Marquette University

From the SelectedWorks of Marta T. Magiera

April, 2010

An Extensive Analysis of Preservice Middle School Teachers' Knowledge of Algebraic Thinking

Marta T. Magiera, *Marquette University*
Leigh A. van den Kieboom, *Marquette University*
John C. Moyer, *Marquette University*

Available at: https://works.bepress.com/marta-magiera/30/

American Educational Research Association

From the

AERA Online Paper Repository

http://www.aera.net/repository

Paper Title An Extensive Analysis of Preservice Middle School Teachers' Knowledge of Algebraic Thinking

Author(s) Marta T. Magiera, Marquette University; Leigh A. Van Den Kieboom, Marquette University; John Moyer, Marquette University

Session Title Studies of Mathematics Teacher Knowledge in Pre- and In-Service Contexts

Session Type Paper

Presentation Date 4/30/2010

Presentation Location Denver, Colorado, USA

Descriptors Mathematics Education, Teacher Knowledge (cultural/content/pedagogical), Teacher Education - Pre-Service

Methodology Mixed Method

Unit SIG-Research in Mathematics Education

Each presenter retains copyright on the full-text paper. Repository users should follow legal and ethical practices in their use of repository material; permission to reuse material must be sought from the presenter, who owns copyright. Users should be aware of the Ethical Standards of the American Educational Research Association.

Citation of a paper in the repository should take the following form: [Authors.] ([Year, Date of Presentation]). [Paper Title.] Paper presented at the [Year] annual meeting of the American Educational Research Association. Retrieved [Retrieval Date], from the AERA Online Paper Repository.

Pre-service Middle School Teachers' Knowledge of Algebraic Thinking

Marta T. Magiera, Marquette University

Leigh A. van den Kieboom, Marquette University

John C. Moyer, Marquette University

Abstract

In this study we examined the relationship between 18 pre-service middle school teachers' own ability to use algebraic thinking to solve problems and their ability to recognize and interpret the algebraic thinking of middle school students. We assessed the pre-service teachers' own algebraic thinking by examining their solutions and explanations to multiple algebra-based tasks posed during a semester-long mathematics content course. We assessed their ability to recognize and interpret the algebraic thinking of students in two ways. The first was by analyzing the pre-service teachers' ability to interpret students' written solutions to open-ended algebra-based tasks. The second was by analyzing their ability to plan, conduct, and analyze algebraic thinking (AT) interviews of middle school students during a concurrent semester-long, field-based education class. We used algebraic habits of mind as a framework to identify the algebraic thinking that pre-service teachers exhibited in their own problem solving, and we asked students to use them to analyze the algebraic thinking of middle school students. The data revealed that pre-service teachers' AT abilities varied across different features of algebraic thinking. In particular, their ability to justify a rule was the weakest of seven AT features. The ability to recognize and interpret the algebraic thinking of students was strongly correlated with the strength of the pre-service teachers' own algebraic thinking. Implications for mathematics teacher education are discussed.

Background

Over the last three decades the mathematics education community has engaged in discussions about the role and the nature of school algebra in the mathematics curriculum. While most mathematics educators advocate for the inclusion of algebra-based topics at the K-8 level, they are by no means calling for elementary and middle school students to be taught algebra in the *traditional* way. Traditional *algebra* focuses on issues related to skills such as manipulating algebraic expressions and solving equations. In contrast, *early algebra* instruction aims to advance students' conceptual knowledge and skills by shifting attention away from symbolic manipulations and equation solving toward analyzing and generalizing patterns using multiple representations (NCTM, 1989, 2000; Silver, 1997; Kieran, 1996; Carpenter & Levi, 2000). The teaching of algebra concepts at the early grades focuses on the development of *algebraic thinking* by providing students with opportunities to examine algebraic ideas in the context of arithmetic. Ideally, algebraic experiences in the elementary and middle grades are designed to allow students see algebra as a network of knowledge and skills rather than as a muddle of isolated concepts. Done in this way, early algebra instruction is much more likely to prepare students for a smooth transition from arithmetic to more formal algebra (Carpenter & Levi, 2000; Silver, 1997; Kieran, 1992, 1996; Kaput, 1998).

Algebraic Thinking

The term *algebraic thinking* has different connotations. For some, algebraic thinking closely relates to what Cuoco, Goldberg, and Mark (1996) defined as habits of mind, useful ways of thinking about mathematical content. For example, Driscoll (1999) used this term to signify thinking about quantitative situations in ways that make the relationships between variables obvious. He conceptualized algebraic thinking as thinking habits focused on Building Rules to

Represent Functions (BRRF), making generalizations by abstracting from computations, and doing and undoing procedures and operations. The algebraic habits of mind encapsulate thinking processes that, for example, focus on recognizing and analyzing patterns, investigating and representing relationships, generalizing beyond specifics of an example, analyzing how processes or relationships change, or seeking arguments for how and why rules and procedures work.

Others (e.g. Kieran and Chalouh, 1993) use the term *algebraic thinking* to connote the ability to build meaning for the symbols and operations of algebra in terms of arithmetic. Although Kieran and Chalouh's connotation is somewhat different than Driscoll's, it is not inconsistent with it. Kieran (1996) further refined this perspective, interpreting algebraic thinking as the ability to use a variety of representations to analyze quantitative situations in a relational way. Swafford and Langrall (2000) interpreted algebraic thinking as the ability to think about unknown quantities as known. Kieran (2004) summarized that algebraic thinking in the early grades can be developed

> …within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. (p. 149)

A natural consequence of the call for algebra reform is a concern that effective early algebra instruction cannot occur without a more adequate preparation of elementary and middle school teachers. Recent reports published by the U.S. Department of Education's Mathematics Panel (2008) and the National Council on Teacher Quality (Greenberg & Walsh, 2008) recommend strengthening teachers' understanding of the algebra-based ideas taught at the middle school level. The early introduction of algebraic ideas provides challenges. Teachers'

own experiences with traditional school algebra often strongly influence and limit their views of algebraic thinking and, in turn, counter their efforts at mathematics education reform. For example, both practicing and pre-service teachers' understanding of algebraic topics often consists of the fragmented knowledge of a disconnected system of symbols and procedures (Ball, 1990). Teaching that is informed by such limited knowledge short-circuits the algebraic-thinking goals of early algebra instruction because, unless elementary and middle school teachers understand the ideas behind algebraic thinking, they are unable to provide, recognize, or take advantage of opportunities to engage students in algebraic thinking. Teachers need to understand how students develop algebraic thinking in order to capitalize on students' reasoning in a way that helps students develop an understanding of algebraic ideas and make connections among them.

Teacher Knowledge

Teachers' knowledge has been identified as an important factor that influences the outcome of their practice (Borko & Putman, 1996). Sowder and Schappelle (1995), and Hill, Rowan, and Ball (2005) documented how students' achievement closely relates to teachers' mathematical knowledge. Prospective teachers need to learn how to provide elementary and middle school students with opportunities to see algebra as a study of patterns and structures, and how to use elementary and middle school students' informal pre-existing knowledge to facilitate the transition from arithmetic to the more abstract and formal ways of thinking needed in algebra. In order to do so, pre-service teachers must not only be able to themselves think algebraically, they also must be able to identify it in students. Therefore, in our research we broadly define pre-service teachers' "knowledge of algebraic thinking" as the ability to think algebraically, coupled with the abilities to engage students in algebraic thinking and to recognize

and interpret algebraic thinking in students.

Despite Ball's (1990) stated concerns about pre-service teachers' limited and procedural knowledge of the K-12 mathematics curriculum, few research efforts have focused on pre-service teachers' knowledge of algebraic thinking. An understanding not only of pre-service teachers' ability to think algebraically, but also of its relationship to the teachers' ability to recognize and interpret the algebraic thinking of students is very much needed. This understanding is paramount for the design of strong teacher education programs that successfully prepare teachers to introduce early algebra concepts and foster algebraic thinking in their K-8 students. To prepare prospective teachers for the challenges of early algebra instruction, mathematics teacher educators need to have a strong understanding of pre-service teachers' knowledge of algebraic thinking, broadly defined.

Goal

The goal of this study is to address the need in teacher education research to provide insight into pre-service middle school teachers' broadly defined knowledge of algebraic thinking. We are seeking to understand the nature of pre-service middle school teachers' knowledge of algebraic thinking through the analyses of (1) the strength of pre-service teachers' algebraic thinking, (2) pre-service teachers' awareness of opportunities to engage students in algebraic thinking, and (3) the relationship between the strength of pre-service teachers' algebraic thinking and their ability to recognize and interpret middle school students' algebraic thinking.

Conceptual Framework

Building Rules to Represent Functions (BRRF)

For this research, we conceptualized algebraic thinking in a way consistent with Kieran (1996), Driscoll (1999, 2001), Swafford and Langrall (2000) and used the taxonomy of algebraic

habits of mind (Driscoll, 2001) as a framework. We focused our investigation on the aspects of algebraic thinking identified as Building Rules to Represent Functions (BRRF) and used Driscoll's description, presented in Table 1, of the different features of BRRF, as our operational definition.

