Dissipative shallow water internal waves and their acoustical properties

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DISSIPATIVE SHALLOW WATER INTERNAL WAVES 
AND THEIR ACOUSTICAL PROPERTIES

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Abstract - Zhou and colleagues [J. Acoust. Soc. Am., 44, 2042-2054 (1991)] showed that nonlinear shallow water internal waves (IW's) have the frequency dependent property of enhancing bottom interaction of sound. At preferred frequencies, internal waves can induce the coupling of lower-order waterborne modes to higher order modes, which, in turn, can penetrate more deeply into lossy ocean bottom sediments. In a follow up study Broadhead [J. Acoust. Soc. Am., Submitted (1995)] analyzed a simplified example of a shallow water waveguide in which internal waves were incorporated through the use of KdV solitons. A frequency-dependent enhancement of acoustic energy transfer from below the thermocline into the mixed layer was observed from the simulations. Analysis revealed that mode coupling induced by the presence of the internal wave was responsible for the energy transfer. This paper extends work by including dissipative effects on the IW's, and their concomitant effects on acoustic propagation. Simulations of the effect of dissipation on IW's was accomplished by numerical solution of the KdVB equation. Especially of interest is the effect of IW pulse broadening on the mixed layer resonance. This led to an increase in the resonant acoustic frequency.

INTRODUCTION

Theory and experiment have been under development for a number of years in the study of shallow water internal waves in coastal environments (refer, for example, to the bibliographies in Ostrovsky and Stepanyants [1] and Zhou et al. [2]). It is well known that shallow water nonlinear internal waves (IW's) that possess the characteristics of solitary wave trains have been observed and theoretically described. These internal solitary waves have been observed in numerous coastal regions around the world and share some of the following general properties: they a) tend to be highly spatially and temporally localized, b) are not described by linear theories such as a Garrett-Munk spectrum, c) tend to be generated by the interaction of barotropic tidal components with bathymetric features, d) tend to propagate uni-directionally, shoreward e) can, when shoaling near shore, develop instabilities that contribute to mixing via turbulent dissipation, f) tend to be characterized by length scales on the order of hundreds of meters, speeds on the order of 0.5 m/s, and amplitudes on the order of tens of meters.

For many years, the main emphasis of studies into the effects of deepwater IW’s on acoustic propagation was on spatial and temporal fluctuations. However, it is now known, through the work of Zhou and colleagues [2-3] that shallow water IW’s can affect the mean sound level as well - in some cases by as much as 25 dB. The primary focus of recent low frequency (< 1000 Hz) ocean acoustics research on shallow water solitary IW’s has been the anomalous drop in sound level (or, increase in transmission loss (TL)) at a resonant (or preferred) frequency [2-4].

We should first mention some of the conditions necessary a) for the presence of internal waves and b) for a noticeable acoustic effect. For a), a density contrast is needed, which is provided in summer months by the establishment of a stable mixed layer of warm, less dense water overlying colder, denser water. Temperature contrasts may be as large as 10°C, while density contrasts are usually on the order of one part per thousand in g/cm³. The pycnocline coincides with the thermocline and the temperature contrast provides a sound speed contrast. The displacement of a higher sound speed undulation into the surrounding lower sound speed layer gives rise to the acoustic effects.

Zhou et al. analyzed acoustic transmission data from the Yellow sea [2-3] where, from the explosive spectrum over 0 - 2000 Hz, it was noticed that, around 600 Hz, there was an anomalous, sharp drop-off in sound level. Their hypothesis was that interaction of the sound with a series of shallow water internal wave packets caused the resonant loss. Their acoustic propagation modeling led them to the conclusion that, in the preferred frequency band, the
IW's induced mode coupling, which in turn, transferred energy to higher order modes that more easily penetrate into the lossy bottom, causing a frequency-dependent energy removal from the water column. This is termed resonant bottom interaction. These calculations and conclusions were further supported by work by Chin-Bing and colleagues [4]. It should be noted that the strong thermocline conditions of the Yellow Sea data, combined with the bottom type, conspired to produce such a large effect.

Broadhead [5] did a follow up simulational study to that of Zhou et al. by including IW effects with a Korteweg-de Vries (KdV) solitary wave. Shallow water IW's tend to be weakly dispersive, but retain their form while propagating due to a balance struck between dispersion and weak nonlinearity. This is the defining characteristic of solitary waves. For a two-layer density stratified fluid (a good approximation to Yellow Sea Summer profiles), treatment of the finite amplitude internal wave problem at the density interface leads, up to first order nonlinear effects, to solitary waves and the KdV equation (see bibliography in Reference [5]).

