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Views and Understandings about Mathematical Reasoning and Proving in Elementary School Mathematics Evident in Pre-service Teachers' Reflections

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In the context of integrated mathematics content and pedagogy with field experience courses, this study examined pre-service teachers' learning about mathematical reasoning and proving in elementary school mathematics. Four reflective journals of 36 pre-service teachers (PSTs) completed at the beginning and at the end of the semester, and papers in which PSTs analyzed their practice of engaging an elementary student in justifying a conjecture, were analyzed for evidence of PSTs' views and understandings about teaching elementary school mathematics with a focus on reasoning and proving. The data revealed PSTs' thinking about the nature of reasoning and proving and about teaching with a focus on reasoning and proving. The data also documented how reflecting on multiple experiences in the integrated courses builds PSTs' awareness of different aspects of their knowledge, and helps them develop a more complex view of teaching elementary school mathematics.

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Mathematical Reasoning and Proving in K-8 Mathematics

The activity of reasoning and proving is integral to the construction of mathematical knowledge, and as such, a desired aspect of students' mathematical experiences (Hannah & DeVilliers, 2008; National Council of Teachers of Mathematics, [NCTM], 2000; National Governor's Association & Council of Chief State School Officers [CCSSM], 2011; Stylianou, Blanton, & Knuth, 2009). With the development of mathematical reasoning, students recognize that mathematics makes sense and can be understood. Reasoning skills facilitate learning of mathematics with understanding: "students cannot understand mathematics unless they acquire and employ the forms of argument used to justify mathematical prepositions, and they cannot appreciate the power of these forms of thinking unless they are deployed in learning important mathematical ideas" (Valentine, Carpenter, & Pligge, 2005, p. 96).

Despite the increased curricular emphasis on teaching school mathematics with a focus on reasoning and proving (NCTM, 2000, 2009; Yackel & Hanna, 2003), a growing body of research shows that students have difficulties when asked to create or evaluate mathematical arguments (e.g., Healy & Hoyles, 2000; Knuth, Chopping, Slaughter, & Sutherland, 2002; Harel & Sowder, 2007). Research also documents that many practicing teachers (often K-8 teachers) have limited mathematical knowledge of reasoning and proving (e.g., Goulding, Rowland, & Barber, 2002; Martin & Harel, 1998; Stylianides, Stylianides & Philippou, 2007). They often see reasoning and proving as a specific topic of study rather than an integral aspect of mathematics teaching and learning (Knuth, 2002). They also hold limited beliefs about students' reasoning and proving capabilities (e.g., Chazan & Sandrow, 2007; Knuth 2002). Research shows that teachers frequently value empirical arguments, and consider empirical arguments as more convincing and easier for students to understand in contrast to general arguments and deductive

proofs. Thus, they rely on the use of empirical arguments over deductive ones in their classroom practice (Knuth, 2002; Knuth, Choppin, & Bieda; 2011). Even when using curriculum materials intentionally designed to promote reasoning and proving, they frequently struggle to support students in developing these mathematical practices (Bieda, 2010; Martin & McCrone, 2003).

Central within the mathematics education community is the question of effective preparation of teachers, including pre-service teachers (PSTs), to foster K-8 students' learning of mathematics with an emphasis on mathematical argumentation, justification, reasoning and proof. Given the curricular emphasis on teaching K-8 mathematics with a focus on reasoning and proving, efforts focused on preparation of PSTs and strengthening their understanding of mathematical reasoning and proof in K-8 mathematics are timely. Presented here research is rooted in two areas of inquiry: research on (1) the knowledge of reasoning and proving teachers need in order to engage students in reasoning and proving activities, and (2) reflective thinking in teacher education. This specific goal was to examine grades 1–8 PSTs' learning about mathematical reasoning and proving, in the context of their experiences in the integrated mathematics content and pedagogy (with field experience) courses.

Research Questions

Most of what is known about teacher knowledge in the area of mathematical argumentation, reasoning and proving concerns secondary mathematics teachers (mostly practicing teachers, e.g., Harel, 2002; Knuth, 2002; Stylianides & Stylianides, 2006). The existing research on K-8 PSTs' knowledge of reasoning and proving has been largely limited to the study of PSTs' competencies in writing and evaluating mathematical arguments that can be accepted as proofs (Boyle, Bleiler, Yee, & Ko, 2015; Felton, 2007; Martin & Harel, 1989; Morris, 2007; Stylianides & Stylianides, 2009; Stylianides, et al., 2007). The practices of

constructing and critiquing mathematical arguments are strongly emphasized throughout the entire K-12 mathematics curriculum, however, research has yet to focus on prospective K-8 teachers' conceptions and understanding of mathematical reasoning and proving. A research-based understanding of grades 1-8 PSTs' models and conceptions of mathematical reasoning and proving can guide efforts of the mathematical and pedagogical preparation that enhances PSTs' abilities to engage K-8 students in reasoning and proving.

The goal of study presented here was to identify views and understandings PSTs reveal in their reflections on their learning in the integrated mathematics content and pedagogy set of courses in a teacher preparation program. The specific focus of the analysis was on PSTs' mathematics content and pedagogical knowledge and understandings of mathematical argumentation, justification and proof for K-8 mathematics. Recognizing what PSTs emphasize as they reflect on their learning and their practice of engaging students in justifying a mathematical statement can support instructional interventions in teacher education. To this end, work presented here was guided by the following two research questions:

1. What views and understandings about mathematical reasoning and proof in elementary school mathematics emerge from the analysis of PSTs' reflections on their learning experiences in the integrated mathematics content and pedagogy with field experience courses? How do these views evolve over the course of the semester?
2. What understandings about their practice of engaging elementary school students in reasoning-and-proving activity do PSTs develop, as evidenced in their reflections on their learning experiences?