Derry, Wilsman, and Hackbarth (2007) make the case that complex concepts, such as those related to algebraic thinking, cannot easily be explained or taught using rule-bound instruction. They believe that teachers develop their knowledge of algebraic thinking by being immersed in situations that elicit different aspects of algebraic thinking. With this idea in mind, we sought to create an instructional approach that would engage pre-service teachers in algebraic thinking in the context of situations that encouraged them to recognize and reflect on different forms of BRRF (Driscoll, 1999; 2001) in their own thinking and in the thinking of students. Thus, throughout the narrative of this paper we use the term *algebraic thinking* (AT) with reference to ways of thinking that are useful for BRRF, unless otherwise specified.

Table 1
Features[1] *of Algebraic Thinking (BRRF) Examined in This Study*

Features of Algebraic Habits of Mind	Description of Thinking Exemplified
Organizing Information	Ability to organize information in ways useful for uncovering patterns, relationships and the rules that define them
Predicting Patterns	Ability to notice a rule at work and make sense of how a rule works
Chunking Information	Ability to look for repeating chunks in information about a pattern
Different Representations	Ability to think about and try different representations of the problem to uncover different information about the problem.
Describing a Rule	Ability to describe steps of a procedure or a rule
Describing Change	Ability to describe change in a process or a relationship
Justifying a Rule	Ability to justify why a rule works for *any* number

[1] Adapted from Driscoll (2001)

Multi-Tier Design

Our goal was to capture pre-service teachers' knowledge of algebraic thinking in diverse situations; therefore, we used a multi-tier design (Lesh & Kelly, 2000) to conduct our study. We conceptualized pre-service teachers' knowledge of algebraic thinking as (a) their own AT competencies interpreted as the ability to use different features of algebraic thinking in their own solutions and explanations, (b) their ability to recognize opportunities to engage middle school students in different features of algebraic thinking interpreted as their ability to analyze algebra-related problems for their potential to elicit different features of algebraic thinking, and (c) their ability to recognize and interpret features of algebraic thinking in the work of students. The first tier of this research focused on (a) pre-service teachers' AT competencies as demonstrated in their own solutions and explanations. The second tier of this research focused on (b) pre-service teachers' analysis of problems for their potential to elicit different features of algebraic thinking and (c) pre-service teachers' investigations of the algebraic thinking exhibited by students. We analyzed pre-service teachers' ability to interpret problems and their ability to recognize and interpret middle school students' algebraic thinking in order to gain an understanding of how the strength of their own algebraic thinking related to their ability to recognize and interpret the algebraic thinking of students.

Method

Participants

Participants in this study included 18 undergraduate pre-service teachers (grades 1-8 teaching certification candidates) at a large private Midwestern university. Sixteen of the participants were female and two were male. All participants were juniors or seniors. The seniors were in their final semester prior to their student teaching experience. They were enrolled in an

integrated mathematics content and field experience course designed for pre-service teachers. The content course component was taught in the mathematics department, and it addressed topics in middle school algebra. The goal was to help pre-service teachers develop the ability to interpret, compare, see connections, and generalize across multiple topics within the middle school mathematics curriculum. It engaged pre-service teachers in activities that solicited multiple solutions and representations of mathematical tasks, and encouraged sharing, explaining, comparing, and making interpretations of various representations and reasoning. The field component was taught in the College of Education. It consisted of two weeks of university classroom instruction followed by weekly classroom observations of middle school mathematics instruction and one-on-one tutoring sessions conducted by each pre-service teacher with a selected middle school student. The emphasis of the field component was to engage pre-service teachers in activities that involved analyzing the algebraic thinking of middle school students in authentic classroom situations.

Data Sources and Data Collection

To investigate the nature of the pre-service teachers' knowledge of algebraic thinking we examined several different kinds of data collected during the semester-long study:

- *Class assignments and tests*: Each pre-service teacher completed 130 algebraic tasks selected to elicit their own algebraic thinking and/or encourage them to recognize and interpret the algebraic thinking evident in the artifacts of written students' work.
- *AT interviews:* Each pre-service teacher conducted and transcribed two 45-minute audio-recorded clinical interviews with one middle school student.
- *Debriefing interviews:* We conducted a 30-minute video-recorded semi-structured debriefing interview (Ginsburg, 1997) with each pre-service teacher following each algebraic-thinking

interview with his or her selected middle school student. The goal of the debriefing interviews was to probe pre-service teachers' interpretations of middle school students' algebraic thinking and explore the evidence pre-service teachers used to interpret student thinking. During the interviews, the pre-service teachers' reflections concerning the middle school student's algebraic thinking were stimulated by artifacts of their interviews with their selected middle school student, viz. transcripts of the interviews and the middle school student's written work.

- *AT analysis papers*: After finishing the two AT interviews and both debriefing interviews, each pre-service teacher submitted a paper that provided the pre-service teacher's written interpretation of the selected middle school student's algebraic thinking.

The video- and audio-recordings were transcribed, and all written artifacts of pre-service teachers' work were digitalized for use with the NVivo software.

Data Analysis

Because we were interested in characterizing pre-service teachers' knowledge of algebraic thinking in various contexts (their own work, their ability to analyze mathematical tasks, and their work with students), we selected a combination of qualitative and quantitative methods as the most promising mode of inquiry. We analyzed the data in two phases: a content phase (using the class assignment and test data) and a field phase (using the AT interview data, the debriefing interview data, and the AT analysis paper data). A complete summary of the data analysis process can be found in Appendix A.

Content Phase. We divided the content phase of our data analysis, which comprised qualitative and quantitative analyses of pre-service teachers' written work on assignments and tests, into two stages: (a) pre-service teachers' own AT work and their ability to analyze AT

tasks; and, (b) pre-service teachers' ability to recognize and interpret algebraic thinking in examples of student written work. We used our operational definition of algebraic thinking (as illustrated in Table 1) to identify the features of algebraic thinking encouraged by each task and to code pre-service teachers' solutions to each task.

Stage 1 of the Content Phase. In the first stage of the content phase, we analyzed the pre-service teachers' solutions and explanations to each task with a goal of identifying the features of algebraic thinking exhibited in them. We followed up with a qualitative rating of the strength of algebraic thinking exhibited in each task. We scored the strength of algebraic thinking for each identified feature, as (3) proficient, (2) emerging, or (1) not evident. Finally, we quantified the strength each teacher's algebraic thinking on each feature of BRRF by computing the average of the teacher's ratings on that feature across all analyzed tasks. We also quantified the strength of each pre-service teacher's overall algebraic thinking by computing the average of the teacher's ratings across all analyzed tasks.

Proficient. We rated a pre-service teacher's thinking as (3) proficient on an identified feature if the answer was *correct*, and if the solution articulated thinking *characteristic* of that feature (e.g., if the problem solution showed evidence that the participant "…organized information in ways useful for uncovering patterns, relationships and the rules that define them"), and at the same time provided *clear links* to the context of the problem.

Emerging. We rated a pre-service teacher's thinking as (2) emerging on an identified feature if the answer was *correct*, and if the solution articulated thinking *characteristic* of that feature, but *without clear links* to the context of the problem. We also rated the pre-service teacher's thinking as (2) emerging on an identified feature if the answer was *incorrect*, but the

solution articulated thinking *characteristic* of that feature with *clear links* to the context of the problem.

Not evident. Finally, we rated the strength of a pre-service teacher's thinking as not evident on an identified feature if the problem explicitly encouraged using the feature but the solution did not articulate thinking *characteristic* of that feature (e.g., the problem statement explicitly asked the student to find a formula that could be used to predict a given pattern but such a formula was not included in the solution).

Stage 2 of the Content Phase. In the second stage of the content phase, we examined tasks in which the pre-service teachers were asked to recognize and interpret the algebraic thinking of students. We rated the pre-service teachers' ability to recognize and interpret algebraic thinking in the work of students using the system described above (proficient, emerging, and not evident). We used the 3-point scale to quantify the strength of the pre-service teachers' ability to recognize and interpret students' algebraic thinking with respect to different features. Furthermore, we used pairs of averages overall and for each feature to examine the relationship between pre-service teachers' strength of algebraic thinking and their ability to recognize and interpret the algebraic thinking of students.

Field Phase

In the field phase of our data analysis we analyzed (a) the transcripts of the clinical interviews that the pre-service teachers conducted with their selected middle school student, (b) the debriefing interview transcripts, and (c) the pre-service teachers' AT papers. First, we analyzed the debriefing interview transcripts to identify pre-service teachers' descriptions of their chosen interview tasks' potential to engage students in algebraic thinking. We used our operational definition of algebraic thinking to code the features of algebraic thinking identified in

the pre-service teachers' descriptions, and we used open coding (Miles & Huberman, 1994) to identify patterns in pre-service teachers' perceptions of task potential.

