2017

Endogenous Formation of Security Exchanges

Marta Faias, *Universidade Nova de Lisboa*
Jaime Luque, *University of Wisconsin - Madison*

Endogenous formation of security exchanges

Marta Faias & Jaime Luque

Economic Theory

ISSN 0938-2259
Volume 64
Number 2

Econ Theory (2017) 64:331-355
DOI 10.1007/s00199-016-0989-9


Volume 64 · Number 2 · August 2017

Economic Theory

RESEARCH ARTICLES

Equilibrium prices and trade under ambiguous volatility
P. Beissner 213

Incentives and justice for sequencing problems
P. De · M. Mitra 239

Coordination via correlation: an experimental study
J. Duffy · E.K. Lai · W. Lim 265

Consumption habits and humps
H. Kraft · C. Munk · F.T. Seifried · S. Wagner 305

Endogenous formation of security exchanges
M. Faias · J. Luque 331

Cap-and-trade under transactions costs and factor irreversibility
R. Singh · Q. Weninger 357

Further articles can be found at link.springer.com

Instructions for Authors for *Econ Theory* are available at www.springer.com/199

Springer

Springer

Econ Theory (2017) 64:331–355
DOI 10.1007/s00199-016-0989-9

RESEARCH ARTICLE

Endogenous formation of security exchanges

Marta Faias[1] · **Jaime Luque**[2]

Received: 1 August 2014 / Accepted: 18 June 2016 / Published online: 5 July 2016

Abstract We use club theory for the first time to provide a model of securities exchange (SX) formation. We think of a SX as a local public good that allows its traders to diversify risk by trading their securities with other SX members. In our two-stage equilibrium setting, traders evaluate SXs depending on their risk-sharing possibilities and, given these evaluations, choose the SX they want to join. Security prices can differ among SXs and traders may value SX memberships differently. We establish continuity properties in both stages and show that equilibrium exists for a generic set of economies.

Keywords Endogenous securities exchange structure · Security prices · Risk sharing · Membership prices · Equilibrium · Club theory

JEL Classification D52 · D53 · G12 · G14 · G15 · G18

We thank comments and suggestions from Nizar Allouch, Antonio Cabrales, Jean-Eduard Colliard, Gionvani Nicolo, Mario Pascoa, Erwan Quintin, Mark Ready, Tim Riddiough, Marzena Rostek, Myrna Wooders, Randy Wright, and participants at the annual meetings of the APET 2010, EWGE 2011, SAET 2011, SEA 2011, and PEA 2012, and seminars at Indiana University, University of Miami, Purdue University, and University of Wisconsin–Madison. Luque gratefully acknowledges the Spanish Ministry of Education and Science for financial support in 2010 under Grant SEJ2008-03516. Faias gratefully acknowledges that this work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through UID/MAT/00297/2013 (CMA).

✉ Jaime Luque
jluque@wisc.edu

Marta Faias
mcm@fct.unl.pt

1 Universidade Nova de Lisboa, FCT and CMA, Lisbon, Portugal

2 University of Wisconsin - Madison, 5259 Grainger Hall, 975 University Avenue, Madison, WI 53706, USA

1 Introduction

Security exchanges (SXs henceforth) have origins distant in history. Any group of agents who agree to trade their securities in fact constitutes a SX. Over time, SXs have evolved, and large trading organizations have emerged. By the twentieth century, SXs were linked to particular countries, and outsiders were charged high trading fees for access. The evolution of SXs has never been as dramatic as in the past decade. Changes in regulation and the adoption of electronic trading technologies are driving our traditional SXs out of business, replacing them with new global trading organizations. Recent examples of changes in regulation are the Markets in Financial Instruments Directive (Mifid) enacted by the European Commission in 2007 to facilitate competition of trading platforms across the region. Similarly, the USA has encouraged fragmentation with the Regulation National Market System of the Securities and Exchange Commission (SEC)—see "Concept Release" of the SEC (2010).