Conceptual Framework

Defining Mathematical Reasoning and Proof

The terms mathematical arguments, justifications, reasoning and proof often have various connotations within the mathematics education literature. Thus, it is necessary to establish their meanings in effort to provide foundation for discussion about K-8 PSTs' knowledge and learning about teaching mathematical argumentation, justification and proof. The research described here draws largely on Stylianides (2007) who, in reference to K-8 mathematics, defined mathematical arguments as sequences of assertions that (1) are built on mathematical facts or understandings that are already accepted, and therefore do not require further justifications within the mathematics classroom community; (2) use forms of reasoning that are valid, familiar to the students, or within their conceptual reach; and (3) are presented in developmentally appropriate ways that are within students' conceptual reach. Stylianides (2008) contrasted the above description of an argument that provides a proof, with description of an argument that does not provide a proof. Namely, a valid mathematical argument that constitutes a proof is rooted in accepted facts and mathematical principles and provides conclusive evidence in support (or refutation) of a mathematical claim. The non-proof argument does not validly support the assertion for *all* cases in the problem domain. Stylianides's definition was attractive for the study with PSTs for two reasons. First, his definition carefully attends to the disciplinary tenets of mathematical proofs by highlighting the role of the argument foundations, logic, and generality. At the same time, it allows someone to think about mathematical arguments in the context of K-8 mathematics developmentally. It allows one to examine how well mathematical arguments generated by students and PSTs alike approach the standard of proof.

Teacher Knowledge of Reasoning and Proving

The rich body of research on knowledge mathematics teachers need to be effective in their work offers multiple conceptualizations and frameworks which embrace the complexity of

the knowledge that supports the work of teaching (e.g., An, Kulm, & Wu; 2004; Ball, 1991; Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; Schulman, 1987; Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008). The existing conceptualizations of teacher knowledge largely build on, and extend, the conceptions of subject matter knowledge and pedagogical content knowledge offered by Schulman (1986, 1987). Drawing on these conceptualizations, work presented here is grounded in the research concerned uniquely with teacher knowledge for teaching reasoning and proving (e.g., Stylianides & Ball, 2008; Stylianides & Stylianides, 2006; Lesseig, 2016).

Mathematical Content Knowledge for Teaching Reasoning and Proving. With a specific focus on the mathematics content knowledge (MKT) for teaching reasoning and proving, Stylianides and Ball argued that teachers need to understand the (1) logico-linguistic nature of proof, and have understanding of (2) situations for proving. They interpreted the former broadly as the ability to recognize which arguments provide a proof and which do not, to recognize the logical structure of an argument, and to recognize different forms and modes of presenting mathematical arguments. They explicated the latter as the ability to recognize different kinds of proving tasks, and the relationship between proving task and proving activity.

Drawing on the existing mathematics education literature related to teachers' understanding of reasoning and proving in school mathematics, and their ability to construct and evaluate mathematical arguments, Lesseig (2016) further described and expanded the aspects of MKT proposed by Stylianides and Ball. Lesseig described mathematical content knowledge for teaching reasoning and proving with a focus on common (CCK) and specialized (SCK) content knowledge (see Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; König et al., 2011; Schulman, 1987; Tatto, Schwille, Senk, Ingvarson Peck, & Rowley, 2008 for descriptions of different aspects of MKT). She operationalized the CCK for teaching reasoning and proving in terms of a

teacher's ability to construct arguments that prove, demonstrate an understanding of the (using Stylianides and Balls' terminology) logico-linguistic nature of proof, and demonstrate an understanding that a proof allows to establish the validity of a mathematical statement and serves to communicate and systematize mathematical knowledge. She defined the SCK for teaching reasoning and proving in terms of teachers' understanding of statements and modes of argumentation that can be accepted as proofs. She also viewed SCK for mathematical reasoning and proving as teachers' understanding of any additional functions proofs and proving might serve.

Pedagogical Content Knowledge for Teaching Reasoning and Proving. Lesseig (2016) delineated teacher pedagogical content knowledge (PCK) for teaching reasoning and proving with a focus on knowledge of content and students (KCS) and knowledge of content and teaching (KCT). Lesseig operationalized KCS in terms of teachers' knowledge of conceptions and misconceptions students have as related to reasoning and proving. She argued that to effectively engage students in reasoning and proving activities teachers need to know and be able to recognize the conceptions, misconceptions, and proof schemes of their students. This type of knowledge supports teachers' ability to interpret and apprise student-generated arguments and make instructional decisions that are developmentally appropriate to support, and advance, students' reasoning skills.

Finally, Lesseig (2016) operationalized the KCT for teaching reasoning and proving in terms of teachers' ability to represent, explain and connect proof-related ideas in a way accessible to students. She operationalized KCT in terms of how teachers respond and question students, as well as focus students' attention on the key characteristics of arguments that constitute proof. This is the type of knowledge teachers need to extend student thinking towards

generalizations and further justifications. Operational descriptions of MKT and PCK for teaching mathematical reasoning and proving, outlined by Lesseig (2016), are summarized in Figure 1. This framework served as a guide for this work.

MKT for Teaching Reasoning and Proving	
Common Content Knowledge (CCK)	**Specialized Content Knowledge (SCK)_**
(a) Ability to construct valid proof demonstrated through · Recognition and use of stated assumptions, definitions, and previously established results · Building a logical progression of statements · Analyzing situations by cases, when appropriate · Using counterexamples, when appropriate (b) Understanding of the nature of proof exemplified by recognition that · A theorem has no exceptions · A proof must be general · A proof is based on previously established facts · The validity of proof depends on its logic (c) Understanding that proof serves to establish the truth of a statement, communicate and systematize mathematical knwoledge	(a) Knowledge of proof components demonstrated by understanding of · Useful definitions or theorems · Language and defined terms · Visual and symbolic methods for generating a general argument · Recognition of situations in which different methods of constructing an argument (e.g. proof by exhaustion or counterexample) are effective · Characetristics of empirical and deductive arguments (b) Understanding that proof provides insights into why a statement must be true, builds mathematical understanding, or has explanatory power
PCK for Teaching Reasoning and Proving	
Knowledge of Content and Students (KCS)	**Knowledge of Content and Teaching (KCT)**
(a) Knowledge of student proof schemes demonstrated by the ability to · Recognize characteristics of different proof schemes (e.g., external, empirical, deductive) · Recognize and demonstrate awareness of student tendency to rely on authority or empirical evidence (b) Knowledge of the developmental nature of proof exemplified by the ability to · Recognize definitions and theorems accessible to students · Recognize forms of presenting mathematical arguments that are accessible and appropriate for student level · Understanding the relationship between the mathematical and everyday language in reference to argumentation and proof	(a) Understanding instuction which has the potential to deepen and foster student learning of mathematics with a focus on valid proof schemes · Responding to students in a way that helps them move away from the reliance on authoritarian or emirical proof schemes · Eliciting justifications and encouraging thinking about general cases · Selecting and implementing instructional strategies that scaffold student thinking with a focus on key ideas of proof (b) Knowledge of proof connections · Supporting students in making connections among various modes of presenting mathematical arguments · Ability to help students to link an argument structure to accepted definitions and to produce a general argument from numerical examples or specific diagrams

