Comments on "A Nonlinear Model of Internal Tide Transformation on the Australian North West Shelf"

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1. Introduction

Recently, Holloway et al. (1997) compared simulations and measurements of the nonlinear evolution of internal tides on the Australian North West Shelf. The model they used for simulations was based on a Korteweg-de Vries (KdV) type equation, generalized to include dissipative and range varying effects (henceforth referred to as the gKdV equation). The purpose of this present note is to discuss certain issues concerning their inclusion of dissipation, of which there were two types: 1) turbulent horizontal eddy viscosity, represented by the coefficient $\nu$ (m$^2$ s$^{-1}$), and 2) quadratic bottom friction, represented by the dimensionless coefficient $k$.

The authors pointed out that, for the value of $\nu$ they used ($\nu = 2 \times 10^{-4}$ m$^2$ s$^{-1}$), no apparent effect of eddy viscosity was observed. In considering the question as to what effects would have been produced had the value of $\nu$ been significantly larger, and in investigating this issue further, we made several observations.

1) The explicit finite difference algorithm presented in Holloway et al. (1997) for solving the gKdV equation appears to behave as if unconditionally unstable for levels of eddy viscosity large enough to produce noticeable effects.

2) For internal tide characteristic scales, and the value of $\nu$ used by the authors, the bottom friction term in the gKdV equation overwhelms the effect of the eddy viscosity term by seven orders of magnitude (hence, effectively excluding it).

3) For solitary internal wave characteristic scales, the effect of eddy viscosity can become significant in the presence of bottom friction, but the tidal component is still relatively unaffected.

In addressing observation one in this paper, we discuss a possible modification to their finite difference numerical scheme to allow larger values of eddy viscosity. The modified scheme is used to briefly study the relative contributions of the eddy viscosity and bottom friction terms for a simple (constant coefficient) simulation case, the results of which suggest observations two and three. Also provided is a scaling analysis supporting the same. We should remark that this short note is hoped to be a useful addendum to the more comprehensive study presented in Holloway et al. (1997)—a timely and significant addition to the literature on this subject.

2. Generalized KdV equation

The gKdV equation proposed by Holloway et al. (1997) is given as follows:

$$\xi_x + A \xi \xi_x + B \xi_{uu} + C \xi \xi_u - D \xi_u = 0,$$

where

$$A = \alpha c^2 \sqrt{Q},$$

$$B = \beta c^2,$$

$$C = \kappa c \beta \sqrt{Q},$$

and

$$D = \nu c.$$ 

The subscripts refer to partial differentiation, and the independent variables are the horizontal distance $x$ and a transformed time variable $s$, defined by

$$s = \int_0^t \frac{dx}{c(x)},$$

where $t$ is time. The dependent variable $\xi$ is related to the pycnocline displacement $\eta$ by

$$\xi = \eta \sqrt{Q(x)},$$

where $Q$ is an amplitude factor to incorporate slowly varying depth and horizontal variability in background conditions.
density and shear flow. The remaining coefficients are identified as $\alpha(x)$: nonlinearity (advective), $\beta(x)$: dispersion, $\alpha(x)$: linear long wave phase speed, $k$: bottom friction, and $\nu$: horizontal eddy viscosity. See Holloway et al. (1997) and Pelinovsky et al. (1994) for further details and discussion.

Note that there is a switch between the usual roles of space and time in Eq. (1) since it is desired to simulate time series at different station locations. Hence, $x$ (space) is the evolutionary variable, and the initial condition is given as a function of time

$$\xi(s, x = 0) = a_0 \sin(\omega t),$$

where $a_0$ is a user-specified amplitude, $\omega = 2\pi/P$, and $P \approx 12$ h (tidal period).