Following a recent wave of demutualization, traders can now move freely from national SXs to more preferred ones; they have only to pay a membership fee (see Grant 2011 for anecdotal evidence). Given the importance of such institutions in the market, we provide an equilibrium model of endogenous securities exchange formation. We consider a setting with one good, several securities, three periods, and several states of nature in the third period. Traders are endowed with units of the good in each period and state, and also with units of an exogenously given set of securities that pay in the different states of the third period.

The timing is as follows. In period 0, traders form SXs and pay the corresponding SX membership fee. In period 1, they trade their securities on the SX they belong to, under a no-short-sale-type constraint that says that a trader cannot sell more of a security than what he possesses. In period 2, there are several states of nature where securities pay in units of the good. Individuals who do not belong to any SX remain in autarky and consume their commodity endowments and the returns of their security endowments.

An equilibrium for this economy consists of (i) a partition of the set of traders into SXs and a vector of SX membership prices, and (ii) for each SX, an allocation of securities and commodities for all its traders and the corresponding vector of market clearing prices. This equilibrium concept involves two stages. In the second stage, traders *evaluate* SXs depending on the risk-sharing possibilities in periods 1 and 2. Then, in the first stage, traders take these evaluations as given and choose the SX they want to belong to. For the second stage, we use the standard notion of a two-period competitive financial equilibrium, possibly with incomplete markets. The notion of equilibrium in the first stage is similar to the concept in Allouch and Wooders (2008)—AW hereafter—of a price-taking equilibrium in club economies with communication costs, where clubs are unrestricted in composition and size (possibly as large as the entire population).

These "communication costs" (in AW's terminology) consist of small frictions in period 0 and affect not only traders' opportunities to change security exchange memberships, but also opportunities to purchase a different amount of the commodity. Communication costs have been widely used in the local public goods literature to establish the existence of a price-taking equilibrium through decentralization of outcomes in approximate cores (e.g., models with indivisibilities, non-convexities, coalition production, or multiple-sided matching). So far, our application to financial markets and the introduction of a pre-trading stage have not been predicted to our knowledge.

There are fundamental differences between our economy with endogenous security markets and previous club economies with commodity markets. In club theory, the preferences of an individual are typically independent of prices; prices enter through constraint sets. In securities exchange formation, under our framework, preferences for SXs depend on security prices, which ultimately depend on the complementarities in preferences and endowments among the traders who form the SX. SXs are formed endogenously by traders' choices in a context where SX formation is costly, and then, the security equilibrium prices are determined. Another important difference is that in the club literature, the club good (e.g., park or swimming pool) enters directly as a primitive in the individual's utility function, while in our approach, the club good is a facility that permits security trading, and whose equilibrium outcome, referred to here as "securities trading equilibrium", is embedded in the trader's utility function.

There are two important subtleties of our equilibrium existence result that deserve some remarks. First, there can be multiple "securities trading equilibria". In those cases, we consider a measurable selector with which traders evaluate their SX memberships. We also identify some conditions under which the "securities trading equilibrium" is unique. Second, in order to apply AW's result to prove existence of a partition of traders into SXs through a system of decentralized membership prices, we have to make sure that all assumptions in AW hold true for our economy. Yet, some of the AW's assumptions rely on evaluation of the individual's utility function at the club good, which in our model is endogenous. For example, AW's assumption of "continuity with respect to attributes" ensures that traders who are similar in terms of their attributes have similar utility functions. In our model, the "club good" is an endogenous object, captured by the concept of "securities trading equilibrium"that changes when the attributes of the traders in the SX vary. This condition is satisfied in our model because there is a compact subset of a generic set of traders' attributes where we show that this continuity property on attributes holds.

Our work relates to three different literatures in economics: club/local public goods; general equilibrium with incomplete markets and restricted participation; and market microstructure.