Figure 1. Aspects of MKT and PCK for teaching mathematical reasoning and proving delineated by Lesseig (2016).

Reflective Thinking in Teacher Education

Reflection is an essential element of teacher preparation (Dewey, 1933; Garcia, Sanchez, & Escudero, 2007) because it has the potential to consolidate teachers‘ knowledge. Schulman (1987) advocated for teacher education programs that support PSTs in connecting theory and practice through reflective opportunities. The educational literature documents that reflective practice supports the development of more sophisticated conceptual structures thus has the potential to enhance teachers’ learning (e.g., Driscoll, 2015; Wheatley, 1992). Teachers who reflect on their knowledge and practice are more effective (Santagata & Angelici, 2010; Star & Strickland, 2008) and have the potential to sustain professional growth after leaving teacher preparation and professional development programs (Garcia et al., 2007).

The literature on reflection in teacher education presents multiple models and strategies for engaging and supporting teachers’ reflective activities (pre-service and practicing alike) in effort to enhance learning and practice (e.g., Chamoso, Cáceres, & Azcárate, 2012; Davis, 2006; Dewey, 1933; Etscheidt, Curran, & Sawyer, 2012; Hatton & Smith, 1995; Huang, 2001; Lee, 2005; Roth McDuffie; 2004; Schön, 1983, 1987; Zeichner & Liston, 1987). While *reflection* and *reflective practice* are not unanimously interpreted within the large body of relevant literature, we view reflection as a means by which PSTs develop a greater level of awareness about educational practices and outcomes, and construct personal understanding with the potential to influence changes in their practice. This type of awareness creates opportunities for their professional growth and development. Davis (2006) described this type of reflective awareness as *productive* because it brings to surface the evidence for claims, alternatives or the questioning of assumptions.

Schőn (1987) viewed reflective activity as (a) reflection-in-action and (b) reflection-on-action. The first, is conceptualized as a more immediate form of reflection that occurs *during* activity, and the second is conceptualized as a form of reflective activity in which one engages *after* instruction or experiences. In both cases one draws on multiple knowledge bases (e.g., personal knowledge and beliefs, professional knowledge, current or past experiences). Of specific interest to research described in this manuscript was PSTs' reflection-on-action where they drew on their knowledge, learning, beliefs and experiences as related to K-8 mathematics; particularly, mathematical argumentation, reasoning and proving in reference to the elementary and middle grades mathematics.

Methods

Participants and Study Settings

The study reported here draws on data from a larger study of K-8 PSTs' learning about mathematical reasoning and proof in teacher preparation program. The larger study, situated in a large Midwestern university in the U.S., examined the knowledge development of PSTs over the course of their teacher preparation program and into their student-teaching with a specific focus on the knowledge of reasoning and proving. This paper is limited to data collected in the second semester of the larger project during which the PSTs were enrolled in a mathematics content course and a concurrent pedagogy course with mathematics-focused field experience. The field component was the first (of two) mathematics-focused field components in the program. Both courses were designed for education majors preparing to teach grades 1-8.

Participants, n = 36, were all elementary education majors concurrently enrolled in a mathematics course and an education course with mathematics-focused field experience. They were in their second or third year of a teacher education program and all previously completed

the first mathematics course in a 3-course mathematics sequence for K-8 education majors. All 36 PSTs volunteered to participate in the study.

Mathematics content course. The course, taught in the Mathematics Department, addressed concepts fundamental to elementary and middle school curriculum such as numbers and operations, place value, whole number operations, fractions and fraction operations, and proportional reasoning. Using reasoning and proof as a vehicle for creating deeper understanding, PSTs were engaged in activities designed to support their knowledge of elementary mathematics and their knowledge of elementary students' mathematical thinking. The course instructional activities were situated in the context of mathematical tasks presented from the perspective of teacher practice. Class activities, together with small group and whole class discussions, provided PSTs with context for generating and critiquing the validity of mathematical arguments.

Pedagogy course with field experience. Concurent pedagogy course with mathematics-focused field experience was taught in the College of Education. The course was designed to provide the PSTs with opportunities to explore methods for teaching elementary school mathematics. The curricula of mathematics and pedagogy courses were coordinated to maximize opportunities for the PSTs to connect their learning of mathematics and their learning of teaching K-8 mathematics with a focus on mathematical argumentation, reasoning and proof. Both courses utilized activities that fostered connections with authentic teaching practice. For example, PSTs were asked to prepare, conduct and audiotape two problem-based interviews with a student selected from their field placement classroom. Each interview was about 20 minutes long. The goal of these interviews was for the PSTs to engage "their" student in justifying a conjecture. Because the PSTs' placements ranged from 1st to 5th grade, the set of ten problems

was proposed by the instructors to help PSTs select an interview task appropriate for the grade level of "their" interviewee. The interview experience provided an authentic context for PSTs' reflections on their practice of eliciting justifications and encouraging student thinking about general cases.

Data Sources

The data for this study included: (1) Four reflective journals; two completed at the beginning and two at the end of the semester, and (2) papers in which PSTs (a) analyzed mathematical arguments generated by "their" students during one-on-one interview sessions, (b) reflected on their *practice* of engaging the student in mathematical reasoning and proving, and (c) reflected on their *learning* about mathematical reasoning and proof in elementary school mathematics. The PSTs completed their papers at the end of the semester.