3. Explicit finite difference scheme

Holloway et al. (1997) used an $O(\Delta x^2, \Delta t^2)$ explicit leapfrog type of finite difference scheme to numerically solve (1):

$$\xi_j^{n+1} = \xi_j^n - \frac{\alpha}{c^2\sqrt{Q(\Delta s)}} \frac{\Delta x}{\Delta s} \xi_j^n(\xi_j^n - \xi_{j-1}^n)$$
$$- \frac{\beta}{c^2(\Delta s)^2} (\xi_{j+1}^n - 2\xi_j^n + 2\xi_{j-1}^n - \xi_{j-2}^n)$$
$$- \frac{2\nu}{\beta \sqrt{Q}} \Delta x \xi_j^n \xi_j^n + \frac{2\nu}{c^2(\Delta s)^2} (\xi_{j+1}^n - 2\xi_j^n + \xi_{j-1}^n),$$

(2)

where $n = 1, 2, \ldots, f \in [1, \ldots, M]$, and periodic boundary conditions are applied. The indices on the coefficients that indicate full generality were omitted for clarity. This scheme was a generalization of that developed by Berezin (1987, p. 38) for the KdV equation, and by Pelinovsky et al. (1994) for the variable-coefficients KdV equation. [Note, Berezin credits Zabusky and Kruskal (1965.)] What appears to be a typographical error in the version published in Holloway et al. (1997) has been corrected here. The stability condition, which does not include the effects of dissipation, is

$$\frac{\Delta x}{\Delta s} \left( \frac{2\sqrt{3} \beta}{2(\Delta s)^2} \right) < 1.$$

4. Stability issues

In their simulations, Holloway et al. apparently used a value of $\nu = 2 \times 10^{-4}$ m$^2$ s$^{-1}$. They reported that the eddy viscosity had no significant effect on the tidal evolution; however, they did find that the bottom friction had significant effect. They also reported a range of values for $\nu (0.2-30$ m$^2$ s$^{-1}$) based on the literature (Liu et al. 1985; Liu 1988; Sundstrom and Oakey 1995). Since this range of values is several orders of magnitude greater than the value they used, the question arose as to what effect these larger values of $\nu$ would have on their simulation results.

Upon implementing their numerical scheme (Holloway et al. 1997) and using the parameter set given below, we found that, for the values of $\nu$ in the above range derived from the literature, the method was unstable at all integration step sizes ($\Delta x$) tested—the smallest step size tested being $8.0 \times 10^{-4}$ m. This should be compared to a step size of 2.0 m, which was adequate for a stable, accurate solution when $\nu = 2 \times 10^{-4}$ m$^3$ s$^{-1}$. Excessive run time discouraged trying smaller step sizes, as integration was being carried out to 100 km.

As a clue to the nature of this problem, set coefficients $A$, $B$, and $C$ to zero in Eq. (1). Then the gKdV equation reduces to

$$\xi_x = D \xi_{xx},$$

which has the same general form as the diffusion (or heat conduction) equation. The leapfrog finite difference scheme (2) now reduces to Richardson's method,

$$\xi_j^{n+1} = \xi_j^n + \frac{2\nu}{c^2(\Delta s)^2} (\xi_{j+1}^n - 2\xi_j^n + \xi_{j-1}^n),$$

(3)

which is unconditionally unstable (e.g., see Anderson et al. 1984, p. 111). (See appendix A for a more complete stability analysis.) If it is true that (2) is unconditionally unstable for $\nu > 0$, it needs to be explained why Holloway et al. were able to use it for the value $\nu = 2 \times 10^{-4}$ m$^2$ s$^{-1}$. This will be discussed below.

5. Modified scheme

Dufort and Frankel proposed a simple modification to the scheme for the diffusion–heat equation (3) that rendered it unconditionally stable (Anderson et al. 1984, p. 114). Their solution was to approximate $\xi_x$ as

$$\xi_x = \frac{1}{2} (\xi_{j+1}^{n+1} + \xi_{j-1}^{n+1}),$$

(4)

giving, in our case,

$$\left( 1 + \frac{2\nu}{c^2(\Delta s)^2} \right) \xi_j^{n+1} = \left( 1 - \frac{2\nu}{c^2(\Delta s)^2} \right) \xi_j^n$$
$$+ \frac{2\nu}{c^2(\Delta s)^2} (\xi_{j+1}^n + \xi_{j-1}^n).$$

It should be remarked that, although unconditionally stable, the Dufort–Frankel method does have certain grid refinement requirements for consistency with the original diffusion–heat equation (i.e., one must be careful to avoid oscillatory, wave-like solutions).

