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Optimal Choice of Portfolio Allocation for Regret-Averse Investors under Uncertainty∗

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Abstract: The decision of portfolio allocation for regret-averse investors confront with a portfolio consists of two different risky assets is studied in this paper by extending existing literatures. First of all, we see anticipated disutility from regret has potent effects on diversification choices. Secondly, by comparing the diversification choice of regret-averse investors with that of risk-averse counterparts, we find the two different type of investors make the coincident behavior if and only if the stochastic returns of the two risky assets are independent and identically distributed. Thirdly, a necessary and sufficient condition for diversification is optimal is given for the regret-averse type investors. The last but not the least, we prove that the diversification choice will change with the weight of regret aversion relative to risk aversion.

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Journal of Economic Literature Classification: D81; D92

Introduction

This paper studies the portfolio allocation behavior of a decision maker. Unlike conventional wisdoms, the decision maker considers both risk and regret instead of considering risk only. It’s reasonable to take regret into account in practice because the behavior of decision makers can be influenced to some extent by the prospect of regret. For example,
investors might regret having allocated a small portion to some asset if the return on this asset turns out to be higher at ex-post. On the contrary, investors might regret having allocated a large portion to some asset if the return on this asset turns out to be lower at ex-post. Therefore, such anticipated disutility from regret is crucial when investors should select an initial portfolio allocation at the beginning of an investment project and cannot adjust their previous choices easily thereafter.

Usually, we use traditional expected utility theory that the utility function involved is risk-averse to represent economic behavior. Many empirical researches, however, show inconsistencies in expected utility theory predictions. For instance, the expected utility theory can neither resolve the Allais’ paradox nor explain the hedge strategy well. In this case, regret theory was developed as an alternative theory by many scholars in the past decades. Within them, Bell [1] and Loomes and Sugden [7, 8] not only presented a formal analysis of regret theory, but also argued regret aversion is an alternative theory as well as a rational choice under uncertainty. Moreover, Sugden [13] axiomatized the regret theory systematically. And so forth.

Recently, regret theory was broadly applied in many fields of finance and economics such as insurance and diversification and so on. For example, regret theory was applied to insurance demand by Braun and Muermann [2], and to risk-sharing and asset pricing in a complete market setting by Gollier and Salanie [5]. Diecidue et.al. [3] modeled a dynamic purchase context in which a consumer is uncertain about the product’s valuation, and considered alternative consumer types to characterize how regret affects their spot purchase decisions as well as what triggers the regret. Laciana and Weber [6] provided an upper and a lower bound respectively for the regret factor that would be useful for empirical applications in future research by presenting a parametrization of an expected utility model with a regret correcting term. Applying regret theory, Michenaud and Solink [9] obtained the best currency hedging choice, Mulandzi et.al. [11] analyzed the optimal allocation between loans and treasuries for a regret-averse bank, Renou and Schlag [12] considered price-setting environments and showed that optimal pricing policy follows a non-degenerate distribution by introducing a new solution concept (minimax regret equilibrium), which allows for the possibility that players are uncertain about the rationality and conjectures of their opponents. In addition, a bank’s optimal loan rate under more stringent capital regulation is examined when the bank is not only risk-averse but also
regret-averse in Tsai [14]. The results find that an increase in bank capital requirement results in an increased margin under risk aversion dominating regret aversion, whereas a reduced margin under regret aversion dominating risk aversion. We declare that the former holds when risk aversion domination stems from increasing risk-averse preference, but not from decreasing regret-averse preference, while the latter holds when regret aversion domination results from either decreasing risk-averse or increasing regret-averse preference. Particularly, risk aversion as such makes the bank more prudent and less prone to risk-taking, but regret aversion as such makes the bank less prudent and more prone to risk-taking. Wong [15] discussed the optimal bank interest margin, namely the spread between the loan rate and the deposit rate of a bank, under the condition that the bank is not only risk-averse but also regret-averse. The conclusions suggest that the presence of regret aversion raises or lowers the optimal bank interest margin than the one chosen by the purely risk-averse bank, depending on whether the probability of default is below or above a threshold value, respectively. Moreover, regret aversion as such makes the bank less prudent and more prone to risk-taking when the probability of default is high, thereby adversely affecting the stability of the banking system. The producer can purchase a coinsurance contract with an endogenously chosen coinsurance rate to insure against the revenue risk, by analysing the behavior of a regret-averse producer facing revenue risk in Wong [16]. The paper argues that the regret-averse producer neither fully insures against the revenue risk even though the coinsurance contract is actuarially fair nor chooses to purchase the actuarially fair coinsurance contract when the regret aversion is sufficiently and the loss probability is high. Especially, Muermann et.al. [10] studied the portfolio’s optimal allocation problem for a regret-averse investor, who confronts with a risky and a risk-free asset at the same time, and proved regret-averse investors would hold less stocks when equity premium is high, but would hold more stocks when equity premium is low.