First, our model requires all SX participants to hold a membership (or trading right) on a particular SX in order to access its liquidity and diversify risk by trading securities with other SX members. Traders may value a SX membership differently, because in our model, traders may have different endowments of securities and the private good, and also a different utility from the consumption of the private good in different periods and states of nature. A SX can thus be seen as a local public good, and its membership be priced accordingly. This conceptualization of such a SX allows us to use the theory of club formation, and in particular the AW model, to price SX memberships and analyze competition among SXs. Thus, we link the field of market microstructure to the pricing theory of local public goods.

Second, our model departs from the literature of restricted participation where markets, defined in terms of participation constraints, are exogenously imposed—see Balasko et al. (1990) for an early contribution to this literature. We propose rather a pre-trading stage where securities exchanges are *endogenously* formed (with some costs). The novelty in our approach is the use of club theory to endogenize the participation of traders in SXs.

Third, to the best of our knowledge, we are the first to analyze the market microstructure of trading on SXs from the perspective of club theory. This viewpoint is novel in the market microstructure literature that analyzes the issues of concentration and fragmentation of trade across markets and the impact of trading costs on trading behavior, e.g., Pagano (1989), Madhavan (2000), Lo et al. (2004), and Biais et al. (2005). Our model can deal with these issues using a club theory approach instead. Our work differs as well from the classic models of vertically differentiated duopolies; see Gabszewicz and Thisse (1979) for a leading work on vertically differentiated oligopolies, and Colliard and Foucault (2012) and Pagnotta and Philippon (2012) for models allowing investors to choose between two platforms that trade one security and exchanges compete *à la Bertrand*. In our model, every possible subset of traders can form a SX, and multiple SXs can form in equilibrium. We also depart from the market microstructure literature in that we do not limit our analysis to one security. Instead, our model allows different security structures, with more than one security, and security structures can possibly be (endogenously) incomplete.

The paper is organized as follows. Section 2 presents the model; Sect. 3 introduces the equilibrium concept; Sect. 4 is devoted to the existence result. There we state the assumptions needed to prove the existence of an equilibrium and discuss the main steps of the existence proof. Section 5 concludes and points out other important issues that can be addressed in future research using the approach developed in this paper.

2 SXs as trading clubs: model

We consider an economy with a large but finite number n of traders and three periods, $t = 0, 1, 2$, with Ξ states of nature in the last period. Denote the set of traders by $\mathbf{I} \equiv \{i\}_{i=1}^{n}$ and the set of possible states by $\boldsymbol{\Xi}$. Our model allows any set of traders S to form a SX.[1] We interpret a SX as similar to a club good that allows traders to diversify risk by trading securities with other SX participants. To carry on security trading activity on a particular SX, all SX members must purchase a membership in that SX.[2]

[1] For notational convenience, we use boldface letters to denote a set, except when we refer to a subset of traders, or exchange, here denoted by S.

[2] As stated in the Hong Kong Exchange (HKEx) rules, any trader intending to operate a brokerage business for products available on HKEx, using the trading facilities of the stock exchange and/or futures exchange, must be admitted and registered as an exchange participant of that exchange, and pay the corresponding membership fee. The acquisition of an exchange membership usually involves a commitment for trading in the exchange for a long period of time—usually, the exchange participant stays in the exchange since it enters. See, for example, the MICEX list of participants: http://www.micex.com/markets/stock/members/list.

