exclude the implausible

one. In Peirce's terms, we are confronted with the problem of induction, but not with the problem of abduction.

The problem of abduction, on the other hand, is the problem *how to generate a representation at all*. How is it possible to obtain new ideas or hypotheses by abduction? Elsewhere, I have tried to show that a deeper understanding of abduction demands a "contextualization" of abductive reasoning. The idea is that any general representation of given particulars is determined not only by these data, but also by contextual constraints which limit the range of representations that are conceived as possible in a certain historical situation. (Hoffmann 2000a, Hoffmann 1998). It is *contextual* knowledge, our knowledge about the world, that excludes certain representations of given data and offers the possibility of hypotheses. Thus, we can say that *in practice*, the old problem of induction can be handled by taking into account *contextual constraints*. In other words, I suggest that a first step towards a solution of the learning paradox is to stop considering inductivism and apriorism as two mutually excluding alternative starting points, and to consider them instead as inherently intertwined within a common context.

Such an approach, I will argue, is developed in Peirce's concept of diagrammatic or schematic reasoning (see May 1995, Hoffmann 2000b, Stjernfelt 2000). According to Peirce, reasoning with the aid of diagrams is the essential feature of mathematics. There is no mathematical reasoning that is not diagrammatic (CP 2.216, CP 5.148, 162).

> For mathematical reasoning consists in constructing a diagram according to a general precept, in observing certain relations between parts of that diagram not explicitly required by the precept, showing that these relations will hold for all such diagrams, and in formulating this conclusion in general terms. All valid necessary reasoning is in fact thus diagrammatic (Peirce CP 1.54).

> Mathematics is the study of what is true of hypothetical states of things. That is its essence and definition....But mathematics, as a serious science, has, over and above its essential character of being hypothetical, an accidental characteristic peculiarity—a *proprium*, as the Aristotelians used to say—which is of the greatest logical interest. Namely, while all the "philosophers" follow Aristotle in holding no demonstration to be thoroughly satisfactory except what they call a "direct" demonstration, or a "demonstration why"—by which they mean a demonstration which employs only general concepts and concludes nothing but what would be an item of a definition if all its terms were themselves distinctly defined—the mathematicians, on the contrary, entertain a contempt for that style of reasoning, and glory in what the philosophers stigmatize as "mere" indirect demonstrations, or "demonstrations that." Those propositions which can be deduced from others by reasoning of the kind that the philosophers extol are set down by mathematicians as "corollaries"....In the theorems, or at least in all the major theorems, a different kind of reasoning is demanded. Here, it will not do to confine oneself to general terms. It is necessary to set down, or to imagine, some individual and definite schema, or diagram—in geometry, a figure composed of lines with letters attached; in algebra an array of letters of which some are repeated.... Accordingly, we may

> say that corollarial, or "philosophical" reasoning is reasoning with words; while theorematic, or mathematical reasoning proper, is reasoning with specially constructed schemata (Peirce CP 4.233).

This distinction between corollarial and theorematic reasoning was elaborated by Peirce as a distinction between two fundamental modes of diagrammatic reasoning (see Hintikka 1983, Shin 1997, Levy 1997). He defined them as two forms of necessary deduction:

> A Necessary Deduction is a method of producing Dicent Symbols by the study of a diagram. It is either *Corollarial* or *Theorematic*. A Corollarial Deduction is one which represents the conditions of the conclusion in a diagram and finds from the observation of this diagram, as it is, the truth of the conclusion. A Theorematic Deduction is one which, having represented the conditions of the conclusion in a diagram, performs an ingenious experiment upon the diagram, and by the observation of the diagram, so modified, ascertains the truth of the conclusion (Peirce CP 2.267; see also CP 7.204).