The idea for writing this paper originates from Muermann et.al. [10]. Nevertheless, we extended their model by considering two different risky assets for a regret-averse investor. We show that both regret and risk-averse investors would choose equal shares for each risky asset when the returns of the two risky assets are independent and identically distributed (Hereafter I.I.D). Moreover, one necessary and sufficient condition for diversification is suggested by relaxing the I.I.D premise for the two stochastic returns. Lastly, we prove that when the two random returns are negatively dependent and with the same
mean, then a regret-averse investor would prefer diversification to specialization.

The remaining of this paper is organized as below. In Section 1, we introduce the model explicitly. Section 2 analyzes the choice of portfolio allocation for investors with regret aversion under uncertain returns. Some illustrations for our model is shown in Section 3. Finally, Section 4 presents the conclusions.

1 Model Setup

In this Section, we will elaborate our model at length.

To begin, we suppose there are two different risky assets $A_1$, $A_2$, with stochastic returns $R_1$, $R_2$, respectively. A regret-averse individual decide to make investments between the two risky assets by his initial wealth $W_0$. The regret-averse individual, in other words, need to make a portfolio allocation between $A_1$ and $A_2$. Of course, the objective of the regret-averse investor is to maximize the expected utility of final gains from the investment by exercising the optimal choice of portfolio allocation.

Subsequently, we adopt a utility function with two attributes (risk aversion and regret aversion) in this framework. Analogous to some existing literatures, the form of the utility function we employed is defined as $u(W, R) = v(W) - \lambda g(R)$. Here, $u_W > 0$, $u_R < 0$, which means investors like wealth $W$ but dislike regret $R$. Furthermore, $u_{WW} < 0$, $u_{RR} < 0$, which implies the utility function $u(W, R)$ is concave on the wealth $W$ and the regret $R$ to reflect risk and regret aversion, respectively. The nonnegative parameter $\lambda$ measures the importance of the attribute of regret aversion relative to the attribute of risk aversion. Obviously, the two-attribute utility function will collapse to the conventional risk-averse utility function when $\lambda = 0$. Additionally, function $v$ is strictly concave that accounts for risk aversion and function $g$ is strictly convex on interval $[0, +\infty)$ with $g(0) = 0$ that concerns about the prospect of regret.

In particular, when we define $R = v(W_{\text{max}}) - v(W)$, where $W_{\text{max}}$ is the ex-post optimal level of final wealth, that is, the level of wealth that derives from the optimal ex-ante allocation had the investor known the realized returns of the two risky assets, then function $g$ measures the amount of regret, which depends on the difference between the value of the ex-post optimal level of final wealth $W_{\text{max}}$ and the value of the actual
level of final wealth $W$.

Following the definitions of $u(W,R)$ and $R$ that shown above, we claim that $u(W,R)$ is a function of $W$ in essence because $W^\text{max}$ is a given value. Thereby, the regret-averse utility function can be rewritten as

$$u(W) = v(W) - \lambda g(v(W^\text{max}) - v(W)).$$

(1)

By expression (1), it’s clear to see that it’s better off to possess $W^\text{max}$ than to possess $W$ and to suffer regret experience as well. As a matter of fact, function $g$ indicates that the more pleasurable the consequence might have been, the more regret will be undergone.

Without loss of generality, using the regret-averse utility function expressed in (1), we develop a one-period portfolio allocation model for the two risky assets. Notice that we presume the individual undertake a one-time investment at the beginning of the period, and the returns are realized at the end of the period.

At the end, the objective of the regret-averse investor is to maximize the $Eu(W)$, namely

$$\max_s Eu(W)$$

(2)

subject to

$$W = (1 + sR_1 + (1 - s)R_2)W_0,$$

(3)

$$0 \leq s \leq 1,$$

(4)

where, $E$ is the linear expectation operator, $s$ and $1 - s$ are respectively the shares that are allocated to the risky assets $A_1$ and $A_2$.