Substituting (4) into the diffusive (last) term only in (2), the following modified finite difference scheme is arrived at:
\[ \left(1 + \frac{2\nu}{c^2 (\Delta s)^2} \right) \xi_{n+1}^r = \left(1 - \frac{2\nu}{c^2 (\Delta s)^2} \right) \xi_{n-1}^r - \frac{\alpha}{c^2 \sqrt{Q}} \frac{\Delta x}{(\Delta s)^2} (\xi_{n+1}^r - \xi_{n-1}^r) \]
\[ - \frac{\beta}{c^2 \sqrt{Q}} (\xi_{n-3}^r - 2\xi_{n-1}^r + 2\xi_{n+1}^r - \xi_{n+3}^r) \]
\[ - \frac{2\nu}{\beta \sqrt{Q}} \Delta x (\xi_{n-1}^r - \xi_{n+1}^r) + \frac{2\nu}{c^2 (\Delta s)^2} (\xi_{n+1}^r + \xi_{n-1}^r). \] (5)

For \( \nu = 0 \), the stability properties of the original [Eq. (2)] and modified [Eq. (5)] schemes are the same. However, the modified scheme remains stable as the value of \( \nu \) is increased (as opposed to the original scheme). It is important to check one's grid choice, however, when using (5). For a proper choice, upon setting \( \alpha = \beta = k = 0 \), one should observe the evolution of an initial state to be purely diffusive, with no oscillations developing.

6. Computational results

The purpose of this section is to provide some simulations showing the relative and combined effects of the two dissipation types on the nonlinear evolution of internal tides. Only the constant coefficient case was done, where the following parameter values were used: \( \alpha = -1.0 \times 10^{-3} \) s\(^{-1} \), \( c = 0.5 \) m s\(^{-1} \), \( Q = 1.0 \), \( \beta = 400.0 \) m\(^2\) s\(^{-1} \), \( \Delta x = 0.2 \) m, \( \Delta s = 72 \) s, \( a_0 = -30 \) m, and \( x = 100 \) km. The value of \( \Delta x \) was chosen conservatively, and in order to avoid relying on the predictions of only one numerical method, a separate algorithm was implemented (see appendix B). All calculations were repeated with this second method for model validation purposes, and there was good agreement between the two methods.

With the above parameters, and using the method in Eq. (2), the \( \nu = 0 \) and \( \nu = 2 \times 10^{-4} \) m\(^2\) s\(^{-1} \) cases were calculated and compared with no bottom friction (\( k = 0 \)). The maximum percentage difference between solutions was 0.04%, which is consistent with the conclusions of Holloway et al. (1997). Using the modified method (5), we also calculated various combinations of dissipation types and values, where \( k = 0 \) or \( k = 0.001 \) and \( \nu = 0, 0.2 \) or 1.0 m\(^2\) s\(^{-1} \).

These results are shown in Figs. 1 and 2. In Fig. 1a, the solution for \( \nu = 0, k = 0 \) (dashed curve) and \( \nu = 0.2, k = 0 \) (solid curve) is shown. In Fig. 1b the solution for \( \nu = 0, k = 0.001 \) (dashed curve) and \( \nu = 0.2, k = 0.001 \) (solid curve) is shown. In Fig. 2a the solution for \( \nu = 0, k = 0 \) (dashed curve) and \( \nu = 1.0, k = 0 \) (solid curve) is shown. Finally, the solution for \( \nu = 0, k = 0.001 \) (dashed curve) and \( \nu = 1.0, k = 0.001 \) (solid curve) is shown in Fig. 2b.