While corollarial reasoning, in principle, can also be performed by logical machines (Peirce MS 318: CSP p. 49 = ISP p. 41), the point of theorematic reasoning is its creativity. Indeed, creativity may be the common link between Peirce's concepts of theorematic reasoning and abduction. The problem with this is that theorematic reasoning is related to *deduction* rather than to abduction, a fact which has sparked a debate on the question whether there really is a clear-cut distinction between what Peirce has called "three absolutely disparate ways of reasoning" (Peirce NEM III, 177; see also Crombie 1997). This problem might possibly be resolved if we assume the following three points:

1. As Crombie (1997) has shown with regard to deduction, and others with regard to abduction (Anderson 1987, Hull 1994, Hoffmann 1999), a distinction ought to be made between the *process* of reasoning and the *logical forms* that can represent such processes.
2. While the logical forms of abduction, deduction, and induction are basically different (see Peirce CP 2.623, and the modification of hypothesis/abduction in Peirce CP 5.189), Peirce seems to consider the processes of reasoning themselves to be structured *recursively*. That becomes evident from Peirce's analysis of abduction in processes of perception, which reveals an infinite and continuous series of further abductive inferences within such processes (Peirce CP 5.181; see Hoffmann 1999, 284f.). Recursiveness also seems to be the clue when Peirce, in one of the manuscripts, distinguishes a corollarial and a theoric part even *within* theorematic reasoning, and claims that this "*theôric* reasoning...is very plainly allied to" what is called normally abduction (Peirce MS 754, ISP p. 8; see Hoffmann 1999, 293).
3. If reasoning processes are recursively structured in this way, it may be concluded that theorematic reasoning involves a combination of *deductive* and *abductive* aspects.

Not only could defining the relation between abduction and theorematic reasoning solve a problem that is being debated by Peirce scholars, but it could also offer a way to bridge the gap between mathematical and other forms of reasoning in Peirce. Certainly, there is enough reason to take his considerations on diagrammatic reasoning—which he elaborated only with regard to mathematics—as a paradigmatic model of knowledge acquisition and creativity in general.

I shall argue the thesis that Peirce's concept of diagrammatization represents an attempt to solve the problem of learning in terms of a *processual interplay of* a priori *given and inductivist aspects*. I shall do this by discussing eight essential points of what Peirce called diagrammatic reasoning:

A. The Roots of Peirce's Diagrammatic Reasoning are to be Found in Kant's Schematism.

Peirce uses the terms "diagrammatic" and "schematic" reasoning synonymously (see CP 2.778; CP 4.233; Rosenthal 1994, 23f.), and in some passages it is obvious that he uses "schematic" in the tradition of Kant's "schemata" as developed in the *Critique of Pure Reason* (CP 2.385, CP 5.531, CP 3.556, NEM IV, 318). For Kant, a transcendental schema is a "mediating representation" between categories and appearances, its task being to make "the application of the former to the latter possible." In order to subsume an object under a concept there must be a schema, that is, a "representation of a universal procedure of imagination in providing an image for a concept" (Kant CPR, A 137f.).

> Indeed it is schemata, not images of objects, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination, in respect to pure figures in space. Still less is an object of experience or its image ever adequate to the empirical concept; for this latter always stands in immediate relation to the schema of imagination, as a rule for the determination of our intuition, in accordance with some specific universal concept. The concept 'dog' signifies a rule according to which my imagination can delineate the figure of a four-footed animal in a general manner, without limitation to any single determinate figure such as experience, or any possible image that I can represent in concreto, actually presents. This schematism of our understanding, in its application to appearances and their mere form, is an art concealed in the depths of the human soul, whose real modes of activity nature is hardly likely ever to allow us to discover, and to have open to our gaze (Kant CPR, A 140f.).

The essential function of Kant's schemata may be seen as that of bridging the gap between the rational and the perceptual part of his epistemology, between his "Transcendental Logic" and his "Transcendental Aesthetic." But, in the eyes of Peirce, this gap was precisely Kant's "greatest fault," for

> ...he drew too hard a line between the operations of observation and of ratiocination. He allows himself to fall into the habit of thinking that the latter only begins after the former is complete; and wholly fails to see that even the simplest syllogistic conclusion can only be drawn by observing the relations of the terms in the premisses and conclusion (Peirce CP 1.35).

For Peirce, there is no dualism between *external* things that are given in "appearances" and the *internal* processes of *a priori* "ratiocination." Instead, the role which perception plays in both areas clearly shows there is a continuity between the two. More importantly, Peirce suggests that Kant's introduction of the schemata was itself an unfortunately belated attempt to remedy his flawed dualistic approach:

> His doctrine of the *schemata* can only have been an afterthought, an addition to his system after it was substantially complete. For if the *schemata* had been considered early enough, they would have overgrown his whole work (ibid.).