Furthermore, we selected five pre-service teachers with high overall AT scores (range 2.58 – 2.82) and five pre-service teachers with low overall AT scores (range 1.93 – 2.34), and used the clinical interview transcripts, debriefing interview transcripts, and AT papers to examine possible qualitative differences in high and low AT pre-service teachers' ability to recognize and interpret algebraic thinking of students in context of their own field practice.

To establish validity and reliability, three mathematics education experts in algebraic thinking research independently applied the coding schemes to different subsets of collected data. They then compared the three sets of independent results and cited specific examples to clarify the coding schemes and negotiate coding agreement to 100%.

Results

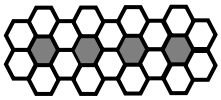
We begin by addressing pre-service teachers' algebraic thinking in the context of tasks they solved in their mathematics content class. We present a detailed examination of sample solutions to a selected task, providing evidence of various features of algebraic thinking identified in these solutions and discussing the strength of the pre-service teachers' ability to use the identified feature. We then follow with a discussion of the strength of the pre-service teachers' algebraic thinking across the collection of tasks and further discuss the nature of algebraic thinking evident in the pre-service teachers' solutions and explanations. Next, we discuss the answer to our second research question, for which we examined the pre-service teachers' ability to interpret algebra-based tasks for their potential to engage middle school students in algebraic thinking. Finally, we present the results of our analysis of the pre-service teachers' ability to recognize and make sense of middle school students' algebraic thinking, and

we discuss the relationship between the strength of pre-service teachers' algebraic thinking and their ability to recognize and interpret the algebraic thinking of students.

Algebraic Thinking Evident in Pre-Service Teachers' Solutions

We use the task presented in Figure 1 as a context for discussion of the different features of algebraic thinking evident in the pre-service teachers' written solutions and explanations to this task**.** Figures 2, 3, and 4 are examples of solutions and explanations that the pre-service teachers' in our study provided for the Flower Bed Task. We use these examples to illustrate how we identified the different features of algebraic thinking and assessed the strength of the pre-service teachers' thinking with respect to each identified feature.

Flower Beds



The city council wishes to create 100 flower beds and surround them with hexagonal paving slabs according to the pattern shown above. (In this pattern 18 slabs surround 4 flower beds)

(1) How many slabs will the council need?

(2) Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

[a]Shell Centre for Mathematical Education, 1984, p. 64.

Figure 1. Example of Tasks Used in the Content Class.

Pre-Service Teacher #10's Solution

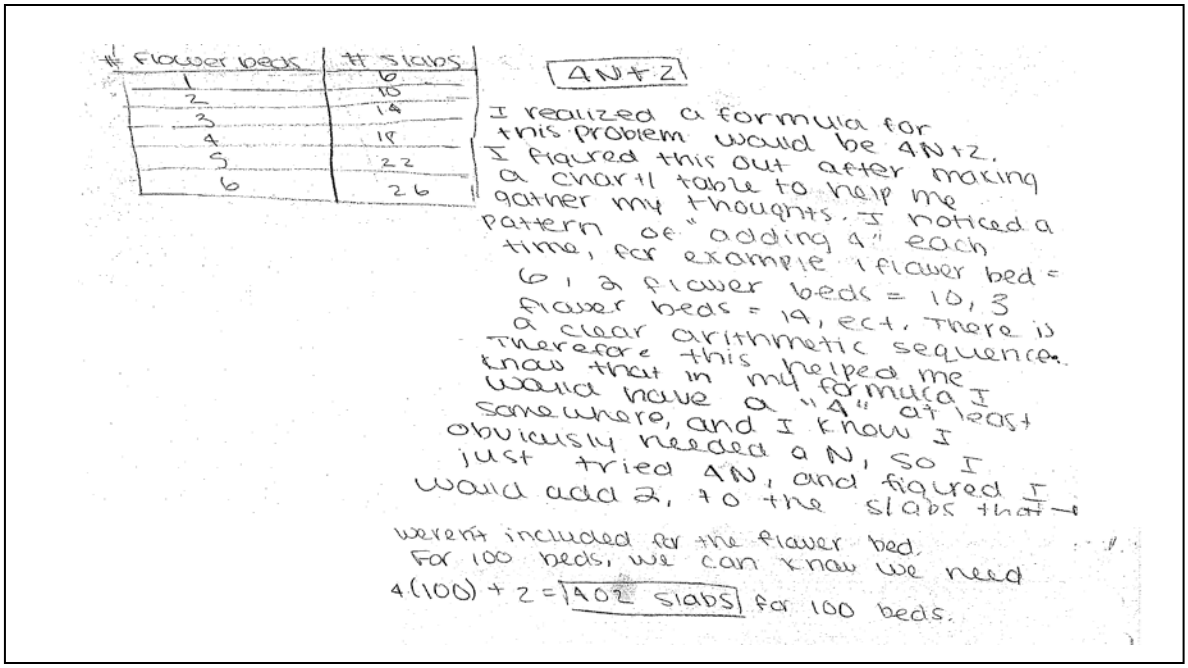
The solution presented in Figure 2 provides an insight into pre-service teacher #10's ability to organize information in a way that is useful for guiding her own thinking about the flower bed pattern. She used the table she constructed to help her understand the relationship between the number of flower beds and the number of slabs. She explained how noticing the

"… pattern of 'adding 4' each time" helped her to develop the rule that corresponded to adding four slabs each time a new bed was constructed.

> …know that I will have a '4' at least somewhere [in the formula], and I knew I obviously needed an N, so I just tried 4N, and figured out I would add 2 for the slabs that weren't included for the first flower bed.

In these ways, pre-service teacher #10 demonstrated proficiency in two of the features of BRRF: Organizing Information ("Ability to organize information in ways useful for uncovering patterns, relationships and the rules that define them"), and Describing a Rule ("Ability to describe steps of a procedure or rule") using the expression $4N + 2$. Thus we rated her solution as *(3) proficient* in Organizing Information and also *(3) proficient* in Describing a Rule.

Her same explanation, however, also clearly reveals that she has little or no understanding of *how* or *why* the predicted rule $4N + 2$ works. That is, her explanation clearly demonstrates her inability to make connections between the predicted rule and the pattern that the rule describes. Thus, we rated the strength of pre-service teacher #10's ability to both Predict Patterns ("Ability to notice a rule and work and make sense of how a rule works") and Justify a Rule ("Ability to justify why the rule works for any number"), in the context of this problem, as *(2) emerging* in accordance with our scoring rubric.



# Flower beds	# slabs
1	6
2	10
3	14
4	18
5	22
6	26

4N+2

I realized a formula for this problem would be 4N+2. I figured this out after making a chart/table to help me gather my thoughts. I noticed a pattern of "adding 4" each time, for example 1 flower bed = 6, 2 flower beds = 10, 3 flower beds = 14, ect. There is a clear arithmetic sequence. Therefore this helped me know that in my formula I would have a "4" at least somewhere, and I know I obviously needed a N, so I just tried 4N, and figured I would add 2, to the slabs that weren't included for the flower bed. For 100 beds, we can know we need 4(100) + 2 = 402 slabs for 100 beds.

Figure 2. Flower Bed Task, PST #10.

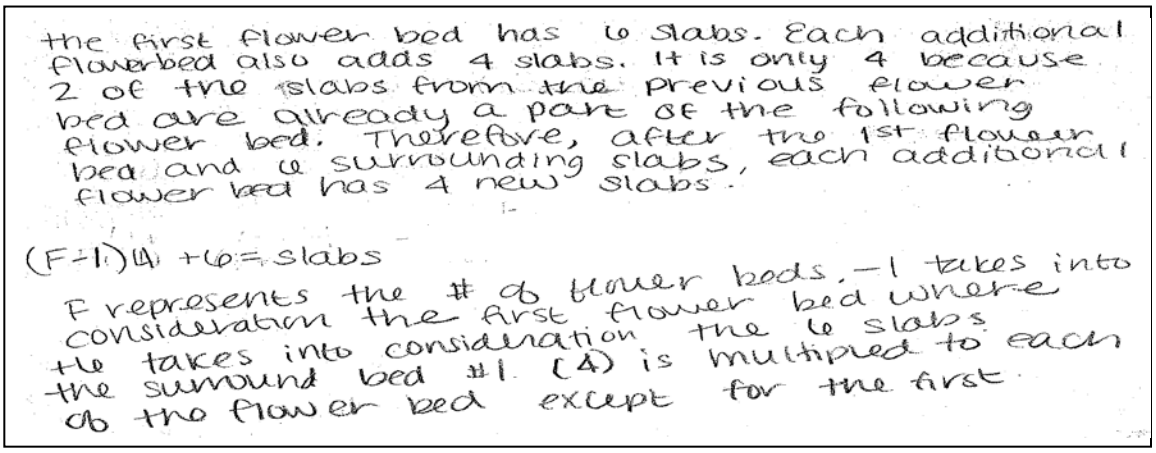
Pre-Service Teacher #11's Solution

The solution presented in Figure 3 is the work of pre-service teacher #11 who, like pre-service teacher #10, recognized the regularity of adding four slabs to construct each additional flower bed. Unlike PST #10, PST #11 did not organize the problem's information in a table. However, her solution clearly shows that she verbally organized all the important information. Therefore, as we did for pre-service teacher #10, we rated #11's ability to Organize Information and Predict Patterns as *(3) proficient*. While we rated PST #10's abilities to Predict a Pattern and Justify a Rule as *(2) emerging* because she was not able to explain *how* or *why* her rule worked for any number of flower beds, we rated PST #11 as *(3) proficient* in both Predicting a Pattern and Justifying a Rule because she was able not only to make sense of how the rule $(F-1)\cdot 4+6=slabs$ works, but also to explain how the observed regularity is seen in the formula and in the context of the problem:

> Each additional flower bed also adds 4 slabs. It is only 4 because 2 of the slabs from the previous flower bed are already a part of the following flower bed. Therefore after the first flower bed and surrounding slabs, each additional flower bed has 4 new slabs…(4) is multiplied to each of the flower bed except the first.