In this economy, there is one consumption good and several securities that pay in the last period in terms of the good. The set of securities available in the economy is denoted by $\mathbf{J} \equiv \{j\}_{j=1}^{J}$. Trader i is endowed with a finite and strictly positive vector of the perfectly divisible private good, $\omega^i = (\omega_0^i, \omega_1^i, (\omega^i(\xi), \xi = 1, \ldots, \Xi)) \in \mathbb{R}_{++}^{2+\Xi}$. Trader i's consumption bundle is denoted by $x^i = (x_0^i, x_1^i, (x^i(\xi), \xi = 1, \ldots, \Xi)) \in \mathbb{R}_{+}^{2+\Xi}$. Traders are also endowed with nonnegative amounts of the available securities. We write e_j^i to denote trader i's endowment of security $j \in \mathbf{J}$. Securities trading occurs in period 1 within each SX. Security payoffs are realized in period 2 and can differ among states of nature. We denote by $a_j(\xi) \geq 0$ the payoff of security j in state ξ. It is convenient to assume that the payoffs of the available securities in the economy are linearly independent (we need this assumption to show that the set of "securities trading equilibria" is a continuous differentiable function of both the commodity endowment and the utility assignment). This assumption is standard in the literature and just requires that there are no redundant securities in the economy (see also Geanakoplos and Polemarchakis 1986).

Denote a *SX structure* with K securities exchanges by $F(\mathbf{I}) = \{S_k\}_{k=1}^{K}$ and the set of all possible SX structures by $\mathbf{F}(\mathbf{I})$. Notice that the set of securities available for trade in a SX coincides with those securities that traders bring to that SX as endowments, i.e., $\mathbf{J}(S) = \{j \in \mathbf{J} : \exists i \in S \text{ with } e_j^i > 0\}$.[3] The cardinality of set $\mathbf{J}(S)$ is $J(S)$. Thus, a SX S has implicitly associated a matrix of security returns $A(S) = \left[a_j(\xi)\right]_{\Xi \times J(S)}$.

A trader can belong to no more than one SX; thus, a SX structure is a partition of the set of traders into securities exchanges. Traders purchase their SX memberships in period 0. Trader i's membership is denoted by $F[i; \mathbf{I}]$. If the trader buys membership in SX $S \in F(\mathbf{I})$, we write $F[i; \mathbf{I}] = S$. We denote trader i's membership price when $F[i; \mathbf{I}] = S$ by $\pi^i(S)$ (alternatively, we also write $\pi^i(F[i; \mathbf{I}])$). A trader with no SX membership will end up consuming his good endowment and the period 2 returns of his security endowments.

In period 1, traders trade their securities in their particular SX.[4] We denote by $q_j(S)$ the price of security $j \in \mathbf{J}(S)$ in SX S expressed in terms of the unit of account. We choose the unit of account to be one unit of the good. Observe that the price of a security j depends on the particular SX where the security is traded. Trader i's security j trading position in SX S is written as $y_j^i(S)$. Because trader i's membership in SX S is denoted by $F[i; \mathbf{I}]$, we also write $q_j(F[i; \mathbf{I}])$ and $y_j^i(F[i; \mathbf{I}])$ below. $y_j^i(S) > 0 (y_j^i(S) < 0)$ means that trader i purchases (sells) security j in SX S.

Each trader faces two constraints in period 1: the budget constraint and the "box" constraint. Given traders' SX memberships and SXs' security prices, trader i's budget constraint in period 1 is

[3] Alternatively, we could assign to each exchange S a subset of *assets* (in the broad sense) $\mathbf{J}(S) \subseteq \mathbf{J}$ that traders in exchange S agree to issue for later trading. We leave this possibility for future research.

[4] For simplicity, we do not model trading fees, which have become substantially less important since implementation of the Mifid regulation (see Colliard and Foucault 2012 for a survey of this literature and for an analysis of the effect of trading fees on the efficiency of the markets); nor do we consider transaction fees (e.g., stamp duty). We focus instead on the pure trading aspects of security markets.

$$x_1 - \omega_1^i + \sum_{j \in J(F[i;\mathbf{I}])} q_j(F[i;\mathbf{I}]) y_j(F[i;\mathbf{I}]) \leq 0. \tag{1}$$

This constraint says that trader i's consumption of the private good net of his good endowment, plus the expenditures minus revenues associated with the trader's security trading activity in SX $F[i;\mathbf{I}]$, must be non-positive.