![Acoustic environment showing the 1-soliton IW simulation.](image)

**FIG. 1.** Acoustic environment showing the 1-soliton IW simulation.

The IW's were represented [5] in the acoustical index of refraction by the displacement of the sound velocity field, where the thermocline in a temperature (sound speed) stratified fluid is coincident with the pycnocline. One-way solutions to the Helmholtz equation were generated with the model KRAKENC [6] for a shallow water (38 m) waveguide over a lossy acoustic halfspace. The frequency and range dependent effects of the simulated IW packets on acoustic propagation were obtained using both adiabatic and coupled mode options. The main result observed was the transfer of acoustic energy from below the thermocline into the mixed layer, at two resonant frequencies (940 Hz and 1005 Hz). A mode coupling analysis revealed that the same type of mechanism was at work in this resonant transfer of energy as in the bottom coupling case analyzed by others.

In this present work, we consider some of the mechanisms for dissipative processes in shallow water internal solitary waves, the simulation of dissipative effects on the IW's, and how these effects may affect acoustic propagation. Simulation of the effect of dissipation on IW's was accomplished by numerical solution of the KdV-B equation. We concentrate on one particular effect - peak broadening, and study its influence on the double resonance structure mentioned above. It was found that there was an upward shift in the resonant frequency of about 20 Hz when the IW width was approximately doubled. We proceed in the rest of this paper by first reviewing results from Ref. [5]. We then present new results for the inclusion of dissipative effects. We close with a discussion and conclusions.

**INTERNAL WAVE PROPERTIES AND SIMULATION**

For summer conditions in shallow water, near the shore, there is typically a mixed layer of warm, less dense water overlaying a colder, denser layer. The density contrast is small (on the order of 10^{-3} g/cm^3) and due mostly to temperature difference. However, because of the low compressibility of water, this small difference is sufficient to support internal waves (interface waves along the pycnocline) of significant amplitude - usually on the order of tens of meters. Perturbation theory for shallow water conditions ($\lambda_{int} \gg H$) leads to the KdV equation [1]. Here, $H$ refers to total fluid thickness and $\lambda_{int}$ is the characteristic length scale of the IW's (refer to Fig. 1). The wave amplitude must be small (but finite) with respect to $H$ for perturbation theory to be valid.

After Ostrovsky [1], we can write the KdV equation in the form appropriate for two-layer conditions:

$$u_t + cu_x + cu_{xx} + \beta u_{xxx} = 0$$  \hspace{1cm} (1)

where $u(x,t)$ is the displacement of the pycnocline from equilibrium level, and

$$c = \left[ \frac{g(\rho_2 - \rho_1)h_2h_1}{\rho_2h_1 + \rho_1h_2} \right]^{1/2},$$  \hspace{1cm} (2)

$$\alpha = \frac{3}{2} \frac{c}{h_2} \frac{\rho_2h_1^2 - \rho_1h_2^2}{\rho_2h_1 + \rho_1h_2}.$$  \hspace{1cm} (3)
\[ \beta = \frac{c h_2}{6} \frac{\rho_1 h_1 + \rho_2 h_2}{\rho_2 h_1 + \rho_1 h_2}. \]  

The KdV coefficients are defined as follows: \( c \) is the linear phase speed, \( \alpha \) is the nonlinear coefficient, \( \beta \) is the dispersion coefficient, \( \rho_1, \rho_2 \) are the densities of upper and lower layers, respectively, and \( h_1, h_2 \) are layer thicknesses. If a traveling wave solution is assumed, along with appropriate assumptions about integrability, then the KdV equation can be directly integrated [7] to yield the famous sech-squared solitary wave solution:

\[ u(x,t) = u_0 \text{sech}^2 \left( \frac{x - Vt}{\Delta} \right), \]

where \( V \) is the soliton velocity, \( \Delta \) is the soliton half width,

\[ V = c + \frac{3\nu_0}{\alpha}, \]

and

\[ \Delta^2 = \frac{12\beta}{\alpha_0}. \]

The solitons are waves of depression if \( \rho_1 h_1^2 < \rho_2 h_2^2 \).

We chose reasonable parameters for a shallow water case (similar to Zhou’s Yellow Sea case). The length scale of the main solitary wave is \( \sim 150 \) m, with an amplitude of \( \sim 10 \) m.