Regarding the ex-post optimal level of final wealth $W^\text{max}$ (Random variable), we have the following consequences

$$W^\text{max} = \begin{cases} (1 + R_1)W_0, & \text{if } R_1 \geq R_2, \\ (1 + R_2)W_0, & \text{if } R_1 < R_2. \end{cases}$$

(5)

2 Impacts of Regret Aversion on Portfolio Allocation

Now that the model has been well constructed in Section 1, then we will analyze the impacts of regret aversion on portfolio allocation in this section. For our purpose, we use
s* to denote the optima corresponding to the share s (Hereafter the same). Nextly, we present our conclusions one by one as below.

**Theorem 1.** Both regret-averse investors (λ > 0) and risk-averse investors (λ = 0) will allocate their initial wealth equally to the two risky assets $A_1$ and $A_2$, i.e., $s^* = 1/2$ for any $λ ≥ 0$, provided that the two stochastic variables $R_1$ and $R_2$ are I.I.D.

**Proof.** The investor’s objective, substituting (1) into (2), can be rewritten as

$$\max_s E\{v(W) - \lambda g(v(W_{\text{max}}) - v(W))\},$$

where, $W$ and $W_{\text{max}}$ are as in (3) and (5), $s$ satisfies the constraint (4).

Differentiating (6) with respect to $s$, we get the first-order condition is

$$\frac{\partial E_u(W)}{\partial s} = E\{u'(W)(R_1 - R_2)W_0\} = 0,$$

and the second-order condition is

$$\frac{\partial^2 E_u(W)}{\partial s^2} = E\{u''(W)(R_1 - R_2)^2W_0^2\} < 0.$$

Here

$$u'(W) = v'(W)\{1 + \lambda g'(v(W_{\text{max}}) - v(W))\},$$

$$u''(W) = v''(W) + \lambda\{g'(v(W_{\text{max}}) - v(W))v''(W) - g''(v(W_{\text{max}}) - v(W))v'(W)\} < 0,$$

since function $v$ is strictly concave with $v'(-) > 0$, $v''(-) < 0$ and function $g$ is strictly convex with $g(0) = 0$, $g'(\cdot) > 0$, $g''(\cdot) > 0$ on interval $[0, +\infty)$. Thus, $0 ≤ s^* ≤ 1$ satisfies (7) is the unique global optimums for optimization problem (2).

Obviously, $E\{R_1v'(W|s=1/2)\} = E\{R_2v'(W|s=1/2)\}$ because $R_1$ and $R_2$ are I.I.D, $W|s=1/2 = (1 + \frac{R_1 + R_2}{2})W_0$. So, $E\{v'(W|s=1/2)(R_1 - R_2)\} = 0$ and $E\{g'(v(W_{\text{max}}) - v(W|s=1/2))v'(W|s=1/2)(R_1 - R_2)\} = 0$. Recalling (7) and (8), we see $\frac{\partial E_u(W)}{\partial s}|_{s=1/2} = 0$ for any $λ ≥ 0$. Therefore, $s^* = 1/2$ is the unique global optimums. Q.E.D.

More generally, we can obtain the following consequence straightforwardly.

**Corollary** For any $λ ≥ 0$, the two type of investors will allocate their wealth averagely to $N(≥ 2)$ risky assets as long as the stochastic returns of all assets are I.I.D.
We now consider the case that the two random variables $R_1$ and $R_2$ are independent but not identically distributed, and show the result in the following Theorem 2.

**Theorem 2.** Supposing the two random variables $R_1$ and $R_2$ are independent but not identically distributed, then for regret-averse investors, i.e., $\lambda > 0$, we get

$$s^*|_{\lambda > 0} > 0, \quad \text{if} \quad E[R_1] - E[R_2] \geq \frac{\text{Cov}(v'(1 + R_2)W_0, R_2)}{E[v'(1 + R_2)W_0]},$$

$$s^*|_{\lambda > 0} < 1, \quad \text{if} \quad E[R_2] - E[R_1] \geq \frac{\text{Cov}(v'(1 + R_1)W_0, R_1)}{E[v'(1 + R_1)W_0]}.$$