In our analysis of SX formation, we focus on securities (i.e., stocks and bonds) and exploit their natural role of "possession" by considering a constraint that keeps track of the amount of securities that a trader brings to the SX. We refer to this constraint as the "box" constraint, which for security $j \in J(F[i;\mathbf{I}])$ is as follows:

$$e_j^i + y_j \geq 0. \tag{2}$$

The "box" constraint says that trader i's endowment of security $j \in J(F[i;\mathbf{I}])$ plus his security trading position in that security must be nonnegative. In other words, this constraint requires the physical amount of security titles that the trader possesses in his "box" to be nonnegative. The term "box" comes from market parlance, originally explicitly recognizing the necessity of keeping security titles in a box. The "box" constraint is used by Bottazzi et al. (2012) to model repo markets. In that paper, a trader can sell a security if he owns it (as security endowment) or if he borrows the security through a repurchase agreement (or repo). Here, we do not allow for repo markets, so the "box" constraint in our context is significantly stronger than in Bottazzi et al. (2012) as it prevents a trader from selling more of a security than his security endowment. Still, this terminology is important in our work because it keeps track of the total of security titles (security endowments) that a trader brings to a SX.

Notice that the "box" constraint may seem like a no-short-sale restriction, but the two restrictions are different. The "no-short-sale" constraint is compatible with "issuance" of a security because its exogenous bound could be chosen to be beyond the trader's security endowment. The "box" constraint, on the other hand, keeps track of the total of the securities and prevents a trader from selling more of a security than he owns. Thus, the main difference between the "box" constraint and the "no-short-sale" constraint in our model is that the "box" constraint precisely identifies the exogenous bound on security sales with the trader's security endowment.

The interaction between the "box" and budget constraints of period 1 can be read as follows: A trader i can sell the amount $y_j(F[i;\mathbf{I}])$ of security j if he is endowed with at least $e_j^i \geq -y_j(F[i;\mathbf{I}])$; if this sale satisfies the box constraint, then the sale is feasible, and the trader would give $\left|y_j(F[i;\mathbf{I}])\right|$ units of the security to the buyer and receive $-q_j(F[i;\mathbf{I}])y_j(F[i;\mathbf{I}]) > 0$ (income in terms of the trader's budget constraint of period 1).

Trader i's budget constraint in period 2 and state $\xi \in \Xi$ is such that his consumption, net of his good endowment, is bounded by the returns on his security positions, i.e.,

$$x(\xi) - \omega^i(\xi) \leq \sum_{j \in \mathbf{J}} a_j(\xi) e_j^i + \sum_{j \in J(F[i;\mathbf{I}])} a_j(\xi) y_j(F[i;\mathbf{I}]). \tag{3}$$

Notice that constraint (3) rules out the possibility of default. Later in Sect. 5, we discuss the institutional aspects of incorporating default in our model as one of the extensions left for future research.

For a SX S to form, a formation cost $z(S) \in \mathbb{R}_+$ must be incurred by its members. Costs include, for example, installation charges, adoption of new trading technology, and acquisition of a trading platform.[5] The membership price that trader i pays to participate in a SX S is denoted by $\pi^i(S)$. Exchange S's profits are given by $\sum_{i \in S} \pi^i(S) - z(S)$.

Finally, the trader i's utility function, denoted by

$$u^i(x_0, x_1, x(1), \ldots, x(\Xi)), \tag{U1}$$

maps the consumption bundle $x \in \mathbb{R}_+^{2+\Xi}$ into a real number, i.e., $u^i : \mathbb{R}_+^{2+\Xi} \rightarrow \mathbb{R}$. For simplicity, we consider a *separable* utility function, which guarantees that the maximum value of $u^i(x_0, \cdot)$ under the trader's constraints does not depend on x_0 for each possible vector of prices. This captures the idea that SX participants are inherently "forward looking": The security trading equilibrium is independent on the private consumption in period 0 (when SXs are formed). Roughly speaking, traders, after their purchase of SX memberships, buy and sell securities in their respective SXs depending *only* on their risk-sharing preferences between periods 1 and 2 and across states of nature.