**LOSS MECHANISMS IN SHALLOW-WATER INTERNAL WAVES**

The subject of internal wave energetics has received considerable attention in the last decade or two (see, for example, the reviews by Gregg and Briscoe, [8], Levine, 1983 [9], Obers, 1983 [10], Müller et al., 1986 [11], as well as their extensive reference lists). The principal reason for studying internal waves is their suspected role in the dynamics of the ocean, especially in affecting the large-scale general circulation and in providing a link in the presumed energy cascade from the large, generation (“source”) scales to the small, dissipation (“sink”) scales. Internal waves are believed to provide an important link in this cascade because of their unique ability to convert two dimensional motions that are prevalent at large scales to three-dimensional motions prevalent at small scales [11]. Here, we are concerned primarily with the loss mechanisms of solitary internal waves, rather than the means of their generation. A complete understanding of the loss mechanisms of solitons is still lacking. However, it is known that shear-induced turbulent dissipation, frictional drag in the bottom boundary layer, radial spreading, and coupling into other scales of wave motion (such as barotropic components) are some of the major sources of energy loss from the solitary wave.

Dissipation is an important end product of the energy cascade manifested in continental shelf areas. In such areas, the interaction of the large scale oceanic surface tide with topographic features generates a shorter scale internal tide. Given enough time, the internal tide evolves into shorter undulations such as internal bores and solitons. Although only a small percentage of the surface tide energy is lost to the internal tide, it is significant for the relatively short scales of the internal tide. Much, if not most, of the internal tide energy is converted to the short waves. The small horizontal scales and large amplitudes of these short waves mean that both wave orbital currents and current shear can become large, which can lead to instability and loss of wave energy by turbulent dissipation (Sandstrom & Oakey, [12]). The shear-induced instability is characterized by the Richardson number, Ri, which is the ratio of the buoyancy forces to the mean vertical shear. When Ri < 1/4 the shear forces may overwhelm the restraining buoyancy forces, leading to instability and turbulent dissipation, and hence to mixing of the water column. This can subsequently lead to a thickening of the pycnocline (Bogucki and Garrett, [13]). Sandstrom and Elliot [14] have pointed out that the IW-associated vertical mixing may act like a “nutrient pump,” supplying nitrates to the euphotic zone in the surface layer.

The dissipation of shallow-water internal waves has been investigated by Sandstrom et al. using their observations on the Scotian Shelf [12, 14]. Repeated samplings of a packet of solitary waves produced the first direct measurements of turbulent dissipation due to the waves in a part of the water column that includes the pycnocline. Enhanced turbulence and mixing were observed to occur in the strongest shear region of the wave packet, consistent with the instability characterized by the Richardson number. It was estimated that 20% of the energy lost by the decaying soliton could be attributed, in the case studied, to turbulent dissipation in the pycnocline. Turbulent dissipation outside the pycnocline was relatively small. Of the remaining 80% of bulk dissipation it was estimated that approximately one-fourth to one-third was lost by bottom friction and the rest to other scales of motion and/or radiation. Sandstrom and Oakey [12] derive the Korteweg-de Vries-Burgers (KdVB) equation (first used in this context by Liu et al. [15]) and use it to comment on the relative effects of nonlinearity, dissipation, and dispersion. They point out that the validity of the use of the KdVB equation as the appropriate 1-D model problem for adding dissipation to the 1W problem is predicated on the assumption that the dissipative time scale is much larger than the dispersive time scale. Otherwise, the large amplitude short waves will be unable to re-adjust their shapes and speeds to the
changing environmental conditions, and hence will lose their soliton form. For example, if the dissipative time scale is comparable to the dispersive time scale, cnoidal rather than solitary waves will emerge from the internal tide. Sandstrom and Oakey note that in their experiment the mean dissipation time scale was 12 hours, while the dispersive time scale was less than 1 hour. They note further that both time scales decreased as the soliton moved onshore.

In an investigation of nonlinear internal wave evolution in the Sulu Sea, Liu et al. [15] examined the relative importance of variable bottom topography, radial spreading, and dissipation on soliton decay. They extended KdV solitary wave theory to include radial spreading and dissipation. Using measured data and a numerical parametric study, they demonstrate that the variable depth effect is appreciable, and that radial spreading and dissipation are significant. The direct effect of bottom friction on soliton dissipation was small. They note also, that because eddy viscosity is not a property of a fluid, it may be a function of depth: that is, strong turbulent mixing in shallow water due to shoaling effects can increase the local value of the eddy viscosity coefficient.