Journals. Journal prompts were open-ended, designed to elicit PSTs' views about mathematical argumentation and proof in K-8 mathematics. At the beginning and the end of the semester, each PST was asked to define "mathematical reasoning and proof" as related to K-8 mathematics, and to describe the characteristics of classroom practices which they consider to be effective in engaging elementary and middle school students in reasoning and proving. Two times during the semester, PSTs were also asked to characterize mathematical tasks that have the potential to engage K-8 students in reasoning-and-proving activity and explain how the reasoning-and-proving tasks might differ from tasks that do not encourage mathematical reasoning and proving. As a follow up, PSTs were asked to propose (select or design) a task with a goal of using this task to engage elementary school students in their field classroom in a reasoning-and-proving activity. They were also required to explain why "their" task has the potential to engage a class of students in reasoning-and-proving activity.

Interview Analysis Papers. As a culminating part of the PSTs' experiences in both courses, they were asked to analyze and reflect on their ability to engage their interviewee in justifying a conjecture, and reflect on their own learning about reasoning and proof in elementary school mathematics. To prepare, PSTs were asked to transcribe audiotapes of each interview they conducted and use these transcripts in conjunction with any artifacts of "their" student's work to support their written analyses. PSTs' final papers served as data in this research.

Data Analysis

The analysis utilized qualitative and quantitative methods. Qualitative content analysis method supported the generation of themes, categories, and patterns from the data (Davis, 2006; Marshall & Rossman, 1999). Several cycles of data analysis were conducted. First, the data were carefully examined to delineate meaning segments (phrases, sentences, paragraphs) which provided insights into the specific views or understandings about mathematical argumentation and proof as revealed by each PST. Guided by previously described framework for teacher knowledge for teaching reasoning and proving (Lesseig, 2016) and open coding (Miles & Huberman, 1994) the goal of the analysis was to identify relevant views and understandings documented in the identified segments. To illustrate, consider the following comment delineated from PST #A4's Journal:

> There are certain areas of elementary math that are more suitable for mathematical reasoning and proving…word problems and more in depth problems would be more suitable than simple addition and subtraction because there is more to work with and to think about… The solving process for addition problems is very simple and when students are asked questions to make them reason they struggle because it is so simple. I think word problems would be more suitable for fostering mathematical reasoning and justification because it challenges them [students] to think outside of the box rather than repeating memorized math facts.

This segment was identified as reflecting the view of teaching with a focus on reasoning and proving (aspect of PCK), and further labeled as representative of a curricular understanding.

Quantitative analyses followed, which included tabulating and comparing code frequencies, to allow for identifying patterns in PSTs' learning about mathematical argumentation, reasoning and proof.

Results and Discussion

Research Question 1: *What views and understandings about mathematical reasoning and proof in elementary school mathematics emerge from the analysis of PSTs' reflections on their learning experiences in the integrated mathematics content and pedagogy with field experience courses? How do these views evolve over the course of the semester?*

Summarized in Tables 1 and 2 are themes that describe understandings of mathematical reasoning and proof in K-8 mathematics that emerged from the analysis of reflective journals PSTs completed at the beginning and the end of the semester.

Table 1. Views and understandings of mathematical reasoning and proving revealed at the beginning of the semester

Theme	Sub-Theme	#PSTs (%)* (n = 36)	Descriptions of subthemes
Views on the nature of reasoning and proving	Meaning of reasoning and proving	26 (72%)	Explaining (Explaining *how* one arrived at their conclusion; sharing different ways of thinking; presenting rationales)
		9 (25%)	Convincing (Using evidence and reasons to support one's mathematical statement; supporting claims with data and reasons; stating *why* things do, or do not, work in mathematics)
Views related to teaching with a focus on reasoning and proving	Curricular Focus	13 (36%)	Some curricular topics *do not facilitate* opportunities for engaging students in mathematical argumentation, justification and proving (e.g., topics in numbers and operations, study of computational algorithms)
		14 (39%)	All curricular topics in elementary mathematics can provide context for engaging students in mathematical argumentation, reasoning and proving
		12 (33%)	Specifically contextual problems can provide good context for engaging students in mathematical argumentation, justification, reasoning and proving
	Instructional Focus and Expectations	22 (61%)	Students share their reasoning and explain *how* they arrived at their results
		16 (36 %)	Students are engaged in challenging each other's ideas and they question each other mathematical thinking

		11 (31 %)	Emphasis is placed on mathematical *evidence* and *reasoning* that supports answers.
	Classroom Culture	15 (42%)	Culture of asking "why" questions, students are comfortable to share/evaluate, and critique each other's thinking

* % in each category do not add to 100% because not all *or* more than one theme/subtheme was identified in PSTs' journals

Table 2. Views and understandings of mathematical reasoning and proving revealed at the end of the semester

Theme	Sub-Theme	#PSTs (%)* (n = 36)	Description of subthemes
Views on the nature of reasoning and proving	Meaning of reasoning and proving	8 (22%)	Providing rationales behind answers and procedures, rather than just answers
		18 (50%)	Mathematical argument reflects understanding; Correct answer does not signify understanding
		16 (44%)	Arguments that prove stand to any challenge, are general, address *all* possible cases in the problem domain
		9 (25%)	Arguments that prove build on known/established facts and definitions
Views related to teaching with a focus on reasoning and proving	Curricular Focus	17 (47%)	Not all curriculum materials/tasks/activities may foster student engagement in mathematical argumentation and proof
		19 (53%)	Reasoning and proving tasks prompt for insights about *why* something is true (or not), foster thinking about generality, prompt for reasoning rather than single number answer, foster generating counterexamples, allow opportunity for divergent ways of thinking
	Instructional Focus and Expectations	30 (83%)	Good questions promote reasoning-and-proving activity
		15 (42%)	Instructional focus on mathematical argumentation and proof benefits students and teachers; Student generated arguments give teachers insights into students' mathematical thinking and understanding
		14 (39%)	Instructional focus on reasoning and proving requires careful planning/providing students with tools they can use to make arguments (e.g., manipulatives, pictures); modeling what a good argument looks like, planning questions
		17 (47%)	Instructional emphasis on reasoning and proving and expectations to justify need to be consistent/aligned with assessment
	Classroom Culture	6 (16%)	Classroom community of inquiry supports engaging students in mathematical reasoning and proving
		13 (36%)	Classroom that values different ways of thinking supports engaging students in mathematical reasoning and proving