An economy is thus defined by a set of traders that are characterized by their utility and their commodity and securities endowments, $\mathcal{E} = \{(u^i, \omega^i, e^i) : i \in \mathbf{I}\}$.

3 Equilibrium

Before giving the formal definition of equilibrium below, we provide here a brief description of our notion of equilibrium. It involves two stages. In the first stage, traders form SXs, and in the second stage, they trade securities in their respective SXs. To form SXs, traders rely on their evaluations of the risk-sharing possibilities associated with periods 1 and 2, captured by the utility function evaluated at the "securities trading equilibrium"corresponding to a given SX.

The notion of "securities trading equilibrium" that we consider in the second stage corresponds to the standard two-period competitive financial equilibrium (an allocation of securities and the commodity and a vector of security and commodity prices satisfying individual optimality and market clearing). Notice that when there are multiple security trading equilibria, we prove that the set of measurable selectors is non-empty and we consider that traders evaluate their SX memberships using one of these measurable selectors.

The notion of equilibrium in the first stage is borrowed from Allouch and Wooders (2008). It corresponds to a price-taking equilibrium in club economies with communication costs, where clubs are unrestricted in composition and size (possibly as large

[5] See the HKEx security trading infrastructure at http://www.hkex.com.hk/eng/market/sec_tradinfra/CMTradInfra.htm.

as the entire population). In our model, a price-taking equilibrium of the first stage consists of a partition of the set of traders into SXs, an allocation of the private good, and a membership price system (notice that because we consider only one private good and normalize this price to 1, we do not include it in this definition). This notion takes into account a small communication cost of deviating from a given outcome. Allouch and Wooders (2008) demonstrate that, providing that most consumers have many close substitutes, if an economy is sufficiently large, then a price-taking equilibrium with communication costs and possibly some frictions, captured by the presence of an exceptional set of consumers, exists and is in the core.[6] We demonstrate that the assumptions that guarantee existence of an equilibrium in Allouch and Wooders (2008) are satisfied in our economy, and therefore, we can rely on their existence result for our first stage. See Sect. 4 for additional explanations of the main subtleties of these proofs. We present next the formal definitions of equilibrium in each stage. We start by defining stage 2, because its equilibrium solution is needed to define the utility function of stage 1.

3.1 Stage 2: exchange evaluation

For notational convenience, we write $x_0^I \equiv (x_0^i : i \in \mathbf{I})$, $x_1^I \equiv (x_1^i : i \in \mathbf{I})$ and $x^I(\xi) \equiv (x^i(\xi) : i \in \mathbf{I})$ to denote traders' consumption bundle in periods 0 and 1, and state ξ in period 2, respectively, and $x^I = (x^i : i \in \mathbf{I}) \in \mathbb{R}_+^{I(2+\Xi)}$ to denote traders' consumption bundles in the three periods. We write $y^I = (y^i \in \mathbb{R}^{J(F[i;\mathbf{I}])} : i \in \mathbf{I})$ to denote the vector of security positions that traders have in their respective SXs and $q(S) \in \mathbb{R}_+^{J(S)}$ to denote the vector of security prices in SX S (notice that we allow security prices to differ among SXs).

Definition 1 (*Securities trading equilibrium for a given SX structure*): Given the SX structure $F(\mathbf{I})$, a "securities trading equilibrium" consists of a system $(x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, q(S_1), \ldots, q(S_K)(F(\mathbf{I})))$, such that:

(D1.i) Given a trader's SX membership $F[i;\mathbf{I}]$, the trader optimally chooses his period 1 and 2 consumption and securities trading, i.e., $(x_1^i, x^i(1), \ldots, x^i(\Xi), y^i)(F(\mathbf{I})) \in \arg\max u^i(x_0, x_1, x(1), \ldots, x(\Xi))$, subject to constraints (1), (2), and (3).