The relative importance of shoaling and dissipation was examined further by Liu, based on an investigation of solitary internal waves in the New York Bight [16]. Liu uses an extension the KdVB equation to also include an estimate of the effect of shoaling. He found that the relative balance between dissipation and shoaling effects was crucial to the detailed evolution of the wave packet. Shoaling, which depends on the shelf slope, density, water depth, and shear, can have a significant effect in increasing the amplitude of solitons on the continental shelf. On the other hand, dissipation in the thin pycnocline, which Liu parameterizes in terms of an eddy viscosity, was shown to erode the sharp peaks of the large solitons, reducing their amplitudes and increasing their apparent widths at the same time. Radial spreading effects will also be important if the generation area is relatively confined. Therefore, Liu notes, the shoaling effect eventually could be suppressed by the dissipation and radial spreading. Thus, shoaling can, depending on the particular environmental conditions, either suppress turbulent dissipation or enhance it.

The effect of bottom friction on the propagating soliton also appears to be highly dependent on the environment. In some measurements (e.g., off the Scotian Shelf) it was an important loss mechanism, whereas in others (e.g., in the Sulu Sea) it was insignificant. Sandstrom and Elliot [14] suggest that the relatively deep water in the Sulu Sea may explain the smaller role of bottom boundary layer friction in that case. In particular, they point out that loss in the bottom boundary layer is calculated by integrating, over the wave extent, the product of the bottom current speed and the stress just above the boundary layer. Since the stress depends on the square of the current speed, the bottom friction is strongly affected by bottom current speed, which is increased in shallow water shelf areas.

An investigation by Sanford and Grant [17] has shed some additional light on the question of bottom friction. They present a method for theoretically estimating the magnitude of bottom frictional dissipation under fairly general circumstances. They claim that dissipation of internal wave energy in the bottom boundary layer is of first-order importance for the internal wave energy balance on the continental shelf. Environmental factors shown to affect the relative importance of bottom frictional energy dissipation include the vertical density structure, the bottom slope/shelf width, the relative strength of current components at higher and lower frequencies, and the order of magnitude of the bottom roughness.

ACOUSTICAL SIMULATIONS: BACKGROUND

For linear acoustic wave propagation due to a harmonic point source in an azimuthally symmetric ocean waveguide, the appropriate wave equation governing the excess pressure $p$ is the two-dimensional Helmholtz equation

$$
\rho \frac{\partial}{\partial r} \left( \frac{r \frac{\partial p}{\partial r}}{\rho} \right) + \rho \frac{\partial}{\partial z} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} \right) + \frac{\omega^2}{c^2} p = \frac{-\delta(z-z_0)\delta(r)}{2\pi r}
$$

where, in general, the density $\rho = \rho(r,z)$ is a function of range and depth, as is $c$, the sound speed. The point source, located at the $r$-origin at depth $z_0$, has harmonic time dependence $\exp(-i\omega t)$, where $\omega = 2nf$, and $f$ is frequency in Hz. The normal mode model KRAKENC [6] was used to obtain one-way (outgoing) solutions to Eq. 12) for the environmental parameters given in Fig. 1. Two methods were employed: 1) adiabatic normal modes and 2) one-way coupled modes. The adiabatic normal modes solution is valid when the environmental parameters change sufficiently slowly, i.e., when mode coupling can be ignored. It is given by

$$
p(r,z) = \frac{i}{\sqrt{8\pi r}} e^{-i\pi/4} \sum_{m=1}^{\infty} Z_m(z_0)Z_m(z,r) e^{i\int_0^r k_m(s)ds} / \sqrt{k_m(r)},
$$

where $Z_m(z,r)$ are the local modes, and $k_m(r)$ are the horizontal wave numbers.
Note, in practice, we subdivide the environment into a set of discrete ranges, where the modes are calculated in a range independent fashion, and then KRAKENC interpolates the modes and wave numbers to evaluate the complex pressure field.

\[ p^i(r, z) = \sum_{m=1}^{M} \left[ a^i_m H_{0}^{(1)}(k_m r) + b^i_m H_{0}^{(2)}(k_m r) \right] Z^n_m(z). \]  

where \( H_{0}^{(1)}, H_{0}^{(2)} \) are zeroth order Hankel functions of the first and second kind. What remains is to use boundary conditions to match the solutions at adjoining segment interfaces - i.e., continuity of pressure and radial particle velocity, respectively. Also needed are the source condition \((r=0)\) and the Sommerfeld radiation condition \((r \to \infty)\). This leads to a banded block structure system of equations for the amplitudes \(a^i_m, b^i_m\). This global approach is cumbersome, and can often be replaced, without significant degradation, with a one-way (no backscattering) coupled modes formulation that allows a range marching implementation [6].

![Graph 2](image2.png)

**FIG. 2.** Comparison of TL for 1-soliton environment at 940 Hz: range independent (topmost figure), adiabatic modes (middle), and coupled modes (bottom).