7. Discussion of computational results

Figure 1 shows that, in this case, the effect of eddy viscosity is primarily to dampen the amplitudes of the high frequency oscillations (emerging solitary waves) without changing the energy level of the internal tide as a whole. Bottom friction also contributes to oscillation damping, but its primary effect is to significantly reduce the energy of the tide [an order of magnitude decrease in amplitude over the integration range (100
be controlled by bottom friction, and additional oscillation damping is contributed by horizontal eddy viscosity. It should be noted that these conclusions are based on a specific, constant coefficient case. In the next section, a scale analysis gives some further insight into the computational results presented above.

8. Scale analysis

To analyze the relative contributions of the various terms in Eq. (1) for different characteristic scales, we proceed by transforming to dimensionless variables:

\[ x^* = \frac{x}{L}, \quad \xi^* = \frac{\xi}{\xi_0}, \quad x^*_k = \frac{x_k}{L}, \quad \xi^*_k = \frac{\xi_k}{\xi_0}, \]

where \( L, L/c, \) and \( \xi_0 \) are characteristic length, time, and amplitude scales, respectively. After normalizing by the evolution term and omitting the asterisk, (1) can be rewritten as

\[ \xi_0 + \sigma_1 \xi_0^* + \sigma_2 \xi_0^* + \sigma_3 \xi_0^* = 0, \]

where the dimensionless coefficients \( \sigma_1, \ldots, \sigma_4 \) are given by

\[ \sigma_1 = \frac{\alpha \xi_0}{\sqrt{Q}c} \] (advection term)
\[ \sigma_2 = \frac{\beta}{cL^2} \] (dispersive term)
\[ \sigma_3 = \frac{k\xi_0 L c}{\sqrt{Q} \beta} \] (bottom friction term)
\[ \sigma_4 = \frac{\nu}{cL} \] (eddy viscosity term).

Noting that \( \xi_0 = \eta_0 \sqrt{Q} \), let \( \alpha = 1.0 \times 10^{-3} \), \( \eta_0 = 30 \) m, \( \beta = 400.0 \) m s\(^{-1}\), \( \xi_0 = 1 \times 10^{-3} \), and let \( \nu \) vary to take on the values

\[ \nu = 2.0 \times 10^{-4} \text{ m}^2 \text{ s}^{-1} \]
\[ = 1.0 \text{ m}^2 \text{ s}^{-1} \]
\[ = 30.0 \text{ m}^2 \text{ s}^{-1}. \]

Given a fixed wave speed \( c = 0.5 \text{ m s}^{-1} \), let \( L = 21.6 \) km \((L/c = 12 \text{ h})\) for internal tides, and let \( L = 300 \) m
TABLE 2. Ratio of bottom friction term to eddy viscosity term in Table 1. When $\nu \gg 1$, eddy viscosity effects start to become important for internal waves.

<table>
<thead>
<tr>
<th>Eddy viscosity</th>
<th>Internal tide</th>
<th>Internal wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.0 \times 10^{-5}$ m$^2$ s$^{-1}$</td>
<td>$4.4 \times 10^{-3}$</td>
<td>$3.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>$1.0$ m$^2$ s$^{-1}$</td>
<td>$8.7 \times 10^{-3}$</td>
<td>$1.7$</td>
</tr>
<tr>
<td>$30.0$ m$^2$ s$^{-1}$</td>
<td>$2.9 \times 10^{-3}$</td>
<td>$5.6 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

$(L/c = 600$ s$)$ for internal waves. The results are summarized in Tables 1 and 2. For internal tide scales, nonlinearity and bottom friction effects dominate over dispersion and eddy viscosity (even for the largest value of $\nu$ in the range). For internal wave scales, the dispersive and eddy viscosity effects become more important, but for $\nu = 3 \times 10^{-5}$ m$^2$ s$^{-1}$, eddy viscosity effects are still swamped by bottom friction. This scale analysis only tells us generally what to expect; however, the results are consistent with the details of the numerical experiments shown earlier.

9. Conclusions

The numerical scheme presented in Holloway et al. (1997) appears to be unconditionally unstable. However, they were able to proceed with its use since the value of the eddy viscosity coefficient they chose was so small as to conspire with the scaling properties of the diffusion term to effectively eliminate that term from the scheme—and it is precisely the diffusive term that causes the difficulties with stability, since they were using a leapfrog-type method.

