RELATIONSHIPS AMONG FEATURES OF PRE-SERVICE TEACHERS' ALGEBRAIC THINKING

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In this study we examined 18 pre-service middle school teachers' ability to use algebraic thinking to solve problems. The data revealed that pre-service teachers' AT abilities varied across different features of the algebraic habit of mind Building Rules to Represent Functions, and that significant correlations existed between 8 pairs of the features. The ability to justify a rule was the weakest of seven AT features, and it was correlated with the ability to predict patterns and the ability to chunk information. Implications for mathematics teacher education are discussed.

BACKGROUND

Over the last three decades the mathematics education community has engaged in discussions about the role and the nature of school algebra in the mathematics curriculum. While most mathematics educators advocate for the inclusion of algebra-based topics at the K-8 level, they are by no means calling for elementary and middle school students to be taught algebra in the traditional way. Traditional algebra focuses on issues related to skills such as manipulating algebraic expressions and solving equations. In contrast, early algebra instruction aims to advance students' conceptual knowledge and skills by shifting attention away from symbolic manipulations and equation solving toward analyzing and generalizing patterns using multiple representations (NCTM, 2000; Silver, 1997; Kieran, 1996; Carpenter & Levi, 2000). Ideally, algebraic experiences in the elementary and middle grades are designed to allow students to see algebra as a network of knowledge and skills rather than as a muddle of isolated concepts. Done in this way, early algebra instruction is much more likely to prepare students for a smooth transition from arithmetic to more formal algebra (Carpenter & Levi, 2000; Silver, 1997; Kieran, 1996; Kaput, 1998).

Algebraic Thinking

The term algebraic thinking has various connotations (Swafford & Langrall, 2000; Kieran & Chalouh, 1993; Kieran, 1996). For some, algebraic thinking closely relates to what Cuoco, Goldberg, and Mark (1996) defined as habits of mind: useful ways of thinking about mathematical content. For example, Driscoll (1999) used the term algebraic thinking to signify thinking about quantitative situations in ways that make the relationships between variables obvious. He conceptualized algebraic thinking as including three habits of mind, one of which is *Building Rules to Represent Functions*. According to Driscoll, Building Rules to Represent Functions is a habit of mind that enables thinking processes that, for example, include recognizing and analyzing patterns, investigating and representing relationships, generalizing beyond

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specific examples, analyzing how processes or relationships change, or seeking arguments for how and why rules and procedures work.

Building Rules to Represent Functions

For the research reported here, we used Driscoll's (1999, 2001) taxonomy of algebraic habits of mind as a framework, and focused our investigation on the Building Rules to Represent Functions habit of mind. Thus, unless otherwise specified, throughout this paper we interpret *algebraic thinking* in terms of thinking processes characteristic to Building Rules to Represent Functions. Our operational definition of algebraic thinking is based on Driscoll's identification of features that characterize the algebraic habit of mind Building Rules to Represent Functions: (1) Organizing Information, (2) Predicting Patterns, (3) Chunking Information, (4) Different Representations, (5) Describing a Rule, (6) Describing Change, and (7) Justifying a Rule.

Teacher Knowledge

The call for the early introduction of algebraic ideas has many inherent challenges. For example, teachers' own experiences with traditional school algebra often strongly influence and limit their views of algebraic thinking and, in turn, counter their efforts at mathematics education reform. Both practicing and pre-service teachers' understanding of algebraic topics often consists of the fragmented knowledge of a disconnected system of symbols and procedures (Ball, 1990; Ma, 1999). Teaching that is informed by such limited knowledge short circuits the algebraic-thinking (AT) goals of early algebra instruction.

It is commonly accepted by mathematics educators that elementary and middle school teachers must understand the ideas behind algebraic thinking in order to take advantage of opportunities to engage students in algebraic thinking. Effective early algebra instruction requires a more adequate preparation of elementary and middle school teachers than currently exists (National Mathematics Advisory Panel, 2008; Greenberg & Walsh, 2008). Teachers' knowledge has been identified as an important factor that influences the outcome of their practice (Borko & Putman, 1996; Mewborn, 2003; Sowder & Schappelle, 1995; Hill, Rowan, & Ball, 2005).

Despite Ball's (1990) stated concerns about pre-service teachers' limited and procedural knowledge of the K-12 mathematics curriculum, few research efforts have focused on understanding the breadth and depth of pre-service teachers' algebraic thinking (AT) ability. An in-depth understanding of the relationships among various

features of pre-service teachers' algebraic thinking would contribute to our ability to successfully design programs that prepare teachers to introduce early algebra concepts and foster algebraic thinking in their K-8 students.

Research Question

Guided by the need for an in-depth understanding of pre-service teachers' AT ability, we designed this study to answer the following question: Which features of pre-

service teachers' own algebraic thinking support and strengthen one another, and which features develop independently?