**Proof.** Evaluating $\frac{\partial E(u(W))}{\partial s}$ at $s = 0$ and $s = 1$, respectively, under the condition that $\lambda > 0$ (Regret aversion). It’s not difficult to obtain

$$\frac{\partial E(u(W))}{\partial s} \bigg|_{s=0} > W_0(1 + \lambda g'(0))E\{v'(1 + R_2)W_0)(R_1 - R_2)\}, \quad (9)$$

$$\frac{\partial E(u(W))}{\partial s} \bigg|_{s=1} < W_0(1 + \lambda g'(0))E\{v'(1 + R_1)W_0)(R_1 - R_2)\}. \quad (10)$$

Straightforwardly, by (9), we see that if $E[R_1] - E[R_2] \geq \frac{\text{Cov}(v'(1 + R_2)W_0, R_2)}{E[v'(1 + R_2)W_0]}$, then $s^* > 0$ for $\lambda > 0$. Analogously, from (10), we know that if $E[R_2] - E[R_1] \geq \frac{\text{Cov}(v'(1 + R_1)W_0, R_1)}{E[v'(1 + R_1)W_0]}$, then $s^* < 1$ for $\lambda > 0$. Q.E.D.

**Remark 1.** For investors with regret aversion ($\lambda > 0$) that make portfolio allocation between two risky assets, we notice that an explicit economic interpretation is disclosed by Theorem 1. That is, regret-averse investors will allocate a fraction of their initial wealth to some risky asset as long as the difference between the expected return of the one and of the other is no less than some threshold value, but regardless of the their volatilities. Put differently, regret-averse investors won’t leave out any opportunity to grasp the higher return, because they would not like to undergo regretful experience at ex-post.

**Remark 2.** For regret-averse investors, Theorem 2 can not always assure diversification is the optimal behavior because $s^*|_{\lambda > 0} > 0$ can rise to 1 when the difference between $E[R_1]$ and $E[R_2]$ is sufficiently large, whereas $s^*|_{\lambda > 0} < 1$ can fall to 0 when the difference between $E[R_2]$ and $E[R_1]$ is large enough.

To give a necessary and sufficient condition for diversification for regret-averse investors, we present Theorem 3 as below.

**Theorem 3.** Assuming the two stochastic returns $R_i (i = 1, 2)$ are independent but not identically distributed, then diversification ($0 < s^*|_{\lambda > 0} < 1$) is optimal for regret-averse
investors ($\lambda > 0$) if and only if \[ \frac{\text{Cov}(v'((1+R_i)W_0),R_2)}{E v'((1+R_i)W_0)} \leq ER_1 - ER_2 \leq \frac{\text{Cov}(v'((1+R_1)W_0),-R_1)}{Ev'((1+R_1)W_0)}. \]

**Proof.** As we assumed, $v$ is a concave function, then $v'((1+R_i)W_0)$ is decreasing with $\mathbb{R}_i$, which implies that $\text{Cov}(v'((1+R_i)W_0),\mathbb{R}_i) < 0$, $i = 1, 2$. Thus, $\frac{\text{Cov}(v'((1+R_1)W_0),R_2)}{Ev'((1+R_2)W_0)} < 0$ and $\frac{\text{Cov}(v'((1+R_1)W_0),-R_1)}{Ev'((1+R_1)W_0)} > 0$ due to $Ev'((1+R_i)W_0) > 0$, $i = 1, 2$. Applying Theorem 2, we can obtain the necessary and sufficient condition for diversification directly. Q.E.D.

**Remark 3.** Here, notice that $ER_1 = ER_2$ is allowed in Theorem 3. In other words, the necessary and sufficient condition for $0 < s^*|_{\lambda > 0} < 1$ is still valid for the two independent but not identically distributed random variables $\mathbb{R}_i(i = 1, 2)$ with the same mean.

Mention that we have relaxed the premise that random variables $\mathbb{R}_i(i = 1, 2)$ are identically distributed in the previous paragraphs. In the sequel, we shall abandon the premise that $\mathbb{R}_i(i = 1, 2)$ are independent. To this end, we shall recall the following conventional definition to measure the dependence of two random variables.