One possible solution was suggested—a simple modification of their scheme using an approximation due to Dufort and Frankel. This modified scheme was then used to investigate briefly the nature of the effect of including larger levels of eddy viscosity, noting the relative effect with respect to tidal evolution, SWF formation, and interaction with the bottom friction term. However, the effects of eddy viscosity appear to be largely secondary to bottom friction. One might well ask whether both terms should be included.

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APPENDIX A

Stability Analysis

Presented here is a stability analysis of a linearized version of (2). Let $k = 0$, $\xi(t) = \xi(x, t)$, $\xi(x, t) = \xi_{1,1}^{*} + \xi_{2,2}^{*}$, where $\xi$ is an appropriately chosen constant, and $\xi_{1,1}^{*} = \lambda^{*} \exp(\mu \omega_{s} t)$, where $s_{1,2}^{*} = s_{1,2} + \Delta s$, etc. Substituting the above into (2), one can finally obtain

\[
\lambda_{1,2} = \frac{1}{2} \left( \xi_{1,1}^{*} \pm \sqrt{\xi_{1,1}^{*2} + 4} \right), \quad (A1)
\]

where

\[
\xi = \xi_{1,1}^{*} + i \xi_{2,2}^{*},
\]

such that

\[
\xi_{1,1}^{*} = 2 \xi_{1,1}(\cos \theta - 1),
\]

\[
\xi_{2,2}^{*} = 2 \sin \theta \xi_{1,1} + 2 \xi_{1,1}(\cos \theta - 1),
\]

and

\[
\xi_{1} = \frac{\alpha}{c^{2} \sqrt{\Omega}} \frac{\Delta x}{\Delta s},
\]

\[
\xi_{2} = \frac{\beta}{c^{4} \Delta s^{2}},
\]

\[
\xi_{3} = -\frac{2 \nu}{c^{2} \Delta s^{2}},
\]

\[
\theta = \omega \Delta s, \quad \theta \in [0, 2\pi].
\]

Let

\[
|\lambda| = \max(|\lambda_{1}|, |\lambda_{2}|).
\]

For grids $(\Delta x, \Delta s)$ such that $|\lambda| > 1$, the scheme is said to be unstable. As the grid is refined, $|\lambda|$ tends to zero. For small $|\lambda|$ we can expand the radical in (A1) to first order and write

\[
|\lambda|^{2} \approx 1 + |\xi|^{2}.
\]

Hence, for $\nu > 0$,

\[
|\lambda_{1,2}|_{\text{max}} = \frac{8 \nu}{c^{2} \Delta s^{2}} > 0,
\]

and $|\lambda| > 1$, irrespective of grid size. This implies that the finite difference scheme (2) is unconditionally unstable.

APPENDIX B

Implicit Finite Difference Scheme

Let

\[
F = \frac{1}{2} \xi^{2}
\]

and consider the following notation for differencing operators (similar to Fletcher 1991):

\[
\Delta \xi^{*} = (\xi_{1,1}^{*} - \xi_{0}^{*})
\]

\[
L_{0} \xi_{0}^{*} = (\xi_{1,1}^{*} - \xi_{2,2}^{*})/(2 \Delta s)
\]

\[
L_{0} \xi_{1,1}^{*} = (\xi_{1,1}^{*} - 2 \xi_{1,1}^{*} + \xi_{2,2}^{*})/(\Delta s)^{2}
\]

\[
L_{0} \xi_{2,2}^{*} = (\xi_{1,1}^{*} - 2 \xi_{1,1}^{*} + 2 \xi_{1,1}^{*} - \xi_{2,2}^{*})/(\Delta s)^{2}.
\]
Then we can write the following implicit finite difference scheme for (1) (in conservative form):

$$\frac{\Delta \xi^{n+1}}{\Delta x} + \frac{1}{2} L_n (F^n_F + F^n_F') + \frac{1}{2} B_{nn} (\xi^n + \xi^n_F)$$

$$+ \frac{1}{2} C (\xi^n_F \xi^n + \xi^n_F \xi^n_F') - \frac{1}{2} D_{nn} (\xi^n + \xi^n_F') - \frac{1}{2} = 0,$$

where we used Crank–Nicolson averaging over the $n$ and $n+1$ terms. This leads to a nonlinear system of equations to solve.