When mode coupling cannot be ignored, as is the case in examples shown later, coupled mode theory can be used [6]. The environment is approximated by subdividing it into range-independent segments. For the \(j\)th segment, the (two-way) normal mode solution can be written as

![Graph 3](image3.png)

**FIG. 3.** Horizontal TL slices at receiver depth of 5 m corresponding to Fig. 2. Curves for range independent, adiabatic and coupled modes are shown.

We should briefly remark on KRAKENC's handling of absorption. The "C" refers to "complex". That is, rather than a perturbational treatment of the imaginary part of the horizontal wave numbers, the full complex eigenvalue problem is solved. An additional benefit is achieved through the choice of allowed phase speeds, enabling KRAKENC to include continuum approximations by making use of the leaky-mode approximation. These terms can be important when mode-coupling re-stimulates continuum contributions at potentially large source-to-receiver ranges \(r\).
SIMULATION OF SOUND PROPAGATION THROUGH INTERNAL WAVES

In Fig. 1, the acoustic environment parameters are given, as well as the displacement of the thermocline that simulates the presence of a solitary internal wave [5]. Note, we have used a 38 m water depth, and a 13 m mixed layer (ML) depth. The parameters are generic, and not representative of a specific area.

The given environment was input into KRAKENC and TL calculations were performed at a number of frequencies. We noticed various interesting phenomena, but concentrated on what occurred at 940 Hz.

Figure 2 shows the result of this calculation, where the TL is displayed for 3 km of source-receiver separation, for a source depth of 25 m, which is below the thermocline [5]. The topmost figure, represents the normal mode solution in the wave guide in the absence of an IW packet. When the IW is present, we have two choices of range dependent calculation with our model: 1) adiabatic normal modes and 2) coupled modes, which are displayed in the middle figure and bottom figure, respectively.

The propagation feature we call to the reader's attention is the enhanced propagation in the mixed layer (ML, hereafter), due to the presence of the IW (coupled modes case). Actually, we should be more precise - due to the nature of this case (as is clear form the figure) energy does not stay trapped in the ML, but tends to be confined between the ML and the bottom. This is because there is no surface duct to trap energy in the ML. Subsequently, the high angle rays penetrate through the thermocline, on into the bottom and "drop out" of the problem. The lower grazing angle rays have trouble getting back into the ML due to the increase in reflection coefficient at the thermocline. Finally, there is the subsequent decay of the energy left in the ML due to geometrical spreading. These above issues are all well understood. Hence, what is really being observed is a re-introduction of energy into the ML by the IW. It should be made clear that this energy will eventually "leak" out again if no surface ducting is present.

Thus, for the scenario studied, the coupled modes solution predicts an enhanced transfer of acoustic energy from below the thermocline into the ML at least for propagation frequencies around 940 Hz and 1005 Hz. The fact that this phenomenon is missing
in the adiabatic modes calculation is initial evidence that the IW is inducing mode coupling, which is, in turn, responsible for the energy transfer phenomenon. We will analyze this in more detail as we proceed. We should point out the similarity of the adiabatic normal modes and range-independent solutions for the cases presented. In Fig. 3, we take a more detailed look at the calculations by considering a horizontal slice through the 2-D TL surfaces in Fig. 2. The energy transfer already mentioned is evident. We point out the obvious fact that the three curves must overlay until the soliton is reached in range (due to our use of one-way calculations).

In Fig. 4(a), we display the results of calculations for TL vs. frequency. The main feature to point out is the peak at 940 Hz for the coupled modes case, and the decay of the energy transfer on either side of the peak [5]. This is made more evident by the difference curve in Fig 4(b) (where the peak has been shifted a bit to 945 Hz). Note, a second resonance at 1005 Hz is present, and keep in mind that we are considering a rather narrow portion of the spectrum.

FIG. 6. Mode functions 1, 3, 5, and 8 for receiver range 1500 m. Note, modes 1, 3, and 5 are mainly confined to below the mixed layer, while mode 8 has a larger concentration of energy in the mixed layer.

The doublet nature of the spectral structure over the bandwidth shown will not be pursued further in this paper, though it is currently under investigation. We shall, rather, concentrate on a frequency of 940 Hz. We now proceed with the methodology and results of a mode coupling analysis.

**MODE COUPLING ANALYSIS**

To recover the modal amplitudes, we use the following expression

\[
a = U^T p
\]

where \( p \) is a vector of pressures at a fixed range, \( a \) is a vector of modal amplitudes and \( U \) is a matrix whose columns are the appropriate (local) mode functions. These amplitudes can be used to explore the nature of possible mode coupling that is playing a role in the propagation.