* % in each category do not add to 100% because not all *or* more than one theme/subtheme was identified in PSTs' journals

Views on the Nature of Reasoning and Proving

PSTs' reflections about the meaning of mathematical reasoning and proving provided some insights into their understanding of these concepts. At the beginning of the semester (Table 1), PSTs' understanding of mathematical reasoning and proving was largely that of *explaining* the *how* behind one's thinking or strategies, or explaining what one does to solve a problem. The vast majority of the PSTs discussed mathematical arguments with an emphasis on classroom discussions and describing the "what" aspect of one's mathematical activity. Only a quarter of PSTs (9 of 36, 25%) discussed mathematical arguments explicitly in terms of presenting *evidence* and *reasons* to support a mathematical claim.

At the end of the semester (Table 2), PSTs shifted their interpretations of reasoning and proving in elementary school mathematics from thinking about mathematical arguments as explaining how one arrived at their conclusion to viewing arguments and proofs in terms of students' mathematical understanding. Half of the PSTs explicitly shared that the ability to produce the correct answer does not signify understanding, but the ability to provide a valid argument does. Forty-four percent of PSTs (16 of 36) explicitly discussed the issue of generality as a condition for an argument to prove, that is, to stand up to any challenge. Nine (25%) PSTs explicitly commented that mathematical arguments need to build on established facts and definitions.

Views on Teaching with a Focus on Reasoning and Proving

PSTs' reflections which addressed their understanding of pedagogical aspects of teaching elementary school mathematics with a focus on reasoning and proving in elementary school mathematics were grouped into three themes: (a) mathematics curriculum, (b) instruction, and (c) classroom culture.

Views on mathematics curriculum. At the beginning of the semester (Table 1), about a third of the PSTs shared that curricular topics in elementary mathematics, specifically topics in numbers and operations, are not rich enough to engage students in reasoning and proving. Those PSTs viewed the study of numbers and operations as memorization of facts and procedures. For example, PST #26 shared in her journal: "Elementary mathematics has a memorization component … flashcards are frequently used to help students memorize their facts. This memorization piece of math makes it less suitable for mathematical reasoning and argumentation." She then went on to say how situating the study of computational procedures in the context of word problems can provide better opportunity for engaging students in reasoning and proving because the students can "defend the realistic-ness of their answers." In their beginning of the semester journals, PSTs frequently used their own experiences with elementary school mathematics to say that topics that address computations and computational procedures provide fewer opportunities for engaging students in reasoning and proving. Only about 39% of the PSTs (14 of 36) at the beginning of the semester considered that students can engage in reasoning and proving within *all* curricular topics of elementary school mathematics.

At the end of the semester (Table 2), PSTs shifted their focus from thinking about curriculum topics, to thinking about the potential of curriculum materials in engaging elementary school students in reasoning-and-proving activity. Forty seven percent of PSTs (17 of 36) commented at the end of the semester that not all curriculum materials effectively engage students in reasoning and proving. Furthermore, 53% (19 of 39) of the PSTs identified in their end of the semester reflections at least one of the following as the necessary condition for an instructional material to facilitate a reasoning-and-proving activity: (a) whether the task prompts for insights about *why* something is true, (b) prompts for reasoning rather than single number

answer, (c) facilitates thinking about generality of results or a solution strategy, (d) provides opportunity to generate counterexamples, or (e) allows for divergent ways of thinking.

Views on instructional focus and expectations. Considering the views PSTs shared at the beginning of the semester about instructional focus and expectations for teaching elementary school mathematics with emphasis on reasoning and proving, two thirds of the PSTs (22 of 36, 61%) expressed their understanding with a reference to engaging students in explaining how they arrived at their results. PSTs' instructional expectations at the beginning of the semester appeared to be consistent with understanding they articulated about the meaning of reasoning and proving in K-8 mathematics (see Table 1). Only about a third of the PSTs (11 of 36, 31%) in their beginning of the semester journals, discussed the focus on evidence and reasoning as a highlight of instruction which emphasizes reasoning and proving.

At the end of the semester PSTs presented a wider range of views. Eighty three percent of PSTs (30 of 36) discussed the quality of questions that engage students in reasoning and proving, 42% (15 of 36) discussed how instruction focused on reasoning and proving benefits students and teachers. Close to a half of the PSTs (17 of 36, 47%) considered consistency and alignment with assessment as a characteristic of instruction that promotes reasoning and proving, and 39% (14 of 36) discussed instructional planning as a characteristics of instruction that engages students in reasoning and proving.

Views on classroom culture. In their beginning of the semester reflections over 40% (15 of 36) of the PSTs shared that a classroom culture which promotes reasoning-and-proving activity should create a comfortable environment for students to ask questions, share, evaluate, and critique each other's thinking. At the end of the semester, 36% (13 of 36) of the PSTs described classroom culture that promotes reasoning-and-proving activity as one that values

different ways of mathematical thinking, and an additional 16% (6 of 36) of PSTs discussed the culture of inquiry as supportive for fostering elementary students' engagement in reasoning and proving.

Research Question 2: *What understandings about their practice of engaging elementary school students in reasoning-and-proving activity do PSTs develop, as evidenced in their reflections on their learning experiences?*

The analysis of papers, each about 8-10 pages long, in which PSTs examined and reflected on their practice of engaging an elementary school student in justifying a conjecture revealed three major areas of focus. These included: (1) mathematical capabilities of elementary school students, (2) one's own pedagogical practice, and (3) one's broader views about teaching elementary school mathematics.

Developing an Understanding about Mathematical Capabilities of Elementary School Students.