Keeping in mind both the extraordinary significance of Kant's philosophy for Peirce's development, and the central role epistemological questions play in his philosophy, one is tempted to conclude from this quote that Peirce did in fact design his own philosophy as an attempt to reformulate Kant's epistemology on the basis of schematic or diagrammatic reasoning. Certainly, Peirce sees great difficulties with Kant's approach to schemata. Thus, he notes with regard to the "Critique":

> Every detail is left in the rough; and there is no more unfinished apartment in the whole glorious edifice than that devoted to the Schematization of the Categories. Kant says that no image, and consequently we may add, no collection of images, is adequate to representing what a schema represents. If that be the case, I should like to know how a schema is not as general as a concept. If I ask him, all he seems to answer is that it is the product of a different 'faculty' (Peirce CP 5.531).

Indeed, the complex question, whether—and in what sense—schemata or diagrams are "general," and how "it can be that, although the reasoning is based upon the study of an individual schema, it is nevertheless necessary, that is, applicable, to all possible cases," (Peirce CP 4.233) is one of the central questions in Peirce's theory of diagrammatic reasoning. Contrary to Kant's schematism—but in accordance with Kant's emphasis on the role of constructions in mathematics (see Kant CPR, A 716)—Peirce stresses that schemas and diagrams must be observable empirical entities that can be objects of experiments and concrete operations. But in spite of this necessarily *external* character of diagrams, their construction involves a generality which makes diagrammatic reasoning the most promising candidate for making it possible to study the interplay of internal and external aspects in abduction and creative thinking.

Before discussing this in greater detail, we should mention a further important difference between Peirce and Kant, which is that Peirce attempts to overcome the transcendental *a priori* by his *evolutionary* approach (see Hausman 1993). Kant, he says, before asking, "How are synthetical judgments *a priori* possible?," "ought to have asked the more general [question], 'How are any synthetical judgments at all possible?' How is it that a man can observe one fact and straightway pronounce judgment concerning another different fact not involved in the first?" (Peirce CP 2.690; see also CP 5.348; CP 4.92). While Kant conceives the *a priori* forms and categories as a kind of absolute and unchangeable conditions of possible experience and "*Erkenntnis*," the central idea of Peirce's evolutionary philosophy is expressed in his claim that laws, general rules, and forms in themselves are the results of evolution (see Peirce W 4, 544–554). Peirce's generally *processual* orientation is the main feature that distinguishes him from Kant.

B. Diagrammatization of Thought is a Process of Fixing the Indefinite and Vague. It Permits Self-Control of Thought.

While Plato places his forms as *a priori* conditions of all knowledge at the *beginning* of the learning process, Peirce, the first evolutionary epistemologist, considers definite ideas as belonging to *the end* of an evolutionary process. Thus, he says in his *Cambridge Conference Lectures*:

> Looking upon the course of logic as a whole we see that it proceeds from the question to the answer,—from the vague to the definite. And so likewise all the evolution we know of proceeds from the vague to the definite.... The evolutionary process is, therefore, not a mere evolution of the *existing universe*, but rather a process by which the very Platonic forms themselves have become or are becoming developed (Peirce RLT, 258; see CP 6.203f.).

In this respect, Peirce's view sharply contrasts with, for instance, Fodor's view as expressed in his discussion of the learning paradox. An essential assumption in Fodor's argument against the possibility of learning is that the starting point of any cognitive development is a determinate and in principle computable set of propositions. Moreover, his view of the paradox of learning is based on the premise that there is always a clear-cut distinction between the various stages of cognitive development. For Peirce, however, the process of acquiring knowledge appears far more as a process leading from "absolutely undefined and unlimited possibility" (Peirce CP 6.217) to a piecemeal fixation of beliefs. In this continuous process from the vague to the definite, diagrammatization plays a fundamental role. "Let us construct a diagram," Peirce writes invitingly at the outset of his *Prolegomena to an Apology for Pragmaticism*, "to illustrate the general course of thought; I mean a System of diagrammatization by means of which any course of thought can be represented with exactitude" (Peirce CP 4.530). Thus, diagrammatization may be understood as a sort of self-controlled management of one's own thoughts, because the clarity and exactitude that is realized eventually in diagrams, in turn furthers the exactness of cognitive activities. Peirce defined logic as "the theory of self-controlled, or deliberate, thought" (Peirce CP 5.191), and he developed his system of Existential Graphs as a tool in representing thought with the greatest possible exactitude (see Roberts 1973). This System of diagrammatization, he says, is a system which "greatly facilitates the solution of problems of Logic....If logicians