Although the notation she used might be a cause for concern about PST #11's use of variables, her explanation linking the above formula to the pattern clearly showed this pre-service teacher's ability to validate the predicted rule.

Her statement "The first flower bed has 6 slabs. Each additional flower bed also adds four slabs," is evidence that this pre-service teacher was *(3) proficient* at Chunking Information ("Ability to look for repeating chunks of information about a pattern"). Finally, we rated PST #11 as *(3) proficient* at Describing Change ("Ability to describe change in a process or relationship") because her proficiency with respect to this feature of algebraic thinking is evident in her statement "…each additional 1 flower bed has 4 new slabs." This statement is clear evidence that she understands there is a functional relationship between the change in the total number of slabs and a unit change in the number of flower beds.


the first flower bed has 6 slabs. Each additional flowerbed also adds 4 slabs. It is only 4 because 2 of the slabs from the previous flower bed are already a part of the following flower bed. Therefore, after the 1st flower bed and 6 surrounding slabs, each additional flower bed has 4 new slabs.

(F-1)(4) +6 = slabs

F represents the # of flower beds. -1 takes into consideration the first flower bed where the takes into consideration the 6 slabs the surround bed #1. (4) is multipled to each of the flower bed except for the first.

Figure 3. Flower Bed Task, PST #11.

Pre-Service Teacher #9's Solution

Figure 4 shows how pre-service teacher #9 approached the flower bed task. The solution reveals her ability to create different representations (i.e., a verbal description, a formula, a table, and a diagram) to guide her thinking about characteristics of the flower bed design. That is, her solution revealed that she was *(3) proficient* creating Different Representations ("Ability to think about different representations of the problem to uncover different information about the problem") of the flower bed pattern.

Unfortunately, in her verbal representation of the solution, she reasoned proportionately about chunks of four flower beds, (4 flower beds:18 slabs = 100 flower beds:450 slabs). Because the situation in the flower bed task is not proportional in nature, pre-service teacher #9's verbal representation does correctly predict the total number of slabs needed for 100 flower beds. Furthermore, she did not seem to realize that her verbal representation is inconsistent with her other three representations. For example, the answer (450) she gets using the proportional relationship in her verbal description does not agree with the answer (402) she would get by substituting 100 for F in her formula.

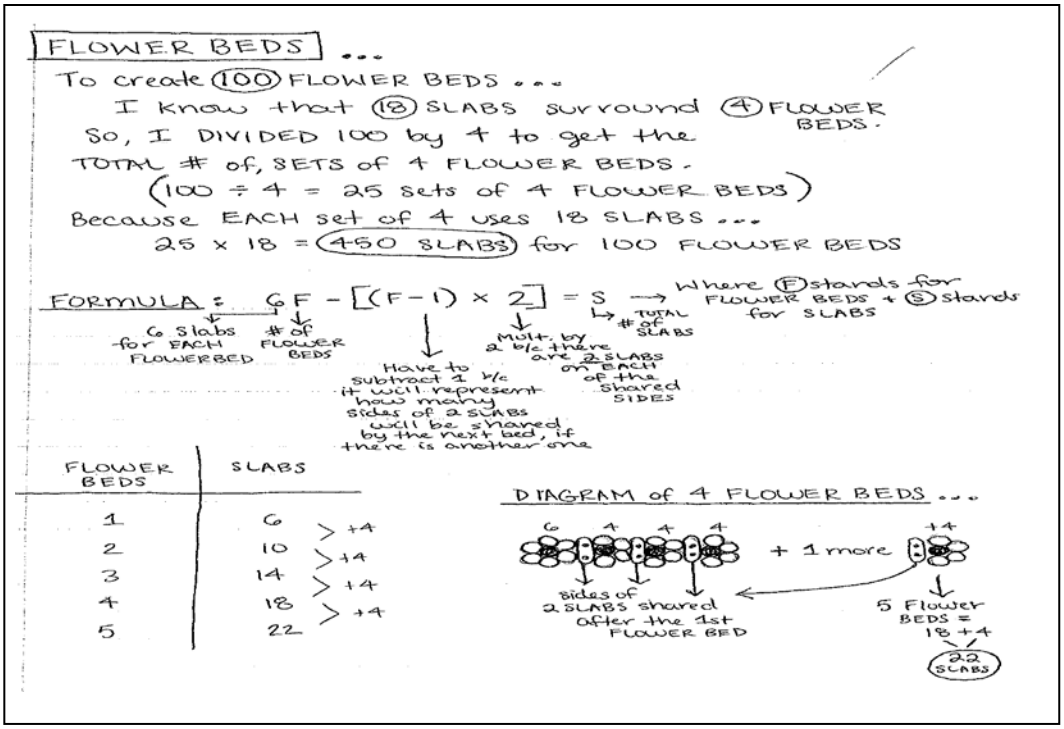


Figure 4. Flower Bed Problem, PST #9.

Turning to pre-service teacher #9's table and diagram, we see that the thinking she revealed in these representations is focused on the change (+4) that occurs in the total number of slabs for each unit (+1) change in the total number of flower beds. As a result, we rated her ability to Describe Change in a relationship as *(3) proficient* based on evidence from both the table and the diagram's characterization of the relationship between the number of flower beds and the total number of slabs. This is in contrast to our "Describing Change" rating of pre-service teacher # 10 (Figure 2), who organized the flower bed information in a similar table. However, her table did not contain any evidence that consideration was given to describing the change relationship that exists between the number flower beds and the total number of needed slabs.

PST #9's diagram also reveals that she analyzed the pattern by focusing on repeated chunks of information in at least three different ways. First, her diagram demonstrates that she recognized the regularity of adding (a chunk of) four slabs for every additional flower bed. Secondly, her diagram provides evidence that she realized that a pair of slabs is shared by each pair of adjacent flower beds (repeating chunks of information about the pattern). Third, the diagram also provides an insight into how pre-service teacher #9's thinking about the two types of chunks led to the formulation of the rule $6F - \left[(F-1) \cdot 2 \right] = s$. Specifically, it is clear from the rule and the accompanying justification that the pre-service teacher realized that the net change in slabs (chunk = 4) for each additional flower bed is the result of subtracting the number of overlapping slabs (chunk = 2, as noted in the diagram) from the number of slabs in a single flower bed (6).

Although she did not realize it, PST #9 predicted two patterns that were not consistent with each other. Her verbal representation led to an incorrect pattern because the sense-making she used to justify the verbal rule was based on an incorrect understanding of proportionality. Her formulaic representation was correct because the sense-making she used to justify the formula was based on a correct understanding of chunking information and describing change. Despite the fact that one of her predicted patterns was correct and one was incorrect, we rated her ability to both predict patterns and describe a rule as *(3) proficient,* but her ability to justify a rule as *(2) emerging.* We did so because we felt that, given the correct assumptions about the underlying structure of a pattern, she would be very proficient at predicting a pattern and describing change. Furthermore, it seems clear that her error, as well as her inability to recognize the inconsistency in the two patterns, stemmed from an inability to justify a rule (rather than predict or describe it), in addition to lacking the inclination to do so.

Strength of Algebraic Thinking

Table 2 summarizes the strength of the pre-service teachers' algebraic thinking across all tasks and features. The results provide a reason to be rather optimistic. The overall mean score for the algebraic thinking, as evidenced in the pre-service teachers' solutions across all tasks and all features, was $\bar{M}$ = 2.455 (max 3) with *SD* = 0.242. Of all the features of algebraic thinking, the pre-service teachers' ability to justify a rule was by far the weakest, as illustrated in Table 2.