(D1.ii) The security market clears for each SX, i.e., $\sum_{i \in S} y_j^i = 0, \forall j \in \mathbf{J}(S), \forall S \in F(\mathbf{I})$.

Denote the set of securities trading equilibria, given $F(\mathbf{I})$, by $E(F(\mathbf{I}))$. The first assumption is needed to prove existence of a "securities trading equilibrium". Assumption (A1) is standard:

(A1) *For every $x_0 \in \mathbb{R}_+$, $u^i(x_0, \cdot)$ is twice continuously differentiable, is strictly monotone, and has the matrix of second derivatives, $D^2u^i(x_0, \cdot)$, negative semidefinite.*

Note that with a strictly monotone utility function, the market clearing equations for security trading in each SX imply the market clearing equations of the good in

[6] Allouch and Wooders (2008) demonstrate that, given communication costs, for all sufficiently large economies, the core is non-empty and the set of price-taking equilibrium outcomes is equivalent to the core.

both periods and states of nature, i.e.,

$$\sum_{i\in\mathbf{I}}(x_1^i - \omega_1^i) = 0 \text{ and } \sum_{i\in\mathbf{I}}\left(x^i(\xi) - \omega^i(\xi) - \sum_{j\in\mathbf{J}} a_j(\xi)e_j^i\right) = 0, \text{ for all } \xi \in \Xi.$$

Lemma 1 below uses Assumption (A1) to prove that a security trading equilibrium exists. We will then use this lemma to prove existence of an equilibrium for our economy with security SXs.

Lemma 1 *Let us assume (A1) holds. Then, for a fixed SX structure, there exists a securities trading equilibrium.*

The proof of Lemma 1 is omitted because it is immediate. To see this, notice that in our model, the SX structure is in fact a partition of the set of traders because there is only one good and traders belong to only one SX. Existence of equilibrium in each SX follows from standard arguments.

To see this, recall that the SX structure with a single SX membership is in fact a partition of the set of traders. Existence of equilibrium in each SX follows from standard arguments.

In some particular cases, the "securities trading equilibrium" can be unique. When markets are complete, our competitive financial equilibrium two-period model of the second stage is equivalent to an Arrow–Debreu economy, and global uniqueness holds under quite general conditions on the primitives. When markets are incomplete, global uniqueness is more difficult to obtain (incomplete markets can arise in our economy if the structure of SXs is such that there are SXs where traders' security holdings are not enough to insure risk in all possible states of nature). Because global uniqueness is a nice feature to work with in a more simplified version of our model (where an equilibrium selector is not needed), we establish, for the sake of exposition, a set of conditions on the primitives that guarantee, given a SX structure, the uniqueness of the "securities trading equilibrium".

Remark (On the uniqueness of the "securities trading equilibrium"): Let us consider that all traders in our economy have utility functions that are functionally separable on x_0 with $u^i(x_0, x_1, x(1), \ldots, x(\Xi)) = u_0^i(x_0)u_1^i(x_1, x(1), \ldots, x(\Xi))$, and let the utility function $u_1^i(x_1, x(1), \ldots, x(\Xi))$ be quadratic as follows:

$$u_1^i(x_1, x(1), \ldots, x(\Xi)) = \sum_{\xi=1}^{\Xi} \rho_\xi^i\left(x_\xi - \frac{1}{2}\alpha^i x_\xi^2\right)$$