Upon carrying out a Taylor series linearization of the advection and bottom friction terms, we can write

$$\frac{\Delta \xi^{n+1}}{\Delta x} + \frac{1}{2} L_n (F^n_F + F^n_F') + \frac{1}{2} B_{nn} (\xi^n + \xi^n_F)$$

and

$$\frac{1}{2} (\xi^n_F \xi^n + \xi^n_F \xi^n_F') \approx \xi^n_F \xi^n.$$

Then,

$$\frac{\Delta \xi^{n+1}}{\Delta x} + \frac{1}{2} L_n (\xi^n \xi^n_F' + \xi^n_F \xi^n_F') + \frac{1}{2} B_{nn} (\xi^n + \xi^n_F)$$

$$+ \frac{1}{2} C (\xi^n_F (\xi^n_F + \xi^n_F') - \frac{1}{2} D_{nn} (\xi^n + \xi^n_F') = 0.$$

Applying the above operator definitions and collecting terms, one obtains

$$\delta_1 \xi^{n+1} + \delta_2 \xi^{n+1} + \delta_3 \xi^n + \delta_4 \xi^{n+1} + \delta_5 \xi^{n+1}$$

$$= \xi^n_F \xi^n_F - \xi^n_F \xi^n_F' + \xi^n_F \xi^n_F' + \xi^n_F \xi^n_F' + \xi^n_F \xi^n_F'$$

where

$$\delta_1 = -\gamma_B$$

$$\delta_2 = -\gamma_{DA} + 2 \gamma_B$$

$$\delta_3 = 1 + \gamma_{DA} (\xi^n_F + \xi^n_F')$$

$$\delta_4 = \gamma_{DA} (\xi^n_F + \xi^n_F') + 2 \gamma_B$$

$$\delta_5 = \gamma_B - \delta_1,$$

$$e_1 = \gamma_B$$

$$e_2 = -(2 \gamma_B - \gamma_D)$$

$$e_3 = (1 - 2 \gamma_B)$$

$$e_4 = (2 \gamma_B + \gamma_D)$$

$$e_5 = -\gamma_B - e_1.$$

and also,

$$\gamma_1 = \frac{1}{4} \Delta x,$$

$$\gamma_2 = \frac{1}{4} \Delta x,$$

$$\gamma_3 = \Delta x,$$

$$\gamma_4 = \frac{1}{4} \Delta x,$$

An example $(j \in [1, M = 8])$ of the tableaux obtained is

$$\begin{bmatrix}
\delta_{13} & \delta_{14} & \delta_{15} & 0 & 0 & \delta_1 & \delta_{12} & \xi^{n+1}\n
\delta_{23} & \delta_{24} & \delta_{25} & 0 & 0 & \delta_2 & \delta_{12} & \xi^n\n
\delta_1 & \delta_{23} & \delta_{24} & \delta_{25} & 0 & 0 & 0 & \xi^{n+1}\n
0 & \delta_1 & \delta_{23} & \delta_{24} & \delta_{25} & 0 & 0 & \xi^n\n
0 & 0 & 0 & \delta_2 & \delta_{24} & \delta_{25} & 0 & \xi^{n+1}\n
0 & 0 & 0 & 0 & \delta_2 & \delta_{24} & \delta_{25} & \xi^n\n
\delta_{24} & \delta_{25} & 0 & 0 & 0 & \delta_1 & \delta_{12} & \xi^{n+1}\n
\end{bmatrix}$$

Note that, except for the corner terms (due to the periodic boundary conditions), the coefficient matrix has a pentadiagonal band structure, which can be exploited for efficient solution.

REFERENCES