Table 2
Pre-Service Teachers' Performance Scores on Different Features of Algebraic Thinking

Features of Algebraic Thinking Demonstrated in Pre-service Teachers' Work	$\bar{M}$	*SD*	*Range*
1. Organizing Information	2.56 1	0.295	2.12 – 2.99
2. Predicting Patterns	2.543	0.293	2.01 – 2.99
3. Chunking Information	2.385	0.389	1.71 – 2.95
4. Different Representations	2.496	0.420	1.46 – 2.99
5. Describing Rule	2.578	0.218	2.21 – 2.90
6. Describing Change	2.456	0.311	1.65 – 2.78
7. Justifying a Rule	2.169	0.408	1.19 – 3.0

Our examination of the different features of algebraic thinking that arose in the pre-service teachers' solutions allowed us to *recognize* and *assess* the strength of the pre-service teachers' ability to employ the features of algebraic thinking. However, it did not give us any information about how the pre-service teachers' abilities to use the individual features relate to each other. We anticipated that uncovering possible associations between the identified features of algebraic thinking might provide additional information concerning the nature of algebraic thinking evidenced in the pre-service teachers' work. Thus, we extended our analysis by examining the associations between pairs of performance scores (strength) on different features of algebraic thinking identified across the collection of tasks.

As shown in Table 4 and the accompanying 3-dimensional diagram (Figure 5), there were significant positive pair wise correlations among the following 5 features: (1) Organizing Information (the ability to organize information in ways useful for uncovering patterns, relationships, and the rules that define them), (2) Predicting Patterns (the ability to recognize a rule at work and make sense of how a rule works), (3) Chunking Information (the ability to look for repeating chunks of information about a pattern), (5) Describing a Rule (the ability to describe steps of a procedure or a rule), and (7) Justifying a rule (the ability to justify why a rule or procedure works for any number). The diagram illustrates that 8 of the 10 possible pairs of these five features were significantly correlated. The only two features that were not significantly correlated are indicated by dotted segments in the diagram, viz. the pair wise correlations of (7) Justifying a Rule with both (1) Organizing Information and (5) Describing a Rule. The heavier weights of four segments in the diagram illustrate that four of the significant pair wise correlations were stronger ($0.72 < r < 0.91$) than the other four significant correlations ($0.48 < r < 0.54$). Interestingly, neither of features (4) Different Representations (the ability to think about and try different representations of the problem to uncover different information about the problem) or (6) Describing Change (the ability to describe change in a process or relationship) was significantly correlated with any of the other abilities.

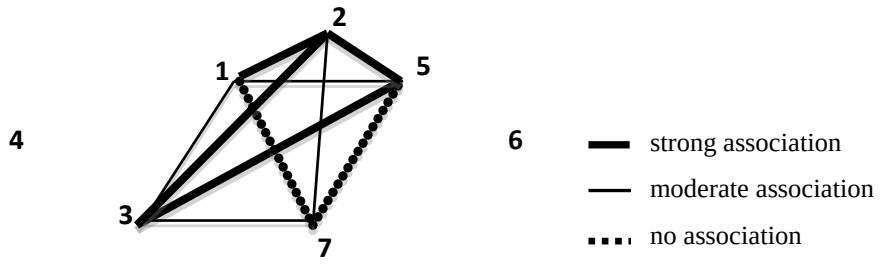


Figure 5. Pair wise correlation patterns among seven AT features

Table 3
Correlations Between Performance Scores on Different Features of Algebraic Thinking

Feature of Algebraic Thinking	1	2	3	4	5	6	7
1. Organizing Information	–						
2. Predicting Patterns	0.717**	–					
3. Chunking Information	0.535*	0.914**	–				
4. Different Representations	0.387	0.466	0.399	–			
5. Describing a Rule	0.508*	0.771**	0.727**	0.277	–		
6. Describing Change	0.462	0.337	0.321	0.122	0.173	–	
7. Justifying a Rule	0.444	0.537*	0.484*	0.324	0.360	0.376	–

* α = 0.05, ** α = 0.01

These results show that there are strong and significant pair wise correlations among the abilities to (2) predict patterns, (3) chunk information, and (5) describe a rule. Significant relationships also exist between these three abilities and (1) the ability to organize information, as well as (7) the ability to justify a rule. However, the relationships are not as strong, and in the case of (7), they are not pair wise complete, i.e. associations between (7) and (1) and between (7) and (5) are not even statistically significant. Finally, the results show that abilities (4) Different Representations and (6) Describing a Rule are not related to any of the other five abilities.

Pre-Service Teachers' Interpretations of a Task's Potential to Foster Algebraic Thinking

We studied participants' awareness of opportunities to engage students in algebraic thinking in the context of the tasks they selected for their AT interviews. Prior to conducting the AT interviews with a middle school student, we asked the pre-service teachers to select two of the seven tasks presented in Appendix B. These tasks were similar to the tasks pre-service teachers solved in their mathematics class. Each task encouraged analyzing a pattern and describing it in terms of a rule or a procedure. During each debriefing interview, we asked the pre-service teachers to reflect on the potential of the selected task to foster the algebraic thinking

of a middle school student. We prompted the participants' description of the thinking that the selected task could elicit by posing the question: "Which features of the algebraic habits of mind did you expect the problem could elicit from the middle school student?" We then followed up with the questions "why?", and "Are there any other features of algebraic habits of mind that you think the task could encourage?"

The overall summary of the pre-service teachers' responses are presented in Table 4.

Table 4
Recognition of Tasks' Potential to Engage Students in Algebraic Thinking

Features of algebraic thinking encouraged by the task	Task (Number of participants using and analyzing task)						
	Task 1 (15)	Task 2 (8)	Task 3 (5)	Task 4 (3)	Task 5 (3)	Task 6 (3)	Task 7 (1)
Organizing Information	7	2	3	3	3	3	1
Predicting Patterns	11	5	3	1	1	1	1
Chunking Information	6	1	2	0	1	2	1
Different Representations	7	3	3	1	0	0	0
Describing Rule	7	4	3	3	3	2	1
Describing Change	6	3	1	2	1	1	0
Justifying Rule	6	3	4	3	3	3	1

Despite extensive discussion and analysis of the features of the BRRF algebraic habit of mind in the content class, the pre-service teachers were able to identify in these tasks only a limited number of the features that underlie BRRF. Only 55% of the pre-service teachers' responses anticipated that a selected task could be used to encourage students to engage in at least four of the seven different features of algebraic thinking.

Our analysis of the pre-service teachers' responses revealed two common characteristics underlying the pre-service teachers' perceptions of task potential: (1) reliance on one's own

mathematical experiences with the task, and (2) literal use of the task description to identify the thinking processes encouraged by the task.

Pre-service teachers frequently referred to their own experiences with the selected task and rarely considered that the task might encourage ways of thinking different from their own. These results indicated that the pre-service teachers' own understanding of the mathematics embedded in each of the tasks greatly influenced their recognitions of task potential for eliciting algebraic thinking, as did their prior experiences with the task. In particular, their interpretations of task potential were closely related to the ways of thinking they exhibited in their own mathematical work. This might explain, at least partially, the reason that the pre-service teachers had such overall limited perceptions of a task's potential to foster algebraic thinking in students.

The debriefing interview excerpts below illustrate how pre-service teachers' awareness of their own thinking while solving a task guided their judgments about the features of algebraic thinking that the task could possibly foster:

> Um, I would say definitely organizing information, cause when I did this problem myself, I wrote out like seven, nine, but then also, drew like squares for the figure, so, I drew kind of a numerical thing (PST 1)

> Going into it I thought organizing information just because I know, for me as a learner, immediately when I did this problem, I did a chart and I just did the figures that way (PST 6)

> When I was doing it originally, um, I think immediately you can create a table. (PST 14)

> Well, definitely organizing information, because I knew one of the first things I anticipated, one of the first things I did was to write this out as a list (PST 18)

> And noticing… I didn't really see that until after I came up with a formula and the diagrams increased… I think the more you can do the formula with the different numbers you plug in, that can prove that the formula is right. (PST 11)

> While doing it in class I thought about chunking information by showing that the different, like it starts with bottom one and then you go by two and just keep increasing by two. Different representations too, by the equation, pattern ([referring to the picture shown], and by explanations. (PST 3)

While the participants referenced above used recollections of their own thinking about a task to identify a task's potential to foster algebraic thinking in students, other participants used the statement of the task itself as a guide. Exclusive focus on the task statement, without considering various alternate ways of thinking about the solution to the task, often limited the pre-service teachers' ability to anticipate features of algebraic thinking that the task might foster, thus leading to a superficial and incomplete judgment of the task's potential to elicit features of algebraic thinking.