**Definition** For any two random variables $Z_i$ and two real numbers $z_i$, $i = 1, 2$, if there holds $P(Z_1 \leq z_1, Z_2 \leq z_2) \geq P(Z_1 \leq z_1)P(Z_2 \leq z_2)$, then $Z_1$ and $Z_2$ are said to be positive quadrant dependent. Otherwise, they are said to be negative quadrant dependent.

Again, we recall a Proposition from Egozuche et.al. [4] as follows.

**Proposition** Let random variables $Z_i(i = 1, 2)$ be positive quadrant dependent, $f$ and $h$ be two real functions. Then we have

(a) $\text{Cov}(f(Z_1),g(Z_2)) \geq 0$ if monotonicity of function $f$ is in line with that of function $h$,
(b) $\text{Cov}(f(Z_1),g(Z_2)) \leq 0$ if monotonicity of function $f$ is opposite to that of function $h$.

Similarly, for any two random variables that are negative quadrant dependent, there yields

(c) $\text{Cov}(f(Z_1),g(Z_2)) \leq 0$ if monotonicity of function $f$ is in line with that of function $h$,
(d) $\text{Cov}(f(Z_1),g(Z_2)) \geq 0$ if monotonicity of function $f$ is opposite to that of function $h$.

At the end, we present an alternative conclusion in Theorem 4.

**Theorem 4.** If the two stochastic returns $\mathbb{R}_i(i = 1, 2)$ are negative quadrant dependent and $ER_1 = ER_2$ as well, then for regret-averse investors ($\lambda > 0$), a diversified portfolio is optimal, i.e., $0 < s^*|_{\lambda > 0} < 1$. 
Proof. For $\lambda > 0$, reminding (9) and (10), we can rewrite them respectively as

$$\frac{\partial E_u(W)}{\partial s} \bigg|_{s=0} > W_0(1 + \lambda g'(0))Cov(v'((1 + R_2)W_0), R_1 - R_2)$$
$$= W_0(1 + \lambda g'(0))\{Cov(v'((1 + R_2)W_0), R_1) + Cov(v'((1 + R_2)W_0), -R_2)\}$$
$$\geq W_0(1 + \lambda g'(0))Cov(v'((1 + R_2)W_0), R_1), \quad (11)$$

$$\frac{\partial E_u(W)}{\partial s} \bigg|_{s=1} < W_0(1 + \lambda g'(0))Cov(v'((1 + R_1)W_0), R_1 - R_2)$$
$$= W_0(1 + \lambda g'(0))\{Cov(v'((1 + R_1)W_0), R_1) + Cov(v'((1 + R_1)W_0), -R_2)\}$$
$$\leq W_0(1 + \lambda g'(0))Cov(v'((1 + R_1)W_0), -R_2). \quad (12)$$

Note that we used the definitions of positive/negative quadrant dependent and $E R_1 = E R_2$ as well as (a) and (b) in the Proposition above to obtain (11)-(12). Clearly, if $R_1$ and $R_2$ are negative quadrant dependent, then according to (c) and (d) in the Proposition above, we see that for $\lambda > 0$, there yields $\frac{\partial E_u(W)}{\partial s} \bigg|_{s=0} > 0$ and $\frac{\partial E_u(W)}{\partial s} \bigg|_{s=1} < 0$ due to the concavity of $v$, which means $v'$ is decreasing. Consequently, $0 < s^*|_{\lambda>0} < 1$. Q.E.D.

**Remark 4.** Under the same assumptions that are given in Theorem 4, then for risk-averse investors($\lambda = 0$), there holds $0 \leq s^*|_{\lambda=0} \leq 1$, which can not assure diversification is the optimal behavior since two specializations($s^*|_{\lambda=0} = 0, 1$) are involved.

Finally, we examine the dynamic relationship between $s^*$ and $\lambda$.