would only embrace this method, we should no longer see attempts to base their science on the fragile foundations of metaphysics or a psychology not based on logical theory" (Peirce CP 4.571). Diagrammatization, in short, allows for a self-controlled process of thought in the fixation of originally vague beliefs.

C. Diagrammatization Creates Clarity by Reducing Complexity.

Peirce defines the "mathematician's business" as considering "the necessary consequences of possible facts," a business which is of interest for "any scientific or other inquiry," for engineers, business companies, physicists, etc. The mathematician does not have the "duty to verify the facts stated. He accepts them absolutely without question." He takes them as *possible* facts, and he is only interested in their *necessary* consequences. But frequently, as Peirce says, it happens

> ...that the facts, as stated, are insufficient to answer the question that is put. Accordingly, the first business of the mathematician, often a most difficult task, is to frame another simpler but quite fictitious problem (Peirce CP 3.559).

The mathematician performs this first task by "skeletonization or diagrammatization of the problem," the "principal purpose [of which it] is to strip the significant relations of all disguise" (ibid.). Thus, like a general who makes use of maps during a campaign, even "when the country they represent is right there," a mathematician makes use of diagrams (Peirce CP 4.530). Like maps, diagrams reduce complexity, thereby enabling concentration on essential relations and revealing those elements that are the most promising candidates for context-dependent interpretations.

D. Representational Systems Disarm the Internal-External Dichotomy.

Diagrams are a product of our intentions and we construct them on the basis of our *given* cognitive means. On the other hand, as Peirce emphasizes with regard to the work of the mathematician, the constructed diagram "puts before him an icon by the observation of which he detects relations between the parts of the diagram other than those which were used in its construction" (Peirce NEM III, 749). By experimenting upon the diagram, and by observing the results thereof, it is possible, as Peirce says, "to discover unnoticed and hidden relations among the parts" (Peirce CP 3.363). Thus, by observation of his diagrams the geometer is able "to synthesize and show relations between elements which before seemed to have no necessary connection. The realities compel us to put some things into very close relation and others less so" (Peirce CP 1.383; see also Kant CPR, A 716f.). Or in the words of Kathleen Hull, 1994: "The diagram becomes the something (non-ego) that stands up against our consciousness....All knowledge begins in experience" (p. 282; see Peirce CP 5.51).

But how is this possible? Do diagrams have a reality of their own, independent of our constructions? Peirce answers this question by making an important

distinction between what is going on in the mind and the compelling power of the chosen system of representation or diagrammatization: any diagram, he says,

> ...necessarily making use of a particular system of symbols—a perfectly regular and very limited kind of language. It may be a part of a logician's duty to show how ordinary ways of speaking and of thinking are to be translated into that symbolism of formal logic; but it is no part of syllogistic itself. Logical principles of inference are merely rules for the illative transformation of the symbols of the particular system employed. If the system is essentially changed, they will be quite different (Peirce CP 2.599).

The interesting point here is that constructing a diagram does not only depend on "internal" means, but also on the rules of the chosen representational system or "system of symbols." Any diagram, Peirce says, is constructed according to a general or abstractly stated "precept" (Peirce CP 1.54; CP 2.216). Representing something in diagrams is possible only by using a certain language, a certain notation or, more generally, a certain representational system which has a rationality of its own, a certain syntax and semantic. These rules and precepts followed in constructing diagrams, *overcome the distinction between the* a priori *given and the things out there.* Representational systems do have a reality independent of ourselves, for it is beyond our power to define their rules arbitrarily (Peirce CP 7.659), but because general precepts and rules are *our* means of constructing diagrams, they are not simply "outside" ourselves like perceivable particulars. Thus, representational means are both private and public: they are tools handed down by a culture which is ours, and represented in our thinking and acting. The rules and means of representational systems overcome the dichotomy between inductivism and apriorism.