> I knew that the student would have to justify how she came up with the rule because that was stated in the series of questions. (PST17)
>
> Well, definitely predicting patterns because pattern is in the title, yeeh, so patterns for sure. (PST4)
>
> Oh, predicting patterns because they have a pattern in the problem here [referring to the statement of the problem] (PST 9)
>
> Let's see, I didn't really have to organize any information per se because it [the task] already gave you the picture. So you did not need to organize information. (PST 5)

Pre-Service Teachers' Interpretation of Algebraic Thinking in Student's Work

Quantitative analysis. The analysis of pre-service teachers' performance on tasks that asked them to recognize and interpret middle school students' algebraic thinking did not reveal differences between the pre-service teachers' own AT proficiency and their ability to recognize and interpret algebraic thinking in the work of students. A paired samples *t* test failed to reveal a statistically reliable difference between the mean of the AT proficiency scores (M = 2.457, SD = 0.242) and the mean of the recognition and interpretation scores (M = 2.433, SD = 0.272; $t(18) = .406$, $p = .690$). Descriptive statistics summarizing the pre-service teachers' ability to recognize and interpret students' written work are included in Table 5.

Table 5

Pre-Service Teachers' Ability to Recognize and Interpret Algebraic Thinking in Student Written Work

Features of Algebraic Thinking Elicited and Interpreted by PSTs'	*M*	*SD*	*Range*
Organizing Information	2.611	0.697	1.00 – 3.00
Predicting Patterns	2.277	0.574	1.00 – 3.00
Chunking Information	2.556	0.615	1.00 – 3.00
Different Representations	2.500	0.485	1.5 0 – 3.00
Describing Rule	2.388	0.530	1.5 – 2.83
Describing Change	2.277	0.391	1.5 – 2.7
Justifying Rule	2.555	0.855	1.00 – 3.00

The recognition and interpretation scores ranged from 1.2 – 2.9, where higher scores identified increased proficiency in recognizing and interpreting algebraic thinking in the work of students. There was a significant positive correlation between pre-service teachers' own AT scores and their recognition and interpretation scores ($r = 0.623$, $p = 0.009$). These results suggest that one's own AT proficiency might be a good predictor of one's' ability to recognize and interpret students' algebraic thinking.

Qualitative analysis. The analysis of AT interview transcripts, debriefing interview transcripts, and algebraic analyses thinking papers provided further insight into the relationship between pre-service teachers' own AT competencies and their ability to recognize and interpret the algebraic thinking of students.

We analyzed the AT interviews that pre-service teachers conducted with their assigned middle school student. We also analyzed the pre-service teachers' papers, which provided us with their written interpretations of the selected middle school students' algebraic thinking abilities in context of the tasks they posed. Our goal was to gain insight into the pre-service

teachers' ability to elicit, recognize, and interpret students' algebraic thinking ability based on interviews that the pre-service teachers planned and conducted. In particular, we sought to examine how pre-service teachers with high (2.58 – 2.82) and low (1.93 – 2.34) AT scores elicited, recognized, and interpreted the algebraic thinking of students. The data provided evidence that, in the context of the interviews they conducted, the pre-service teachers with high AT scores not only consistently elicited algebraic thinking from their interviewees, but they also were able to recognize and interpret students' algebraic thinking when it occurred. The pre-service teachers identified as having low average AT scores, on the other hand, were much less consistent in eliciting algebraic thinking and recognizing situations where students engaged in algebraic thinking. Further, they were limited in their ability to interpret the students' thinking in these situations. Generally, when attempting to analyze student thinking the low AT pre-service teacher group emphasized what the students did during their one-on-one interview sessions, rather than analyze how they thought.

The examples below demonstrate the qualitative differences between the high and low AT pre-service teachers' ability to analyze the algebraic thinking of students in context of their interviews. The first excerpt illustrates how a pre-service teacher (PST #6) in the high group identified and made meaning of a middle school student's thinking in the context of Task 3. The pre-service teacher not only recognized the algebraic thinking behavior of the student, but also identified the observed behavior as exhibiting the ability to chunk information to describe how a pattern works:

> She [the middle school student] was able to predict a pattern. She stated "Like two children go over, one comes back, an adult goes over, then a child comes back, wait, so if two children go over and one comes back and then one adult goes over and child comes back, so that's two go over one comes back and adult goes over the child comes back. Wait, its' the same thing over and over again!" . . . At first she was counting … then she realized that the pattern repeated itself ever four turns and then "plus one" at the end of the problem was the

> two children crossing at the end. It was interesting to see her coming up with a rule 4a + 1 because the plus one is for children coming back. She was thinking in chunks CC C A C and CC C A C. (PST 6)

Another pre-service teacher (PST #17) in the high AT group interpreted how the student sought to predict the V-pattern by focusing on each side of the V-design (Task 1). This pre-service teacher identified how the student engaged in thinking about repeating chunks of information by consistently thinking about the pattern in terms of two groups of blocks:

> He did a good job chunking the information to make more sense of the problem, and in a long run making his development of an equation simpler. He states "there is three on this side [. . .] if you add three to the four you get seven". This statement, along with his usage of the figure, indicates that he is thinking of the figure in two different sections. The one side that is equal to the figure number and the other side that is equal to one less than the figure number. Later when describing another figure he states:" So, there is fourteen on this side not counting this one, and then there is fifteen. (PST 17)

While the pre-service teachers with high AT scores consistently linked the behaviors observed during one-on-one AT interviews to students' algebraic thinking, the pre-service teachers with low AT scores rarely provided such connections. Rather than focusing on students' thinking, the latter group emphasized students' actions by highlighting what the student did during the one-on-one interview.

For example, consider clinical interview excerpt showing how a low AT pre-service teacher (PST #18) engaged a middle school student in solving Task 2:

Student:	If the pattern continues how many of the blocks will be contained in the next letter V? So, there is one in the first, three in the second, five in the third, seven in the sixth, no I mean in the fourth. So… there will be one, two, three, four, five, six, seven, eight, nine blocks.
PST:	How did you solve that?
Student:	Because I figured out you have two more blocks to every V because one has one, that has to be the tip, and then in the second pattern [second letter V] there are two, and in the third pattern [third letter V] there is two more and so on.
PST:	And what did you mean by tip?
Student:	Cause, the letter V has to have a point like right there. . .
PST:	So, does the tip ever change as the pattern goes up?
Student:	No.

In her paper, when she "interpreted" her middle school student's thinking, the pre-service teacher recognized that the student engaged in writing (action) for the purpose of organizing the problem information:

> Within the first problem [Task 1], the letter V, she did begin an interesting organization process: she wrote out the first figure numbers 1 through 15, and then next to it put the number of total blocks in each of these figures. (PST 18)

Her "interpretation," however, failed to link the behavior of the student to the student's thinking about the regularity in the number of tiles needed for each consecutive letter V. In this context, the low AT pre-service teacher failed to interpret the middle school student's recognition that each consecutive V requires two additional blocks as an instance of using the AT feature Chunking Information.

Later in the paper, the same pre-service teacher continued to focus on her student's actions. Further, when she finally attempted to connect that action to the AT feature Describing Change, she did so in a naïve and superficial way:

> She [the student] saw in both problems [Task 1 and Task 2] that the figures changed each time. She used counting to figure out changes that were occurring from one figure to figure. She stated " . . . there is one in the first, three in the second, five in the third, and seven in the fourth" in reference to the change in the number of blocks in the letter V problem. She knew [that] change was occurring and used counting skills to distinguish the differences in figures. (PST 18)

Another example of a missed opportunity to interpret student's algebraic thinking was demonstrated by a pre-service teacher in the low group (PST 5). Like PST 18, this pre-service teacher described the actions of the student and failed to link them to specific features of algebraic thinking that the student employed. Specifically, in an obvious attempt to reference the Different Representations feature of algebraic thinking, PST #5 focused on the middle school student's actions of making diagrams, charts, and an equation, but she failed to discuss any

specifics about the student's "[a]bility to think about and try different representations of the problem to uncover different information about the problem:" .

> While working through the first problem [Task 2], she paid a lot of attention to the blocks in the middle of the letter I. This is when she noticed the increase in blocks. She drew out [diagrams] for all the size lengths probably because she is a visual learner. After making diagrams she made charts, and a rule or equation. Each of these were like a step in her process of getting equation. The diagram helped her build the chart and the chart helped her create an equation. (PST 5)

Overall, our findings provide reasons both to be encouraged and to be discouraged about our pre-service elementary teachers' broadly defined knowledge of algebraic thinking. We are encouraged because the PSTs own solutions and explanations to algebra-based problems demonstrated some reasonably high ability to think algebraically. Also, we are encouraged because the PSTs demonstrated the ability to recognize various features of algebraic thinking in students' written work. On the other hand we are discouraged because the PSTs' analyses of the algebraic thinking of students in the context of their clinical interviews with students were much weaker. Their awareness of opportunities to foster different features of algebraic thinking in the context of clinically administered, algebra-based tasks was also limited.