**Theorem 5.** Supposing the two random variables $R_1$ and $R_2$ are independent but not identically distributed, then for regret-averse investors($\lambda > 0$), we have

$$\frac{ds^*}{d\lambda} > 0, \quad \text{if} \quad E R_1 - E R_2 = \frac{Cov(v'((1 + R_2)W_0), R_2)}{E v'((1 + R_2)W_0)},$$
$$\frac{ds^*}{d\lambda} < 0, \quad \text{if} \quad E R_2 - E R_1 = \frac{Cov(v'((1 + R_1)W_0), R_1)}{E v'((1 + R_1)W_0)}.$$

**Proof.** The optimal fraction $s^*$ satisfies the first-order condition $\frac{\partial E_u(W)}{\partial s} \bigg|_{s=s^*} = 0$. Taking the total differential of the first-order condition with respect to $s$ and $\lambda$, we have

$$\frac{\partial^2 E_u(W)}{\partial s^2} \bigg|_{s=s^*} \cdot ds + \frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*} \cdot d\lambda = 0, \quad \lambda > 0.$$

Then, it yields

$$\frac{ds^*}{d\lambda} = -\frac{\frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*}}{\frac{\partial^2 E_u(W)}{\partial s^2} \bigg|_{s=s^*}}, \quad \lambda > 0.$$
Obviously, since \( \frac{\partial^2 E_u(W)}{\partial s^2} \bigg|_{s=s^*} < 0 \), then
\[
\text{sign} \left( \frac{ds^*}{d\lambda} \right) \text{ is the same as sign} \left( \frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*} \right).
\] (13)

In the sequel, we are examining the sign of \( \frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*} \) step by step.

As a matter of fact, we see
\[
\frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*} = E\{g'(v(W^{\text{max}}) - v(W^*))v'^*(W^*)(R_1 - R_2)W_0\},
\]
where, \( W^* = (1 + s^*R_1 + (1 - s^*)R_2)W_0 \). Remembering the first-order condition, we get
\[
\frac{\partial E_u(W)}{\partial s} \bigg|_{s=s^*} = E\{v'(W^*)(R_1 - R_2)W_0\} + \lambda \frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*} = 0.
\]

Since \( \lambda > 0 \), we obtain
\[
\text{sign} \left( \frac{\partial^2 E_u(W)}{\partial s \partial \lambda} \bigg|_{s=s^*} \right) \text{ is the same as sign} \left( -E\{v'(W^*)(R_1 - R_2)W_0\} \right) \quad (14)
\]

Notice that \( v(W) = u(W) \) in the case of \( \lambda = 0 \), thus
\[
E\{v'(W^*)(R_1 - R_2)W_0\} = \frac{\partial E(u(W)|\lambda = 0)}{\partial s} \bigg|_{s=s^*|\lambda > 0}.
\] (15)

Applying the same argument of Theorem 2 to the risk-averse type investors, that is, \( \lambda = 0 \), it’s easy to see \( s^*|\lambda = 0 = 0 \) if \( E\{v'(W^*)(R_1 - R_2)W_0\} = \frac{\text{Cov}(v'((1+R_1)W_0), R_1)}{E(v'((1+R_2)W_0))} \). Taking Theorem 2 into account together, there yields
\[
\frac{\partial E(u(W)|\lambda = 0)}{\partial s} \bigg|_{s=s^*|\lambda > 0} < \frac{\partial E(u(W)|\lambda = 0)}{\partial s} \bigg|_{s=s^*|\lambda = 0} = 0. \quad (16)
\]

Then for \( \lambda > 0 \), taking (13)-(16) into account, we see \( \frac{ds^*}{d\lambda} > 0 \). Analogously, we can get \( s^*|\lambda = 1 = 1 \) if \( E\{v'(W^*)(R_1 - R_2)W_0\} = \frac{\text{Cov}(v'((1+R_2)W_0), R_2)}{E(v'((1+R_2)W_0))} \). Therefore
\[
\frac{\partial E(u(W)|\lambda = 0)}{\partial s} \bigg|_{s=s^*|\lambda > 0 < 1} > \frac{\partial E(u(W)|\lambda = 0)}{\partial s} \bigg|_{s=s^*|\lambda = 0 = 1} = 0.
\]

Then for \( \lambda > 0 \), considering (13)-(15) and the last expression, we see \( \frac{ds^*}{d\lambda} < 0 \). Q.E.D.

### 3 Illustrations

In this Section, we will give some examples to illustrate our findings proposed in Section 2. Specifically, Example 1 is to illustrate Theorem 1, Example 2 is to simulate Theorems 2 and 5, and Example 3 is to interpret Theorem 3, respectively.
To begin, and without of generality, we assume that the concave function \( v(x) = x^{1/2} \), the convex function \( g(x) = x^2 \), so \( u(x) = x^{1/2} - \lambda (x^{\max})^{1/2} - x^{1/2} \). Meanwhile, for simplicity, we normalize the initial wealth \( W_0 \) to one unit, that is, \( W_0 = 1 \).