In Fig. 5, we display the modal amplitude calculated from the vertical pressure at the 1500 m range, for 940 Hz, and a 1-soliton IW packet [5]. There are several points to note: 1) three modes, modes 1, 3, and 5, dominate the propagation in the range independent case. 2) These three modes carry less energy when the IW is present (and mode coupling allowed), whereas the higher modes are enhanced (especially mode 8). This suggests a coupling of modes 1, 3, and/or 5 to at least mode 8. These four modes are the key players. Further confirmation of this can be seen in Fig. 6(a) - (d), where we display the modes in question. Modes 1, 3, and 5 are largely confined to the wave guide region between the ML and the water bottom. However, mode 8, which is stimulated by the IW at the expense of the other three, has a large component in the ML.

FIG. 7. Length scale bounds associated with the main solitary wave.

Following a procedure analogous to that used by Zhou et al. [2-3], we can use the mode coupling theory to tie the characteristic spatial scales of the IW packets to the normal mode spatial scales represented by horizontal wave numbers. This theory predicts significant coupling between modes that obey the relation

\[
k_{\text{int}} = k_n - k_m
\]
where \( k_n \) - Horizontal wave number corresponding to \( n \)th mode, and \( k_{\text{int}} \) - Wave number corresponding to appropriate length scale of internal wave.

The length scales that bracket the main solitary wave in Fig. 7 correspond to the following wave number interval [5]:

\[
[k_{130m}, k_{60m}] = [0.042, 0.105] \tag{13}\]

The horizontal wave numbers obtained from KRAKEN for modes 1, 3, 5 and 8 obey:

\[
\Delta k_{i,8} = 0.084, \tag{14a}
\]
\[
\Delta k_{3,8} = 0.071, \tag{14b}
\]
\[
\Delta k_{5,8} = 0.045, \tag{14c}
\]

which fall within the above wave number range for the IW. This indicates a general consistency with mode coupling theory for our main solitary wave. It also confirms the role that coupling of modes 1, 3, and 5 to mode 8 plays in the effect.

![Graph](image)

**FIG. 8.** Time evolution of KdVB equation for \( \mu = 0.01 \) and \( N = 3 \) in the initial condition. Amplitude decay rate is a function of amplitude. \( t = 0.69 \)

### INCLUSION OF DISSIPATIVE EFFECTS

As mentioned, the well-studied shallow, finite amplitude, 2-layer internal wave problem leads to the KdV equation. One begins with the equations for inviscid, incompressible flow (Euler’s equation and continuity equation) and, through the perturbative method of multiple scales, arrives to first order nonlinearity, at the KdV equation. If we include viscous effects, then we need the Navier-Stokes equations

\[
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u}, \tag{15}\]

along with the incompressibility condition

\[
\nabla \cdot \mathbf{u} = 0, \tag{16}\]

where \( D \) is the convective derivative

\[
\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla \tag{17}\]

and \( \mu \) is the first coefficient of viscosity. The vector \( \mathbf{u} = [u, v, w] \) is the fluid particle velocity.

These equations are different from the inviscid case only by the dissipative (or diffusive) term \( \mu \nabla^2 \mathbf{u} \). Of course, we also have to use no-slip boundary conditions. In fact, for our purposes, this term and the convective term \( (\mathbf{u} \cdot \nabla) \mathbf{u} \), and their relative interplay will be the most essential features we consider. The simplest 1-D model problem for representing convection/diffusion problems, or from our perspective, nonlinear dissipative wave phenomena, is Burgers equation [7,18]

\[
u_t + u u_x = \mu u_{xx} \tag{18}\]

where we retain \( \mu \) to indicate the dissipation coefficient, but no longer identify it strictly with viscosity. A slightly more general form of Burgers equation (the equivalence can be made through a simple change of variables) is

\[
u_t + u_x + uu_x = \mu u_{xx}. \tag{19}\]

It is instructive to consider the linearized dispersion relation, obtained by substituting a plane wave \( e^{i(kx - \omega t)} \) into Eq. 19 sans the nonlinear term. This gives

\[
\omega = k - i\mu k^2 \tag{20}\]

which, upon substitution back into the plane wave, gives

\[
e^{i(kx -(k-\mu k^2)t)} = e^{i(kx-t)} e^{-\mu k^2 t} \tag{21}\]

The presence of the damping exponential term indicates how \( u_{xx} \) leads to dissipation.