Discussion and Implications

Our study helps to fill a gap in the existing body of literature related to early algebra instruction by investigating an important under-researched area, namely pre-service teachers' knowledge of algebraic thinking, broadly defined. Our work examined pre-service teachers' knowledge of algebraic thinking by identifying (1) their own AT competencies interpreted as the ability to use different features of algebraic thinking in problem solutions and explanations, (2) pre-service teachers' ability to recognize opportunities to engage middle school students in different features of algebraic thinking, interpreted as their ability to analyze algebra-related problems for their potential to elicit different features of algebraic thinking, and (3) pre-service

teachers' ability to recognize, and interpret features of algebraic thinking in the work of students. We also examined the relationship between the strength of the pre-service teachers own AT competencies and their ability to recognize and interpret algebraic thinking exhibited by students.

The first significant finding of our study is a promising one. We found that the pre-service teachers in our cohort were able to competently use many features of algebraic thinking to solve algebra-related problems. Although overall promising, the results indicated that the strength of the pre-service teachers' ability to justify a rule or procedure was weak when compared to the strength of their ability to engage in the other features of the habit of mind Building Rules to Represent Functions. The latter result is consistent with Castro (2004) who also found that pre-service teachers lacked sufficient ability to justify why algebra-based algorithms and procedures work.

Another significant result that the data revealed was the complex nature of the algebraic thinking identified in the pre-service teachers' work. The ability to (2) analyze and predict patterns was positively associated with the ability to (1) organize information, (3) look for repeating chunks of information in the pattern, (5) describe how the rule or procedure works, and (7) generalize how the rule or procedure works. Taken together, these correlations suggest that the pre-service teachers' abilities to (1) organize information, (2) predict patterns, (3) chunk information, and (5) describe a rule support each another in a mutual, symbiotic, and holistic way. However, our research also suggests that while the ability (7) to justify a rule may depend somewhat on (2) and (3), strengthening (2) and (3) is probably not sufficient to support the strengthening of (7) in a significant way. Furthermore, (7) appears to be fairly independent of (1) or (5). Finally, the correlations suggest that the abilities (4) to use different representations

and (6) to describe change are not closely interwoven with any of the other five abilities. The uncovered correlations among the features of algebraic thinking might suggest that rather than targeting learning activities at algebraic thinking in general, helping teachers to become competent algebraic thinkers may be better accomplished by targeting learning activities at specific AT features or groups of features, namely (4) Representations; (6) Change; (2,3,7) Patterns, Chunking, and Justifying; (1,2,3,5,7) Organizing, Predicting, Chunking, Describing a Rule, and Justifying.

Strengthening pre-service teachers' algebraic thinking abilities should be a focus of the entire teacher education program curriculum, and not an exclusive aim of an isolated course. In all aspects of mathematical content pre-service teachers should explicitly be encouraged to consider alternative solutions, in the context of which, they could question, challenge, reason, generalize and justify. To develop and assimilate ways of thinking useful for thinking about mathematical content the pre-service teachers need systematically engage in thinking how different ways of representing or organizing given situation might help to reason about and provide justifications for different mathematical descriptions of that problem-situation.

Secondly, the pre-service teachers demonstrated the ability to recognize the various features of algebraic thinking in the students' written solutions to selected algebra-based problems. Our comparison of the mean AT competency scores with the recognition and interpretation scores did not indicate they were statistically different. In fact, we uncovered a strong positive association between these two groups of scores. This positive relationship suggests that a pre-service teacher's own AT proficiency might be an important factor in the teacher's ability to analyze students' algebraic thinking through their written work.

Our results concerning high and low AT pre-service teachers' knowledge of algebraic thinking shed further light on the relationship between the pre-service teachers' own algebraic thinking and their ability to recognize and interpret the algebraic thinking of students. The pre-service teachers in the high AT group were more consistently able than the teachers in the low AT group to apply the features of algebraic thinking to the tasks posed in the content course, They were also more consistently able to interpret the thinking that students' used to answer the problems posed during the AT interviews. In contrast, the pre-service teachers in the low AT group were less consistent in their identifications of different aspects of algebraic thinking that middle school students showed during the problem-based clinical interviews. The low AT group of pre-service teachers analyzed the students' AT thinking predominantly by recounting students' actions to solve the problem without making connections to students' thinking.

Prior research documents that understanding students' thinking provides teachers with important insights about how students develop mathematical ideas or concepts (Carpenter & Fennema, 1992; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Tirosh, 2000; Vacc & Bright, 1999). When teachers develop a habit of paying attention to students' thinking, they position themselves to determine what their students already know or do not know, and they become better equipped to make appropriate instructional decisions.

Our result concerning the relationship between the strength of our pre-service teachers' algebraic thinking and their ability to recognize and interpret middle school students' algebraic thinking in a clinical setting indicates that the ability to interpret and analyze ways of algebraic thinking exhibited by others might develop independently of the pre-service teachers' own ability to exhibit these same ways of thinking. This finding has implications for teacher preparation programs, suggesting a need to focus on both aspects of broadly defined pre-service

teachers' knowledge of algebraic thinking: (1) pre-service teachers' own algebraic thinking, and (2) their ability to recognize and analyze ways of thinking exhibited by the students.

Finally, our study provides an important window into pre-service teachers' awareness of the potential of algebra-based tasks to engage students in algebraic thinking in clinical settings. We found that pre-service teachers' had a rather limited ability to recognize the richness of algebra-based tasks' potential to foster algebraic thinking in students. To effectively engage their future students in algebraic thinking pre-service teachers need to understand the context in which algebraic thinking might arise. While participants of our study were able to anticipate some of the ways their students could exhibit algebraic thinking while solving selected tasks, their ability to anticipate these ways was inadequate. This result suggests that it may prove beneficial to explicitly engage pre-service teachers in discussions of how algebra-based tasks elicit different aspects of algebraic thinking. Such discussions could be orchestrated in the context of analyzing alternative solutions to AT tasks, with a goal of helping pre-service teachers recognize ways of thinking that alternate solutions might encourage. Explicit consideration of alternative solutions, as well as comparison of the algebraic thinking features that generate them, would strengthen the pre-service teachers' own algebraic thinking. At the same time, it could heighten pre-service teachers' awareness of how problem situations can provide rich contexts for engaging students in algebraic thinking and also increase their sensitivity to important issues in early algebra instruction.

Algebraic thinking is at the heart of teaching and learning algebra at the K-8 level. Building pre-service teachers' broadly defined knowledge of algebraic thinking needs to be an important goal for elementary and middle school mathematics teacher education programs. Our study provides additional recognition and understanding of the complexity of pre-service

teachers' knowledge of algebraic thinking. The results can help mathematics teacher educators and mathematics education researchers design programs sensitive to important issues related to the early algebra instruction. We recognize that the results of our study need to be interpreted with caution, given the small number of participants, a lack of comparison groups, and lack of consideration given to other types of courses and/or settings. However, we believe that our results highlight the importance of strengthening pre-service teachers' ability to apply in clinical situations, the knowledge of algebraic thinking they learn in university coursework. We also believe that our results suggest important directions for the mathematics teacher education community to pursue.

References:

Ball, D.L. (1990). The mathematical understanding that prospective teachers bring to teacher education. *Elementary School Journal, 90*(4), 449 – 466.

Ball, D.L. (2003). *What mathematical knowledge is needed for teaching mathematics?* Retrieved October 1, 2005, http://www-personal.umich.edu/~dball/.

Ball, D.L., & Bass, H. (2003). Toward a practice based theory of mathematical knowledge for teaching. *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group,* Edmonton, AB, 3 –14.

Borko, H. & Putman, R.T. (1996). Learning to teach. In R. Calfee & D. Berliner (Eds.). *Handbook of educational psychology*. (pp.673 – 725). New York: Macmillan

Castro, B. (2004). Pre-service teachers' mathematical reasoning as an imperative for codified conceptual pedagogy in Algebra: A case study of teacher education. *Asia Pacific Education Review, 15*(2), 157 – 166.

Carpenter T.P & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades.* National center for improving student learning and achievement in mathematics and science. University of Madison, Wisconsin. Accessed on June 27, 2009 at http://ncisla.wceruw.org/publications/reports/RR-002.PDF

Cuoco, A., Goldberg, P. & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curriculum. *Journal of Mathematical Behavior, 15*, 375 – 402.

Derry, S., Wilsman, M., & Hackbarth, A. (2007). Using contrasting cases to deepen teacher understanding of algebraic thinking. *Mathematical Thinking and Learning, 9*(3), 305 - 329

Driscoll, M. (1999). *Fostering algebraic thinking. A guide for teachers grades 6 – 10*.

Portsmouth, NH: Heinemann.

Driscoll, M. (2001). *The fostering of algebraic thinking toolkit.* Introduction and analyzing written student work. Portsmouth, NH: Heinemann.

Eisenberg, T. (1977). Begle revisited: Teachers knowledge and students achievement in algebra *Journal for Research in Mathematics Education*, 8, 216 - 222.