E. The "Reality" of General Forms and Rules is a Necessary Condition of Diagrammatic Reasoning and of the Possibility of Learning.

One of the fundamental tenets of Charles Peirce's "pragmaticism," as he called his own form of pragmatism from 1905 on, is that general forms, laws of nature, and habits are *real.* He claimed to hold the position of an "extreme scholastic realism," where by "extreme" he meant that he thought it necessary to go even beyond Duns Scotus: "Even Duns Scotus is too nominalistic when he says that universals are contracted to the mode of individuality in singulars, meaning, as he does, by singulars, ordinary existing things. The pragmaticist cannot admit that" (Peirce CP 8.208; see Boler 1963). His argument for the *reality of universals* rests on the consideration that, unless the reality of such universals is assumed, it is impossible to explain our *expectations* regarding future events, which are the mainstay of our everyday experience (see Peirce CP 5.98–101). How could I get the certainty that I shall find my way home tonight, unless I have *a priori* confidence in the reality of my habits, of the world, and of its order? It would indeed be the end of any science if we were to live in the constant fear that our perception, our microscopes, our measurements, our instruments etc., function randomly. And so Peirce concludes: "everybody's actions show that it is impossible to doubt that there is an element of

order in the world; but the moment we attempt to define that orderliness we find room for doubt" (Peirce CP 8.208).

In order to understand what Peirce means by speaking of the "reality" of forms, laws, and habits, it is important to note the distinction he stresses between "reality" and "existence":

> [*R*]*eality* means a certain kind of non-dependence upon thought, and so is a cognitionary character, while *existence* means reaction with the environment, and so is a dynamic character; and accordingly the two meanings, he would say, are clearly *not* the same (Peirce CP 5.503; see also CP 8.191).
>
> The reality of things consists in their persistent forcing themselves upon our recognition. If a thing has no such persistence, it is a mere dream. Reality, then, is persistence, is regularity (Peirce CP 1.175).

Clearly therefore, in Peirce's view, "reality" must be ascribed properly to generals, and "existence" properly to particulars. Existence is of the order of what Peirce calls "secondness," while reality is of the order of what he calls "thirdness" (see also CP 1.23f., CP 5.93f., CP 5.121, CP 5.436, CP 8.330). The existent, for Peirce, is perceived as *hic et nunc*, and it "cannot be generalized without losing its essential character" (CP 2.146). And because an "expectation" of an individual event in the future presupposes a belief in the reality of general laws, rules, or habits "which will probably govern the individual event when it occurs" (ibid.), it presupposes their reality, even if there is no explicit idea of the generals which it involves.

An inevitable implication of Peirce's distinction between reality and existence is that the distinction must be made between the *assumption* of the reality of generals on the one hand, and the explicit *formulation* of such assumptions in the form of laws, concepts, and theories on the other. Because our theories are always fallible, Peirce's point of view must confront the problem that, for instance, there was a time when the famous "phlogiston" was "real," whereas, since Lavoisier, "oxygen" has been real. For the only criterion to distinguish "true" from "false" realities is our developing knowledge. Science determines what is real, and, as the work of Thomas S. Kuhn has shown, changes in science imply a change of our world. This insight constitutes an important aspect of what has been called "Peirce's Evolutionary Realism" (Hausman 1993).

But what does Peirce's realism mean for our question of creativity in diagrammatic reasoning? In the previous section, we have seen that any representational system we choose has its own particular rules. These rules and general precepts determine the construction of diagrams as well as the set of possible and legitimate *transformations* of diagrams. However, the possibility of such a determination depends on *our acceptance* of this determination. That is, we have to acknowledge the *reality* of these rules as norms for our operations with diagrams. Only on this account can it be said that the "realities [of relations between elements of a diagram] compel us to put some things into very close relation and others less so" (Peirce CP 1.383), which Peirce claims to be an essential feature of diagrammatic reasoning. The reality of rules and general forms is the reason for the

"compulsive power" of diagrams over our reasoning, and it explains that diagrammatic reasoning is not governed by chance alone.