Burgers equation is well studied [7] and, through the Cole-Hopf transformation, can be put into the form of the heat equation, which allows one to obtain
analytical solutions. The steady-state shock wave solution can be written [19]

\[ u(x) = -u_\infty \tanh \left( \frac{u_\infty x}{2\mu} \right), \quad u_\infty > 0 \quad (22) \]

This smooth shock profile transition from upstream to downstream conditions points out the balance between the nonlinear term, which tends to a discontinuity, and the diffusive term, which "regularizes" the equation (keeps the derivatives finite).

\[ \text{FIG. 9. Time evolution of KdVB equation for } \mu=0.01 \text{ and } N=3 \text{ in the initial condition (solid) compared to no dissipation (dashed) for } t=0.544. \text{ Packet spreading is inhibited and relative amplitude levels compressed.} \]

Let us now compare the KdV and Burgers equations

\[ u_t + uu_x + u_{xxx} = 0 \quad \text{(KdV)} \quad (23) \]

\[ u_t + uu_x - \mu u_{xx} = 0 \quad \text{(Burgers)} \quad (24) \]

In the former case, the dissipative term is missing, in the latter the dispersive term. A natural generalization to weakly nonlinear, dispersive, dissipative systems is to consider the model problem

\[ u_t + uu_x + u_{xxx} = \mu u_{xx} \quad (25) \]

which is called the KdVB equation (see Ref. [19] for a bibliography of papers concerning this equation). For KdV, we have a balance between convective and dispersive effects, for Burgers, between convection and dissipation.

What does the solution space of Eq. 25) look like? For steady state, it has been shown (see Ref. [19]) that when dissipation dominates, we have a monotone shock wave; as dispersion becomes relatively stronger, the monotone shock becomes an oscillatory shock (good for modeling undular bores). When dispersion dominates, the oscillatory shock becomes a periodic (coidal) wave, which in the limit can become a solitary wave of the sech-squared form.

The accommodation of undular bores and coidal waves in the KdVB model for dissipative IW's is more realistic, ultimately, than approaches that modify the coefficients of the KdV to include viscous effects (e.g., Das and Chakrabarti [20]). However, the energy exchange from internal tides to these and more complex forms of propagation and dissipation (such as turbulence and mixing) are beyond the scope of this study (and, in reality, beyond the serious scope of the KdVB equation).

\[ \text{FIG. 10. a) KdVB evolution for } \mu=0.01 \text{ and } N=1 \text{ in the initial condition. } t=5.6 \text{ b) Overlay of first pulse (solid) and last (dashed). Dashed curve has been shifted and scaled for clarity. Points out pulse broadening.} \]

Hence, we will continue the analysis for the rest of this paper in the spirit of modifying the coefficients of KdV, which means, practically, choosing a time scale that does not allow the KdVB to reach the steady state. We proceed by looking at well-posed initial value problems, and consider the numerical solution of Eq. 25).

Starting with a slightly more canonical form for the KdV part, we write

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To solve this equation, we modified Goda's [21] finite difference scheme for the KdV equation to include the dissipative term, giving the following implicit scheme

\begin{align*}
-\beta u_{m-2}^{n+1} + (\gamma_m - \epsilon)u_{m-1}^{n+1} + (1 + 2\epsilon)u_m^{n+1} + (\delta_m - \epsilon)u_{m+1}^{n+1} + \beta u_{m+2}^{n+1} &= u_m^n \\
27a
\end{align*}

where

\begin{align*}
\alpha &= k/h, \quad \beta = k/2h^3, \quad b) \\
\gamma_m &= 2\beta - \alpha(u_m^n + u_{m-1}^n), \quad c) \\
\delta_m &= \alpha(u_m^n + u_{m+1}^n) - 2\beta, \quad d) \\
\epsilon &= \mu k/h^2, \quad e)
\end{align*}

and \( k \) and \( h \) are time and space steps, respectively.

This scheme leads to a penta-diagonal linear system, to which we applied a banded matrix optimized LU decomposition solver. For stability, \( \epsilon \) must be sufficiently small. To minimize dissipation error, it was found that \( h \) must also be small. These conflicting goals were remedied by making the time step \( k \) sufficiently small, at the expense of a large number of total time steps. Calculations were carried out in double precision on a SUN SPARC II workstation.

\[ u(x, 0) = N(N + 1) \text{sech}^2(x) \quad 28 \]

From well known theory [7], if \( N \) is an integer, the KdV evolution will cause this single pulse to fission into \( N \) solitary waves (solitons), propagating at speeds proportional to their amplitudes, and ranked in amplitude front to back, largest to smallest, in the direction of wave packet propagation. The amplitude of the wave packet descends nonlinearly from largest to smallest, but the pulses are relatively spaced in such a way that a straight line passes through the crests. Also, because of the nonlinearity feature of speed proportional to amplitude, the wave packet will spread out in time.