Fennema, E., Carpenter, T., Levi, L., Franke, M.L., and Empson, S.B. (1999). *Children's mathematics: Cognitively guided instruction*. *Professional development materials*. Portsmouth, NH: Heinemann.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice.* New York: Cambridge University Press.

Greenberg, J., & Walsh, K. (2008). *No common denominator: The preparation of Elementary mathematics teachers by America's education schools.* National Council on Teacher Quality: Washington, DC.

Hill, H. C., Ball, D, & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400.

Hill, H. C., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Education Research Journal*. 42(*2*), 371-406.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In S. Fennel (Ed.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a National Symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.

Carpenter, T. P., & Fennema, E. (1992). "Cognitively guided instruction: Building on the

knowledge of students and teachers." In W. Secada (Ed.), *Curriculum Reform: The Case of Mathematics Education in the United States.* Special issue of *International Journal of Educational Research*, (pp. 457-470). Elmsford, NY: Pergamon Press, Inc.

Kieran, C. (2004). Algebraic thinking in the middle grades: What is it? *The Mathematic Educator*, *8*(1), 139 - 151

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *8th International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Sevilla, Spain: S.A.E.M. Thales.

Kieran, C. & Chalouh, L. (1993). *Prealgebra: The transition from arithmetic to algebra*. In D.T. Owens (Ed.), Research ideas for the classroom: Middle grades mathematics. (pp.179 – 198), NY: Macmillan.

Lesh, R., & Kelly, A. (2000). Multi-tiered teaching experiments. In Kely, A., & R. Lesh (Eds.). *Research design in mathematics and science education.* (pp.197 – 230). Kluwer: Amsterdam, Netherlands.

Miles, M., & Huberman, A. M. (1994). *Qualitative data analysis*. Thousand Oaks, CA: Sage Publications.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM. Author.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM. Author.

National Mathematics Advisory Panel. (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.

Shell Centre for Mathematical Education (1984). *Problems with patterns and numbers*. Manchester: Joint Matriculation Board.

Silver, A. E. (1997). Algebra for All: Increasing Students Access to Algebraic Ideas, Not Just Algebra Courses. *Mathematics Teaching in the Middle School*, *2*(4), 204-207

Sowder, J.T. & Shapielle, B. (Eds). (1995). *Providing a foundation for teaching mathematics in the middle grades*. Albany: State University of New York Press.

Swafford, J.O. & Langrall, C.W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, 31, 81 – 112.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5 – 25.

Vacc, N.N., & Bright, G.W. (1999). Elementary pre-service teachers changing belief and instructional use of children's mathematical thinking. *Journal for Research Mathematics Education*, *39* (1), 89 – 110.

APPENDIX A
Summary of Data Analysis Processes

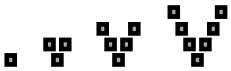
Data Analysis	Purpose	Method
Content Phase:		
Stage 1		
1. Examine each task and code features of algebraic thinking elicited by the task	To identify specific features of algebraic thinking that a given task elicits from a solver	A priori defined codes based on operational definition of algebraic thinking as summarized in Table 1)
2. Code pre-service teachers' solutions to each task for features of algebraic thinking exhibited in that solution	To identify features of algebraic thinking used by pre-service teachers in their solutions	A priori defined codes based on operational definition of algebraic thinking (Table 1)
3. Qualify the strength of each identified feature of algebraic thinking	To identify pre-service teachers' competencies in algebraic thinking with relationship to each feature	A priori defined codes (proficient, emerging, not evident)
4. Quantify the strength of algebraic thinking by features across all tasks	To identify algebraic competency by feature and overall	Compute average scores by feature and overall across all tasks
Stage 2		
1. Code pre-service teachers' interpretations of students' solutions	To identify specific features of AT pre-service teachers' recognize in students written work	A priori defined codes based on operational definition of algebraic thinking (Table 1)
2. Code strength of pre-service teachers' interpretations of students' work	To identify pre-service teachers competency in recognizing and interpreting algebraic thinking in the work of students	A priori defined codes (proficient, emerging, not evident)
3. Quantify the strength of PST's interpretations across all tasks	To identify algebraic competency by feature and overall	Compute average scores by feature and overall across all tasks
4. Searching for patterns with respect to pre-service teachers' AT competencies and their ability to identify and interpret features of algebraic thinking	Identify the relationship between PST's At competencies and their ability to identify and interpret AT in the work of students	Using SPSS examine and compare strength of different features of algebraic thinking within and across pre-service teachers
Field Phase:		
1. Coding transcripts of debriefing interviews conducted with pre-service teachers	(a)Identify features of AT pre-service teachers' recognize in context of the task. (b) Identify patterns in Pre-service teachers' perceptions of task potential.	(a) A priori defined codes describing features of algebraic thinking (Table 1) and (b) open coding (Miles and Huberman, 1984)
2. Coding transcripts of clinical interviews conducted by the pre-service teachers with middle	Identify features of algebraic thinking exhibited by a middle school student	A priori defined codes based on operational definition of algebraic thinking (Table 1)

	school student		
3.	Coding transcripts of debriefing interviews conducted with pre-service teachers	Identify features of algebraic thinking recognized by pre-service teachers in middle school students work	Assigning a priori defined codes based on operational definition of algebraic thinking (Table 1). A priori defined codes (not evident, emerging, and proficient)
4.	Coding pre-service teachers' written analyses of middle school student algebraic thinking	Identify features of algebraic thinking recognized by pre-service teachers in middle school student work	A priori defined codes based on operational definition of algebraic thinking (Table 1). A priori defined codes (not evident, emerging, and proficient)
5.	Searching for patterns in pre-service teachers interpretations of middle school student thinking across all transcripts and written analyses	Identify pre-service teachers interpretations of middle school student algebraic thinking	Comparing and contrasting explanations and evidence provided by pre-service teachers for features of algebraic thinking identified in middle school student work with evidence of middle school student algebraic thinking identified by the researchers.

APPENDIX B

Tasks pre-service teachers analyzed for their potential to engage students in algebraic thinking

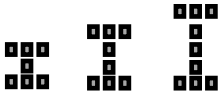
Task 1



1. If the pattern continues, how many blocks will be contained in the next letter V?
2. How many blocks would be in the 15th figure in the sequence? How did you figure out your answer?
3. How could you figure out the number of blocks in any letter V in this pattern?
4. Can you build a letter V that follows that pattern and uses 36 blocks?
5. Would any of the letter V's in this pattern have an even number of blocks? Why or why not?

Task 2

Here is a letter I made in different sizes using small tiles.



Size 1 Size 2 Size 3

1. Describe how the letter grows from one size to the next.
2. How many tiles would you need to make a letter I of: a) Size 6? b) Size 10? c) Size 38? d) Size 100?
3. Write a rule that helps to predict the number of tiles for any size letter I? You may write a rule either in words or using variables.
4. Suppose you had 39 tiles. What is largest size of I that you could make?

Task 3

Eight adults and two children need to cross a river. They have a small boat available that can hold one adult or one or two children. Everyone can row the boat. (a) How many one way trips does it take for them to all cross the river?
(b) What if there were 6 adults and 2 children? 15 adults and 2 children? 3 adults and 2 children?
(c) Describe in words how to figure out the answer for this problem if the group of people to crosses the river includes 2 children and any number of adults? How does your rule work out for 100 adults?
(d) Write the rule for A number f adults and 2 children.
(e) What happens to the rule you wrote if we change he number of children? For example 8 adults and 3 children? 2 adults and 5 children? Any number of adults and 11 children?
(f) One group of adults and children took 27 trips to cross the river. How many adults and how many children were in the group? Is there more than one solution?

Task 4

The shapes shown below are made with toothpicks. Look for patterns in the number of toothpicks in the perimeter of each shape.

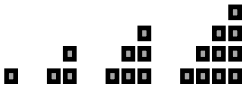


Fig.1 Fig.2 Fig.3 Fig.4

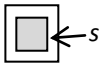
1. Use the pattern from the shapes to determine the perimeter of the fifth figure in the sequence. Clearly explain how you arrived at the answer.2. Write a formula that you could use to find the perimeter of any figure *n*. Explain how you fund your formula.

Task 5

Sally is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has 2 more persons than the group that entered on the previous ring how many guests will have arrived after 20th ring?

Task 6

Below is a picture of an in-ground swimming pool surrounded by a border of square tiles.



1. How many 1-foot square tiles will be needed for the border of a square-shaped pool that has edges length *s* feet?
2. Express the total number of tiles needed in as many ways as you can.
3. How do you know that your expressions are equivalent? Provide convincing arguments that your expressions are equivalent.

Task 7

Each house below was built using pattern-block tiles: triangles and squares.

House 1 House 2 House 3 House 4

1. Determine the total number of tiles needed for each house.
2. Draw a sketch of house 5 and describe what house 5 would look like.
3. Predict the total number of tiles you will need to build house 15. Explain your thinking.
   Write a rule that gives the total number of pieces to build any house in this sequence.