**Example 1.** According to Theorem 1, the two stochastic returns \( R_i(i = 1, 2) \) are independent and identically distributed, so we can assume that \( R_i = 1 \) or \( R_i = 2 \), \( i = 1, 2 \), with equal probability. Then we have

\[
W = \begin{cases} 
2, & R_1 = 1, \ R_2 = 1, \\
2 + s, & R_1 = 2, \ R_2 = 1, \\
3 - s, & R_1 = 1, \ R_2 = 2, \\
3, & R_1 = 2, \ R_2 = 2, 
\end{cases} \quad W^{\text{max}} = \begin{cases} 
2, & R_1 = 1, \ R_2 = 1, \\
3, & R_1 = 1, \ R_2 = 2, \\
3, & R_1 = 2, \ R_2 = 2.
\end{cases}
\]

Under the assumptions, we get

\[ Eu(W) = \frac{1}{4} (\sqrt{2} + \sqrt{3}) + \frac{1}{4} (\sqrt{2} + s - \lambda (\sqrt{3} - \sqrt{2} + s)^2) + \frac{1}{4} (\sqrt{3} - s - \lambda (\sqrt{3} - \sqrt{3} - s)^2). \]

Setting \( \partial Eu(W) / \partial s = 0 \), we obtain

\[ \frac{1}{8} \{(2 + s)^{-1/2} - (3 - s)^{-1/2}\} - \frac{3\lambda}{4} \{(6 + 3s)^{-1/2} - (9 - 3s)^{-1/2}\} = 0. \]

By method of undetermined coefficients, it’s easy to get \( s^* = 1/2 \) regardless of \( \lambda = 0 \) or \( \lambda > 0 \). The empirical result here is in line with Theorem 1.

**Example 2.** In terms of Theorems 2 and 5, we make hypotheses as below.

**Case 1.** Suppose independent variables \( R_1 = 0 \) with probability 1/3 or \( R_1 = 3 \) with probability 2/3, while \( R_2 = 0 \) with probability 1/5 or \( R_2 = 3 \) with probability 4/5.

Obviously, \( R_i, i = 1, 2 \), are not identically distributed. In this case, it’s not difficult to check that \( ER_1 - ER_2 = \frac{Cov(v'(1+R_2)W_0), R_2)}{Ev'(1+R_2)W_0} \), and \( Eu(W) \) can be expressed as

\[ Eu(W) = \frac{1}{15} (17 + 4\sqrt{4 - 3s} + 2\sqrt{1 + 3s}) - \frac{2\lambda}{15} (21 - 3s - 8\sqrt{4 - 3s} - 4\sqrt{1 + 3s}). \]

Letting \( \partial Eu(W) / \partial s = 0 \), we obtain \( s^*|_{\lambda=1/2} = 0.233 \), see Figure 1. Regarding the dynamical relationship between \( s^* \) and \( \lambda \), see Figure 2.

**Case 2.** Suppose independent variables \( R_1 = 0 \) with probability 1/4 or \( R_1 = 3 \) with probability 3/4, while \( R_2 = 0 \) with probability 2/5 or \( R_2 = 3 \) with probability 3/5.

Obviously, \( R_i, i = 1, 2 \), are not identically distributed. In this case, it’s easy to check that

\[ ER_2 - ER_1 = \frac{Cov(v'(1+R_1)W_0), R_1)}{Ev'(1+R_1)W_0} \], and \( Eu(W) \) can be written as

\[ Eu(W) = \frac{1}{20} (20 + 3\sqrt{4 - 3s} + 6\sqrt{1 + 3s}) - \frac{3\lambda}{20} (18 + 3s - 4\sqrt{4 - 3s} - 8\sqrt{1 + 3s}). \]
Setting $\frac{\partial Eu(W)}{\partial s} = 0$, we obtain $s^*|_{\lambda=1/3} = 0.807$, see Figure 3. Respecting the dynamical relationship between $s^*$ and $\lambda$, see Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$s^*|_{\lambda=1/2} = 0.233$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{$\frac{ds^*}{d\lambda} > 0$ for $\lambda > 0$}
\end{figure}