F. Diagrams Enable Experimentation and the Use of Auxiliary Elements.

An essential feature of theorematic reasoning is that its use of diagrams allows for *experimentation*:

> Thinking in general terms is not enough. It is necessary that something should be DONE. In geometry, subsidiary lines are drawn. In algebra permissible transformations are made. Thereupon, the faculty of observation is called into play. Some relation between the parts of the schema is remarked. But would this relation subsist in every possible case? Mere corollarial reasoning will sometimes assure us of this. But, generally speaking, it may be necessary to draw distinct schemata to represent alternative possibilities. Theorematic reasoning invariably depends upon experimentation with individual schemata (Peirce CP 4.233).

As Hintikka (1983) has argued, the "introduction of auxiliary individuals into the argument" is an important aspect of theorematic reasoning, as was already mentioned by Kant (see Kant CPR, A 716f.). In this respect, Hintikka sees a "coincidence" between Peirce's distinction between theorematic and corollarial reasoning and his own distinctions between "non-trivial and trivial logical arguments" on the one hand (ibid.), and between "*definitory* rules" and "*strategic* rules" on the other (Hintikka 1997). The point was summarized by Fernandez as follows:

> Theorematic reasoning characterizes creative mathematical proofs (theorems) and essentially involves the experimental introduction of new elements, such as the auxiliary lines of geometric demonstrations, not *previously given* in the premises. In mathematical demonstrations these auxiliary elements act as catalysts in a chemical reaction: they elicit the process but do not appear in the final result (1993, 236).

G. Diagrammatization yields new Objects of Observation, Communication, and Experimentation.

Diagrams offer the possibility of representing abstractions and generalizations as new objects, that is, in Peirce's terminology, (see Otte 1997, 343f., 355; Otte 1998) as "hypostatic abstractions." As Peirce puts it, in diagrammatization

> ...the greatest point of art consists in the introduction of suitable *abstractions*. By this I mean such a transformation of our diagrams that characters of one diagram may appear in another as things. A familiar example is where in analysis we treat operations as themselves the subject of operations (Peirce CP 5.162).

For Peirce, hypostatic abstraction is at the heart of development in the history of mathematics. The same point is stressed by Salomon Bochner, who speaks of "untrammeled escalation of abstraction, that is, abstraction from abstraction, abstraction from abstraction from abstraction, and so forth" (Bochner 1966, 18).

H. Diagrammatization of a Problem by Means of Different Representational Systems Offers New Possibilities of Perceiving the Problem.

As David Bohm (1974) has shown with regard to Quantum Mechanics, the possible forms of describing, representing, and perceiving scientific facts are crucial for scientific discoveries:

> For example, it was widely believed in the nineteenth century that Newtonian dynamics and Hamilton-Jacobi wave theory of dynamics were 'essentially the same'. Nevertheless, we can now see that the difference between 'wave dynamics' and 'particle dynamics' was potentially of very great relevance in the sense that the former can lead in a natural way to quantum theory, while the latter cannot (1974, 383f.; see also Feynman 1967, 53f.).

Peirce's concept of theorematic or "theoric" reasoning (*i.e.* the creative part of diagrammatic reasoning) is based on just the same idea, which is the idea of "the transformation of the problem, or its statement, due to viewing it from another point of view" (MS 318: CSP p. 68 = ISP p. 225). Peirce takes the term "theoric" from the Greek ϑεωρία (theory) which he translates as "the power of looking at facts from a novel point of view" (ibid., CSP p. 50 = ISP p. 42). For Peirce, the most important discoveries in mathematics are based on such obtaining of new perspectives, as he shows in his 1907 manuscript about "Pragmatism" with the example of the proof of the "ten points theorem" (ibid., see also NEM III, 870f., and Hoffmann 2000a).

In the philosophy of science, the significance of representing the same facts in different ways was emphasized, for instance, by Ian Hacking who suggested that the idea goes back to Heinrich Hertz who in his *Prinzipien der Mechanik* (1894) developed the concept of "scientific image" which later would influence Wittgenstein's "image theory of meaning" in the *Tractatus* (Hacking 1983, 143f.; see also Nersessian 1998, and Putnam 1983, with respect to mathematics).