METHOD

Participants

Participants in this study included 18 undergraduate pre-service teachers (grades 1-8 teaching certification candidates) at a large private Midwestern university. All were enrolled in a mathematics content course designed to help pre-service teachers develop the ability to interpret, compare, connect, and generalize across multiple algebra topics within the middle school mathematics curriculum. In the course, the pre-service teachers engaged in activities that solicited multiple solutions and representations of algebra-based tasks, and encouraged sharing, explaining, comparing, and making interpretations of various representations and reasoning.

Data Sources and Data Collection

To investigate the pre-service teachers' AT ability, we collected the pre-service teachers' written solutions to 125 AT tasks, which they completed for homework, during class, and on performance assessments. The tasks were designed to elicit the pre-service teachers' algebraic thinking.

Data Analysis

The three authors independently coded the data. We rated the pre-service teachers' demonstrated use of a feature of algebraic thinking as (3) proficient, (2) emerging, or (1) not evident. In order to establish validity and reliability, we compared the three sets of independent results and cited specific examples to clarify the coding schemes and negotiate coding agreement to 100%.

First, we identified which features of algebraic thinking are encouraged by each of the 125 AT tasks. For each task we rated the pre-service teachers' written solutions, assessing how well the pre-service teachers' demonstrated the use of each identified feature. Then we produced seven feature scores for each pre-service teacher by averaging all of his/her ratings on each given feature across the collection of tasks. A pre-service teacher's feature score quantified his/her ability to use the given AT

feature. We also calculated an AT proficiency score for each pre-service teacher by computing the mean of his/her seven feature scores. Finally, we used correlation analysis to investigate the associations among the features of the pre-service teachers' algebraic thinking.

RESULTS

In this section, we provide a quantitative summary of the pre-service teachers' performance on each of the AT features and overall. Then, we compare these performances to infer relationships among the features of the pre-service teachers' algebraic thinking.

Each pre-service teacher submitted solutions to 125 AT tasks. Some tasks solicited all seven features of algebraic thinking, but most did not. As described above, we computed seven feature scores for each pre-service teacher. Each assessed the pre-service teacher's ability to use one of the features of algebraic thinking. The means of the pre-service teachers' feature scores are presented in Table 1. Of all the means, the pre-service teachers' mean on the Justifying a Rule feature (2.20) was the smallest. A repeated measures ANOVA revealed that there were significant differences among the seven means ($F(6,102)=4.89$; $p < 0.01$). Bonferroni-adjusted pairwise comparisons of the means confirmed significant differences between the Justifying a Rule mean and three other means: Organizing Information ($p < 0.01$), Predicting a Pattern ($p < 0.01$), and Describing a Rule ($p < 0.01$). The other differences were not significant.

Feature	Frequency	N*	$\overline{M}$	SD
1. Organizing Information	78 (62%)	1404	2.56	0.30
2. Predicting Patterns	58 (46%)	1044	2.54	0.29
3. Chunking Information	38 (30%)	684	2.39	0.39
4. Different Representations	27 (22%)	486	2.50	0.42
5. Describing a Rule	66 (53%)	1188	2.58	0.22
6. Describing Change	71 (57%)	1278	2.46	0.31
7. Justifying a Rule	40 (32%)	720	2.20	0.41

*Number of scores for a given feature across all soliciting tasks and all 18 pre-service teachers.

Table 1: Pre-service Teachers' Mean Feature Scores

We assigned an (overall) AT proficiency score to each pre-service teacher by averaging all seven of his/her feature scores. The AT proficiency score estimates the overall strength of a pre-service teacher's AT ability. The average of all 18 mean feature scores was $\overline{M} = 2.46$ (max 3); $SD = 0.36$, which is an estimate of the overall strength of the algebraic thinking of the pre-service teachers in our study. These finding suggests that the cohort of pre-service teachers in our study had a relatively high overall ability to solve a variety of algebra-based tasks (n =125), but their ability to justify a rule lags behind the others.

Associations Among the Features of Algebraic Thinking

We used correlation analysis to examine the strengths of the pairwise associations among the pre-service teachers' feature scores. The analysis showed that eight of the pairwise correlations were significant (Table 2). Figure 1 illustrates the relative strengths of the eight significant correlations. The heavier weights of four segments in the diagram illustrate that those four significant pairwise correlations are greater ($0.72 < r \leq 0.91$) than the other four significant correlations ($0.48 < r < 0.54$).