Stylianides and Ball (2008) and Lesseig (2016) identified various aspects of mathematical and pedagogical content knowledge teachers need, to be effective in engaging students in reasoning and proving. Assuming that PSTs have the necessary knowledge base, they also need to hold a strong belief and understanding that elementary school students are capable of and have the potential to engage in reasoning-and-proving activity. In their reflections PSTs frequently shared that the interview experience helped them develop such understanding. Over a third of the PSTs (13 out of 36, 36.1%) explicitly discussed how the interview experience changed their belief about the mathematical capabilities of elementary school students. For example, this is how PST #7 reflected on her practice of engaging a fifth grade student in justifying a conjecture:

> I did not think that kids would be able to think in this way and make that argument when I had trouble making those arguments myself. My first interview with Ella [elementary school student] simply reestablished that belief because I was not able to get her to make that argument either. After a while, considering the mistakes I made, and what needed to be fixed, I realized after Interview #1, that maybe it would be possible if I pushed a little harder. After Interview #2 I realized that Ella is capable of thinking, arguing and justifying math and so are other kids. I learned that it is not the kids who have the problem but rather my previous inability to push her enough to make an argument. (PST #7)

She further discussed how during her first interview the student was providing "descriptive reasoning response" meaning, explained *what* she was doing rather than *why*, and went on to say:

> I learned that mathematical justification does not come naturally to kids. I was wrong to believe in Interview #1 that Ella would naturally understand what I wanted from her and validly justify. I was completely wrong. I was wrong not because Ella was not able to justify, I was wrong because I did not understand that I would have to push Ella beyond her natural tendencies [to describe what she was doing] and get her to explain the reasoning behind the *why* [it is true]. (PST #7)

Some PSTs related their expectations about elementary students' ability to engage in mathematical reasoning and to validly justify a conjecture to their own elementary school experiences. For example, PST #20 shared "I did not fully realize this [that elementary students can engage in reasoning and validly justify]. Looking back at me being in elementary school I don't remember having to explain my logic and reasoning." The emerging understanding of PSTs that elementary students *can* successfully generate mathematical arguments and proofs is powerful. Without this belief, it is unlikely to expect that PSTs will be successful in teaching elementary school mathematics with a focus on mathematical reasoning and proving.

Reflecting on her interview experiences with a first grade student, PST #41 commented not only on her developing understanding that the students are able to justify a mathematical statement but also her understanding of ways in which young students can do so. PST #41 shared:

> I believed that proving was a skill that few collegiate undergraduates can do, and cultivating and expecting it from a first-grader was unrealistic. …Yet, my experience this field semester offered me contradictions to most of my initial understandings. Heidi [1st grade student] was able to accurately reason and justify solving the interview task when she could manipulate the cards or move the blocks. Heidi was more engaged in the interview task when she was able to explore and manipulate the blocks on her own. Heidi's [justification] strategies were primarily visual with the help of specific examples working towards more concise reasoning where I wanted Heidi to explain why she can add numbers in whatever order, yet the same does not hold for subtraction. (PST #41).

Finally, some PSTs shared their developing understanding of the developmental nature of reasoning and proving, recognizing that elementary students might not habitually justify their mathematical work. They shared their observations that the reasoning habits need to be systematically developed and nurtured. With this idea in mind, PST #A44 shared:

> What I really learned from this experience is that elementary school students are not very likely to justify and prove. I feel that the part of the reason for this is that they don't know or are not used to being expected to doing more than giving an answer and showing their work. They need some time and guidance to develop these skills.

Developing an Understanding of One's Own Pedagogical Practices

PSTs' reflections addressed multiple aspects of their practice of engaging a student in a one-on-one interview setting in justifying a conjecture. In their interview analyses, over half of the PSTs (19 out of 36, 53%) examined their preparation and commented on their ability to nurture mathematical reasoning and promote justification during their one-on-one interactions with a student. A large group of PSTs (29 out of 36, 80.5%) explicitly discussed their questioning ability, reflecting on the effectiveness of questions they asked. A considerable group (15 of 36 PSTs, 42%) examined their practice through the lens of their ability to anticipate, attend to, and build on student thinking, or recognize opportunities for doing so.

Nurturing and engaging students in justification. The notion of nurturing and engaging students in reasoning and proving was a frequent theme throughout PSTs' reflections. There were, however, differences in how individual PSTs focused on this theme. In the context

of their reflections, many of the PSTs came to the realization that their efforts of fostering mathematical reasoning and engaging a student in providing justification were not always effective. Some PSTs characterized their interactions with a student during the interview as too scaffolding, and thus, discussed how they took away opportunity for their students to generate arguments of their own. For example, PST #2 used this lens to criticize her practice and reflected:

> On paper [while preparing for her interviews] I felt that I will do a good job of engaging a student and fostering justification because my questions [which she planned ahead of the interview] were both logical and precise. However, as I conducted both of my interviews they [the interviews] did not go as well as I had intended…. One of the biggest issues was my tendency to interpret Mikey's [2nd grade student] thinking and speak for him. Rather than expressing his own arguments Mikey agreed with my interpretation of his reasons. I did not give him any opportunity to articulate his mathematical reasoning. For example, during my first interview Mikey was struggling to explain why two odd numbers added together will always give even sum. ….As he explained it to me I noticed his struggle and in an effort to help him I spoke for him. This did not allow Mikey to clarify and make any rationale on his own. The same happened in my second interview. (PST #2)

Commenting on their efforts of fostering mathematical reasoning and proving during their interactions with a student, some PSTs reflected on their own mathematics content knowledge and their preparation for working with a student in one-on-one interview sessions. For example, PST #A6 examined her mathematics content knowledge which she exhibited "in the moment" during her work with a student. In her paper, PST #A6 described that the student (third grader) relied on numerical examples to conjecture about the sum of two odd numbers, the sum of an odd and an even number, and to justify the conjectures. Even though PST #A6 was aware of elementary students' tendencies to rely on empirical arguments, and explicitly stated that her goal was for the student to provide a valid general argument, during the interview she struggled to find an effective way to redirect the student's thinking away from empirical towards a valid generalization. She reflected:

> I did not know how to help the student except of telling her to add an even and an odd number to see what she gets. I had her to do this for adding 2 odd numbers, 3 odd numbers and adding an odd and an even number.