6. An Example of Diagrammatic Reasoning

In order to illustrate the significance of these eight points regarding diagrammatic reasoning, an example may be useful. In a passage from Plato's *Meno*, Socrates attempts to prove the theory of Anamnesis by a discussion with a slave boy whose answers concerning certain mathematical questions are taken as hints at a process of recollection (Plato *Men.*, 82b–86c). It seems to be far more appropriate, however, to interpret this lesson in mathematics as a demonstration of Peirce's claim that "Diagrammatic reasoning is the only really fertile reasoning" (CP 4.571). Socrates asks the following question: Given a square figure of 2 by 2 foot (and an area of 4 square feet), what is the length of the side of a square that is double the size of the original square? The boy's first suggestion is that the side should be 4 foot long.

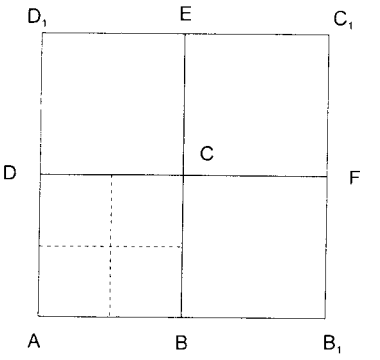

Fig. 1: How is it possible to construct a square double the area of the square ABCD?

Obviously, the boy has but a vague notion of the problem. The system of representation he uses to give an answer is of a rudimentary arithmetical type, for he fails to see the difference between the duplication of areas and the duplication of sides. But what does Socrates do in order to demonstrate that this answer is inadequate? He draws a diagram, thus constructing an external representation of the boy's ideas. By looking at the diagram, it becomes evident at once that doubling the side does not double the original square, but yields a square that is four times its size (see Fig. 1).

As Peirce would say, the constructed diagram "puts before him an icon by the observation of which he detects relations between the parts of the diagram other than those which were used in its construction" (NEM III, 749). On the one hand, the diagram *externalizes* the boy's thought; thus clearly presenting what had been only a vague notion before. On the other hand, however, the diagram *makes visible* what was wrong with the boy's original suggestion, for by drawing the diagram according to the general precepts of geometry some compelling realities become visible. Therefore, by *changing* the representational system, by substituting the geometrical approach for the arithmetical approach, the boy gets the possibility of controlling his own thought, and making it more precise.

But Socrates' question still remains unanswered. However, because the boy sees that the length of the side must be somewhat between the length of the side of the original figure and that of the figure given in his first answer, he suggests that is must be 3 feet long, which is exactly the middle between 2 feet and 4 feet. Again, he has adjusted his representational system; though discussing mathematical problems in terms of arithmetic still seems to be the most familiar to him. Now, how can we solve the problem?

By experimenting a little with the diagram, and by considering that, a square of 8 square feet being half the size of a square of 16 square feet—which was the size of the square of his first answer—he might eventually come to see that it is possible to *construct* the length of the side Socrates had been asking for (see Fig. 2).

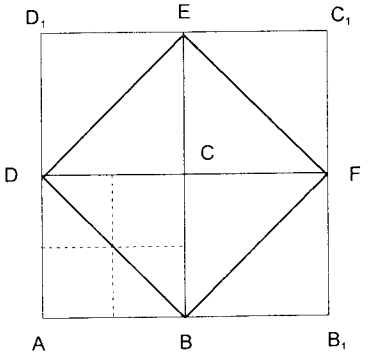

Fig. 2: The duplication of ABCD by experimentation, with hypostatic abstraction of the new concept "diagonal"

The diamond square which appears in the middle of the large square is half the size of the large square, because each of the latter's four parts was cut in half. Thus, the length of the side which Socrates asked for must be the length of the line DB, a line which, as Socrates points out, "the experts call the diagonal" (διάμετρον, p. 85b4). Thus, the process of finding the solution may be seen as a paradigm of what Peirce called "theorematic reasoning," because the solution to the problem needs an "ingenious experiment upon the diagram." The entire process of diagrammatization enables the boy to learn not only some rules of geometry, but also the new concept of "diagonal." The solution is obtained through what Peirce called "hypostatic abstraction," for a part of the diagram becomes an entity of its own, a new object, which can in turn become the subject of further operations.

Thus, the solution to the learning paradox may reside in seeing that learning occurs through the *process* of diagrammatic reasoning, the analysis of which unmasks the dichotomy of apriorism and inductivism as both superfluous and fundamentally ill-conceived.

Notes.

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Process Pragmatism

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