Feature	1	2	3	4	5	6	7
1 Organizing Information	–						
2 Predicting Patterns	0.72**	–					
3 Chunking Information	0.54*	0.91**	–				
4 Different Representations	0.39	0.47	0.40	–			
5 Describing a Rule	0.51*	0.77**	0.73**	0.28	–		
6 Describing Change	0.46	0.34	0.32	0.12	0.17	–	
7 Justifying a Rule	0.44	0.54*	0.48*	0.32	0.36	0.38	–

*$p < 0.05$, ** $p < 0.01$

Table 2: Correlations Between the Pre-Service Teachers' Feature Scores

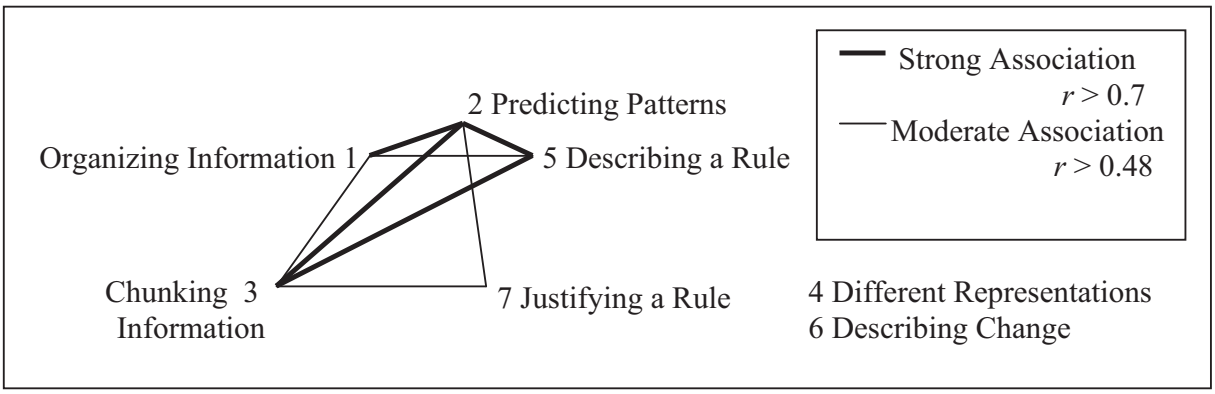

Figure 1: Pairwise Correlations Among the Seven Features of Algebraic Thinking

Significant ($p < 0.05$) pairwise correlations existed among the proficiency scores for features 1 (Organizing Information), 2 (Predicting Patterns), 3 (Chunking Information), and 5 (Describing a Rule). Feature 7 was correlated with features 2 and 3. Neither feature 4 (Different Representations) nor feature 6 (Describing Change) was correlated with any of the other 5 features. The pattern of these correlations indicates that features 1, 2, 3, 5 (and to some extent 7) are associated, perhaps because they develop concurrently, supporting and strengthening one another. On the other hand, since neither of features 4 (Different Representations) or 6 (Describing Change) was associated with the ability to use any of the remaining five features of algebraic thinking, these two features may develop and be used independently of the other five features.

DISCUSSION AND IMPLICATIONS

We found that the pre-service teachers in our cohort were able to competently use many features of algebraic thinking to solve algebra-based tasks. Although promising overall, we found that the pre-service teachers' ability to justify a rule or procedure was weak in comparison to their ability to employ other features of the algebraic habit of mind Building Rules to Represent Functions. This result is consistent with Castro (2004), who also found that pre-service teachers lacked sufficient ability to justify why algebra-based algorithms and procedures work.

Another significant result that our analyses revealed was the complex nature of the algebraic thinking identified in the pre-service teachers' work. Our research revealed correlations between different features of our pre-service teachers' algebraic thinking. Taken together, these correlations suggest that the abilities to organize information, predict patterns, chunk information and describe a rule may develop and support one another in a mutual, symbiotic, and holistic way. However, our research also suggests that the ability to justify a rule may be related somewhat to predicting patterns and chunking information, but not organizing information or describing a rule. These correlations among the features of algebraic thinking might suggest that helping teachers to become competent algebraic thinkers may be accomplished by targeting learning activities at groups of AT features. For example, rather than implementing learning activities designed solely at improving pre-service teachers' ability to justify a rule, it may be more effective to implement learning activities aimed at improving their ability to employ the group of three features: Predicting Patterns, Chunking Information, and Justifying a Rule.

Our study, which was motivated by the need to prepare pre-service teachers for the challenges of early algebra instruction, helps fill a gap in the existing body of teacher preparation literature. Algebraic thinking is at the heart of teaching and learning algebra at the K-8 level. Building pre-service teachers' algebraic thinking ability should be an important goal for elementary and middle school mathematics teacher education programs. The research reported here has investigated pre-service teachers' ability to use features of algebraic thinking to solve problems. It provides a window into the complexity of the relationship between different features of pre-service teachers' AT abilities. Although this is only a single step in the quest to determine effective ways of preparing pre-service teachers to make algebraic thinking a focus of their K-8 instruction, our results can help mathematics teacher educators and mathematics education researchers design programs sensitive to important issues related to early algebra instruction.

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