She continued her reflection by proposing how she instead *should* have worked with the student, and how she should have considered suggesting that the student use drawings of even and odd numbers as a tool that could help the student to articulate his/her reasoning:

> I have to be prepared because I really want my students to visualize and understand that when you add two odd numbers you always get an even number. I want the students to see that when you add two odd numbers each of these numbers will have 1 left over, which can form a group of 2 which makes an even number. So everything in the sum can be a groups of 2 so the answer is always even. …. When you have an even number, you can organize all the boxes in groups of 2 without any leftover. (PST #A6)

Similarly, PST #A17 commented how during the interview she missed the opportunity to guide her student to build an argument by directly drawing on the student's definition of even and odd numbers. Reflecting on her interview experience, PST #A17 shared: "I knew [that] the student knew what even and odd meant due to her defining it at the beginning. She said an even number can be paired up, while an odd cannot." Describing student's difficulty in verbally formulating a general argument for the assertion about the sum of two odd numbers PST #A17 continued:

> I could bring our counters [to the interview] and ask her to pair them up. A student could use them to help her build off [of] her already known information, and thus justify in general applying her information [definitions of even and odd] to all circumstances. (PST #A17)

Finally, considering their ability to nurture and engage students in reasoning and proving, some PSTs also recognized that students need to be given sufficient amount of time to develop, formulate, and defend their arguments. This aspect of PSTs' learning about their practice of nurturing students' mathematical reasoning, is illustrated with an excerpt of PST #A40's reflection:

> I have learned also that I need to give students more time to answer. In the first interview, whenever the student begun to answer, I would jump in and try to help her rephrase her thoughts before she even fully formed them. In my second interview I was conscious of

> giving the student enough time to respond, and also [about] asking questions that prompted her thinking without steering her directly to the answer that I thought was correct…. At other times I just said "OK" in order to let her know I was listening, and she continued on without me having to ask her to continue. As a result, the student was able to articulate *her* reasoning. This was in stark contrast to my first interview where I would try to expand on what she said, explain it back in more depth while adding my own reasoning, and question her "right?" (PST #A40)

Building on student's thinking and reasoning. Commenting on their experiences, PSTs frequently addressed their ability to attend to, and build on, student reasoning in order to facilitate and help students advance their reasoning, clarify ideas, or consider the validity of their arguments. Some PSTs recognized their own ability to build on student ideas as a skill that they need to improve.

PST #A14, for example, shared in her analysis:

> Even now, after the interview is done, I realized that I still don't quite understand his reasoning which was one of my mistakes during the interview. I should commit more thought to his explanation and analyzing potential loopholes which after looking at it [student response] now, I believe do in fact exist.

PST #A43 expressed a similar observation referring to a moment in which "her" student made a statement that "even number divided by 2 is always an odd number" and supported that statement with examples 6÷2=3; 10÷2 =5; 18÷2=9. Silent aspects of her, and many other PSTs' reflections, in which they evaluated their ability to follow up on students' responses, were recognitions of their limited in-the-moment noticing ability. This topic of reflection is illustrated with an excerpt of PST #A43's journal who, reflecting on her practice, shared how she may have inadvertently supported her student's faulty reasoning:

> I should not have just stopped him there, but instead guide him to correct his answer [assertion]. I worry that I might have given him the idea that his reasoning was correct although it was not. It would be beneficial for me to challenge his way of thinking. I should have asked what about 4 or what about 8? Those examples would not have held to Milo's [student] reasoning. If given the chance to conduct this part of the interview again I would challenge his way of thinking by following up his response, by asking to draw a picture to support his general case. (PST #A43)

Frequently, in the context of their analyses, PSTs recognized that at the time of conducting their interview they accepted case-specific numerical arguments their students proposed to support developed assertions. While reflecting on their ability to follow up on student ideas, PSTs offered strategies and suggested ways to advance students' reasoning beyond the reliance on the specific examples. Reflection of PST #A23's below illustrates this aspect of PSTs' learning:

> I wanted my student to give a generalizable rule about the questions [interview tasks]. I selected the Even and Odd task. It was effective because my student was able to make generalizable rule after adding 1+3=4, 5+1=6, 9+1=1, 3+5=8. She then proceeded to say I guess from this it will always be even. I should have asked her then "How do you know without adding them that it will always be an even answer?" …This question would allow my student to think more analytically to prompt her to reason about a general universal rule without examples. …I feel like I missed the opportunity to allow my student to develop her response. I moved right on to the next question that asked "What happens when you add three odd numbers together?" (PST #A23)

Some PSTs articulated how their ability to attend to, and build on, student reasoning improved in the context of their learning. They also shared their developing understanding of the value of this skill. An excerpt from PST #A11's reflection provides an example:

> I know I gained a lot of knowledge about students' mathematical reasoning. One skill I improved was being able to grab on to something a student says, and work with them to expand the idea. I think this is very important skill for a teacher to have… They [students] are not learning from an "outside" perspective.

Asking questions that promote justifications. An analysis of questioning patterns constituted frequent theme throughout PSTs' reflections. Addressing the wide range of issues related to their questioning patterns, PSTs demonstrated awareness of the clarity of their questions, types of questions they asked, question response time, or how effective their questions were in facilitating reasoning and proving.

For example, reflecting on changes in her questioning skills between the first and the second interview, PST #A40 shared:

> My [questioning] skills improved immensely between the first and the second interview. I struggled in the first interview to elicit responses from my student. Because I did not pose questions that were at the correct difficulty level for her, did not phrase them correctly, and asked too many leading questions, the first interview did not provide much opportunity for the student to articulate her reasoning despite its superior length. … The main problem with my first interview was that the questions I asked were too focused on solving [getting an answer] and not focused at all on reasoning and explanations why. … Because of the type of questions I asked, there was no need for her [the student] to show her reasoning, or even do anything other than agree with me.