In Fig. 8, we show an example including dissipation of the non-steady evolution of our IC with \( N=3 \). Note that after the solitons emerge, their amplitudes begin to decay (and they are no longer strictly solitons, as will be discussed). The amplitude fall-off is such that the wave crests at a given time step still fall on a straight line, and the decay rate is clearly a function of amplitude.

Although we will not pursue it here, the fact that the larger amplitudes decay faster than the smaller ones, means that the wave packet spreading is being inhibited, or slowed down. This is shown in Fig. 9, and will have consequences for the acoustic resonance spectral structure. This is currently under further study. There is also a peak broadening effect that will be the main topic of interest for the rest of this paper.

In Fig. 10(a) we show the dissipative effect on a single solitary wave (\( N=1 \)), and in 10(b) an overlay of the first and last pulse in a), where the latter has been scaled and shifted for comparison purposes. The effect we primarily want to point out is the pulse broadening. The shape distortion shows that this solution is no longer strictly a solitary wave. Indeed, in both figures, a pedestal drawing out to the left of the main wave can be observed. This structure is consistent with numerical solutions of the Navier-Stokes equations for surface solitary waves in a canal, obtained by Tang et al. [22]. This effect is inconsistent with Sandstrom and Oakey's [12] hypothesis that the proper description is an equation with coefficients that adjust to preserve solitary waves locally. In the spirit of this idea, we will take, as our model, a broadened version of our solitary internal wave, shown in Fig. 11. Note, we are showing a wave of depression in our ocean model coordinates, as compared to (dashed) the initial solitary wave used.

So far, no effort has been made to tie our simulations to physical scales. We present here a few reasonable numbers to give one a feel for the quantities involved. The energy of a KdV soliton [14] with a 9 m amplitude, where \( \lambda = 150 \) m, and \( \Delta \rho \approx 4 \) kg/m\(^3\), is \( 6.35 \times 10^5 \) J/m. In order for the IW to dissipate to an amplitude of 5.6 m, which is the equivalent percentage loss we found in our KdVB simulation, we need an energy loss of \( 3.87 \times 10^5 \) J/m.

In our case, we need to choose a value of \( \mu \) and a number of time steps to keep us in the non-steady regime. Let us consider the time evolution of the following initial condition (IC)

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig11.png}
\caption{New IW simulation (solid) created by approximately doubling the width of original solitary wave considered (dashed).}
\end{figure}
\end{center}
J/m. If we assume that our solitary wave is traveling at an average speed of about 0.5 m/s, over a distance of ~10 km, and use a dissipation rate of $5 \times 10^{-2} \text{ W/m}^2$, then the desired amplitude reduction will occur over a period of about 5 h.

Now we are ready to pose a specific question as regards the effects of dissipation on the acoustic resonance structure we have identified: how would the peak broadening affect the acoustical resonant structure already identified [5]? In Fig. 11 we have left the amplitude unchanged, so as to isolate one particular effect. Using this new acoustical environment, we repeat the calculations to produce the new resonance curve corresponding to that shown in Fig. 4(b). In Fig. 12, the resonance curve for the broadened IW in Fig. 11 (solid) is compared to the resonance curve from Fig 4(b) (dashed). Note, the structure, slightly altered, is shifted up in frequency by about 20 Hz.

![Resonance curve](image)

FIG. 12. Resonance curve for broadened IW in Fig. 10 (solid) compared to resonance curve from Fig 4(b) (dashed). Note, shifted up in frequency by about 20 Hz.

CONCLUSIONS

Shallow water solitary internal wave packets can have a strong influence on low-frequency sound propagation in water. An important effect is enhanced loss to the bottom due to resonant mode coupling effects. This same mechanism can also enhance energy transfer to the mixed layer for a source below the thermocline. For the particular case studied, the length scale of the internal soliton was critical in determining the resonant structure. The introduction of dissipative effects showed that, acoustically, the relevant features included amplitude reduction, amplitude equalization across a wave packet, packet spreading inhibition, and pulse broadening of individual solitons. The latter effect was studied in terms of its consequences for the resonant structure for enhanced energy transfer to the mixed layer. Simulations were accomplished by numerical solution of the KdVB equation. Especially of interest is the effect of peak broadening on the mixed layer resonance. Given an approximate doubling of the internal solitary wave width, the acoustic effect was to induce an upward shift in frequency of about 20 Hz for the resonance effect under study.

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REFERENCES