**Example 3.** Following Theorem 3, we set independent variables $R_1 = 0$ or $R_1 = 3$ with equal probability, while $R_2 = 0$ with probability $3/8$ or $R_2 = 3$ with probability $5/8$. Obviously, $R_i$, $i = 1, 2$, are not identically distributed. In this case, we see there holds

$$
\frac{\text{Cov}(\nu'((1+R_2)W_0), R_2)}{E\nu'((1+R_2)W_0)} \leq E\mathbb{R}_1 - E\mathbb{R}_2 \leq \frac{\text{Cov}(\nu'((1+R_1)W_0), -R_1)}{E\nu'((1+R_1)W_0)},
$$

and $Eu(W)$ is shown as

$$
Eu(W) = \frac{1}{16}(13 + 5\sqrt{4 - 3s} + 3\sqrt{1 + 3s}) - \frac{\lambda}{16}(55 - 6s - 20\sqrt{1 - 3s} - 12\sqrt{1 + 3s}).
$$

Letting $\frac{\partial Eu(W)}{\partial s} = 0$, we have

$$
3(1 + 3s)^{-1/2} - 5(4 - 3s)^{-1/2} - \lambda\{-4 + 20(4 - 3s)^{-1/2} - 12(1 + 3s)^{-1/2}\} = 0.
$$

For any $\lambda > 0$, it can be proved mathematically that $0 < s^* < 1$ (Indeed, \(\lambda|_{s=0} = -1/12\) and \(\lambda|_{s=1} = -7/20\), which are both contradictory to $\lambda > 0$). Namely, it’s optimal to choose diversification in this case, which is consistent with the conclusion proposed in Theorem 3. The dynamical relationship between $s^*$ and $\lambda$ can be depicted in Figure 5.
Figure 3: $s^*|_{\lambda=1/3} = 0.807$

Figure 4: $\frac{ds^*}{d\lambda} < 0$ for $\lambda > 0$

Figure 5: $0 < s^* < 1$ for any $\lambda > 0$
4 Concluding Remarks

Regret theory (Regret can be viewed as “Opportunity Loss”) is based on the difference between the utility of the actual outcome and the utility that investors would receive if they had made another choice, and supposes that investors are rational and base their decisions not only on expected payoffs but also on expected regret. This axiomatic decision theory can potentially explain many observed violations of the axioms used to build the traditional expected utility approach such as “Preference Reversal” phenomenon, which is in contradiction with transitivity.

In this paper, the choice of portfolio allocation for an investor confronts with one portfolio consists of two risky assets is studied from the regret-averse prospective instead of from the traditional risk-averse perspective, and potent effects of anticipated disutility from regret on diversification choices are shown. We claim that it’s reasonable to consider regret in the process of decision making not only because the decision maker indeed always take regret into account in the process of decision making, but also because regret consideration can explain some interesting economic phenomena in practice. For example, regret can act as one possible explanation for “Allais’ Paradox” (“Common Consequences Effect”), “Certainty Effect” (“Common Ratio Effect”) and so forth.

The major contributions of our paper are that we make a slight further study, above all, by considering a portfolio contains two risky assets instead of considering a portfolio of one risky asset and one risk-free asset in the existing literatures. Secondly, we relax the constraint that the two stochastic returns of the two risky assets are independent and identically distributed, and simultaneously, one necessary and sufficient condition for diversification is optimal is given in the case that the two stochastic returns are independent but not identically distributed. The last but not the least, we make a comparison between the behavior of regret-averse investors and that of risk-averse counterparts. We show that both regret-averse and risk-averse investors will make the same choices when the two stochastic returns of the two risky assets are independent and identically distributed. Nevertheless, the two type of investors will make inconsistent decisions under some conditions. For instance, the risk-averse investors will choose specialization whereas the regret-averse counterparts will choose diversification under some case (See Remark 4 at lengh). Additionally, we mention that with the movement of the weight of regret aver-
sion relative to risk aversion, the optimal choice of portfolio allocation makes changes correspondingly (See Theorem 5 in detail).

In a nutshell, regret theory has appealed to many authors and scholars in recent years because for one thing, it is a normative theory of rational choice under uncertainty, and for another thing, it bears some similarities with prospect theory. Especially, many results of regret theory are consistent with the empirical observations of human behavior that constitute the building blocks of prospect theory. Thus, investors try to anticipate regret and take it into account in their investment decisions is a consistent manner.

References