Developing or challenging one's broader views about teaching and learning elementary school mathematics

The beliefs and views about teaching elementary school mathematics with a focus on reasoning and proving constituted another frequent theme revealed in PSTs' reflections. Within this theme, PSTs articulated their more subjective, personal understandings, which constituted their beliefs and expectations about teaching and learning elementary school mathematics with a focus on reasoning and proving.

Twenty two PSTs (61%) explicitly examined how their views about teaching elementary school mathematics with an emphasis on reasoning and proving evolved over the semester. Those PSTs emphasized that their learning experiences significantly expanded their beliefs about teaching elementary school mathematics. They reflected on the evolution of their thinking about themselves as future teachers and their role as teachers, which many of the PSTs articulated as perceived someone who is "helping students to arrive at a correct answer." PSTs challenged this belief at the end of the semester as illustrated in the excerpt of PST #A26's analysis:

> Before this class and field experience, I thought that the ultimate goal was to get students to answer correctly and quickly, and then objective was met. I have learned that teaching math involves so much beyond memorizing a strategy, method or process, and instead relies on successful understanding of why we do what we do in math. Understanding and formulating

> arguments makes it easier to solve different types of math problems and allows you to apply your knowledge to different concepts. … In my future classroom I want to create an environment that focuses on mathematical argumentation. I want to makes verbal and written explanations integral part of assessments and small group work.

She closed her reflection with a statement:

> This course [referring to her mathematics course] was challenging in the sense that I knew how to solve almost every problem, but explaining it and proving my answers was the hardest part. I have grown to enjoy proving and justifying my work through this class, and want to teach my students that habit as well, so they will be proficient reasoners in mathematics. (PST #A26)

Finally, the developing views about teaching elementary school mathematics with a focus on reasoning and proving were also evident in PSTs' recognition of the value of multiple perspectives and strategies for thinking about mathematics concepts. For example, PST #A23 summarized her observations in the following way: "An elementary classroom should provide a variety of ways to think about the mathematical concepts… This helps students [to] find a way to support their thinking, reason, and create arguments."

Summary and Conclusions

Educational research shows that teachers need the opportunities to construct their own narratives about their learning and their practice (Sparks-Langer & Bernstein-Colton, 1991). In this study, reflective narratives of pre-service teachers were analyzed to reveal their emerging views and understandings about mathematical reasoning and proving in elementary school mathematics. The results of this study serve several purposes. First, they provide insights into the kinds of understandings and views PSTs share about mathematical reasoning and proving in reference to elementary school mathematics at different points of a semester. Second, they provide insights into aspects of PSTs own practice on which they focus, and of which they demonstrate awareness, as they reflect on their ability to engage a student in reasoning-and-proving activity.

First, this study shows that many PSTs associate the meaning of reasoning and proving with explaining *how* one arrived at their answer, rather than, with constructing valid argument for *why* a given answer or assertion is true. PSTs' initial understanding of instruction that facilitates engaging students in reasoning-and-proving activity was also limited to descriptions of students who share their reasoning explaining *how* they arrived at their conclusion. Moreover, PSTs do not consider all mathematics topics presented in elementary school mathematics curriculum to be appropriate for engaging elementary school students in reasoning and proving. Specifically, at the beginning of the semester, prior to their own learning experiences in the mathematics course, PSTs were unable to envision that students can reason and construct mathematical arguments while learning topics in numbers and operations. They regarded that the learning of these topics happens (primarily) through memorization. This result might be similar to that reported for secondary school teachers who view teaching reasoning and proving as a special topic in the mathematics curriculum rather than integral aspect of mathematics teaching and learning (e.g., Knuth, 2002).

At the end of the semester, PSTs shifted their perception of contexts in which reasoning and proving activities can be fostered from thinking about specific topics in elementary school mathematics curriculum to curricular materials. Fifty three percent of participants recognized that tasks or instructional activities which prompt for insights about *why* something is true (or not), foster thinking about generality, prompt for reasoning rather than single number answer, foster generating counterexamples, or allow an opportunity for divergent ways of thinking provide an environment for engaging students in reasoning and proving. Building PSTs' understanding of characteristics of tasks that facilitate reasoning and proving is an important goal of teacher preparation. Research shows that in elementary school mathematics textbooks, tasks

that facilitate reasoning and proving are limited. Analyzing a series of different elementary school mathematics textbooks Bieda, Ji, Drwencke, and Pickard (2013) concluded that students are seldom asked to produce an argument that could be applied to general cases. Thus, helping PSTs develop understanding of tasks that promote student engagement in reasoning and proving is warranted to support PSTs' abilities to enact these practices.

Second, the authentic nature of the experience of engaging a, elementary school student in reasoning-and-proving activity and reflecting on their practice, assisted PSTs in applying essential aspects of the experience to their professional growth. Over a third of the PSTs realized that elementary school students can generate mathematical arguments and engage in reasoning and proving. Many recognized that the practices of reasoning and proving are learned skills thus need to be systematically nurtured. Over 80% of PSTs focused on different aspects of their questioning patterns, expressing greater awareness of the nature of their questioning behaviors and proposing alternatives with a goal of improving their questioning practices. PSTs' reflections also revealed some of the understandings and ideas about teaching elementary school mathematics shared by novice teachers and, at times, provided insights into the aspects of their practice that supported or challenged these understandings. Chapman (2002) articulated that "it has become an accepted view that it is the [mathematics] teacher's subjective school related knowledge that determines for the most part what happens in the classroom," (Chapman, 2002, p. 177). The knowledge of the specific beliefs and understandings PSTs hold in relationship to teaching and learning elementary school mathematics with a focus on reasoning and proving provides insights for mathematics teacher educators. Specifically, it sheds light on factors that can promote or hinder PSTs' learning.

Knowing what matters for PSTs, and what they emphasize in their reflections, allows mathematics teacher educators to understand the extent to which PSTs develop their views of teaching elementary mathematics with a focus on mathematical reasoning and proving. It also provides insights about the effect of the integrated instruction on PSTs' professional development.

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