where $\rho^i \in \mathbb{R}_{++}^{\Xi}$, $\sum_{\xi=1}^{\Xi} \rho_\xi^i = 1$, and $\alpha^i \in \mathbb{R}_{++}^{\Xi}$, for all $i \in \mathbf{I}$. In addition, assume two monotonicity conditions: (i) $1 - \alpha^i\omega_s > 0$ for all $i \in \mathbf{I}$ and all $\xi \in \Xi$ where $\omega = \sum_{i=1}^{I} \omega^i \in \mathbb{R}_{++}^{\Xi}$, and (ii) for all $j \in \mathbf{J}$, $a_j(\xi) > 0$ for at least one $\xi \in \Xi$. Then, if $\rho = \rho^i$ for all $i \in \mathbf{I}$, the securities trading equilibrium is globally unique even when security exchanges are characterized by incomplete markets. The proof follows from Hens and Pilgrim's (2002) Theorem 6.7 and is omitted.

In addition to the conditions of a quadratic utility function and monotonicity stated in our Remark above, Hens and Pilgrim (2002) examine other interesting settings, such as an economy with hyperbolic absolute risk aversion (HARA) utility functions, which underlies much of the finance literature. See Chapter 6 in Hens and Pilgrim (2002) for this and other useful results on global uniqueness of equilibrium in a general equilibrium model with incomplete markets.

3.2 Stage 1: exchange formation

Given a SX structure $F(\mathbf{I})$ and our specification of the utility function u^i, we define trader i's utility via the equilibrium point $\tilde{x}(F[i;\mathbf{I}]) \equiv (x_1, x(1), \ldots, x(\Xi))(F[i;\mathbf{I}])$ as follows:[7]

$$V^i(x_0, F[i;\mathbf{I}]) \equiv u^i(x_0, \tilde{x}(F[i;\mathbf{I}])). \tag{4}$$

Trader i's evaluation of a SX is captured by utility u^i through the trader's SX membership $F[i;\mathbf{I}]$. It is appropriate to refer to function (4) as trader i's "equilibrium SX utility function". Under the standard "rational expectations hypothesis", traders' beliefs about the realization of prices in each state are common and self-fulfilling. Because the set of securities trading equilibria $E(F(\mathbf{I}))$ may have more than one equilibrium, we have to make sure there exists a measurable selector for the equilibrium correspondence $E(F(\mathbf{I}))$. Lemma 2 below addresses this issue and thus guarantees that $V^i(x_0^i, F[i;\mathbf{I}])$ is well defined.

Lemma 2 *There exists a measurable selection* $\tilde{x}^I(F(\mathbf{I})) = (\tilde{x}^i(F[i;\mathbf{I}]) : i \in \mathbf{I})$ *for the equilibrium correspondence* $E(F(\mathbf{I}))$.

The proof is left for the "Appendix." The concept of an equilibrium selector is well known and has been used in different strands of the literature; see for example, Miao (2006) in recursive macroeconomics, Berliant and Page (2001) in public economics, Simon and Zame (1990) in game theory, Faias (2008) and Faias et al. (2002) in financial economics, Stahn (1999) for a general equilibrium model with monopolistic behavior, and Allen (1994) and Mas-Colell and Nachbar (1991) for seminal contributions to the use of equilibrium selectors in economics. In these models, in general, a profile of actions gives rise to a set of equilibrium outcomes. Then, to obtain an equilibrium existence result through well-defined payoff functions, which themselves depend on these profiles, authors use equilibrium selections. For example, this is the case of Cournot–Walras models with a continuous space of actions, where continuous random selections are used. In our model, the set of SX structures $\mathbf{F}(\mathbf{I})$ is finite, and thus, a continuous measurable selector is not needed. Because the set of SX structures $\mathbf{F}(\mathbf{I})$ is a finite set, the equilibrium correspondence $E(\cdot)$ defined in $\mathbf{F}(\mathbf{I})$ is trivially a weak measurable correspondence, and thus, we can use the Kuratowski–Ryll–Nardzewski measurable selection theorem to prove Lemma 2 (see Aliprantis and Border 2006,

[7] In a more sophisticated model with additional benefits of exchange membership (other than risk diversification, e.g., information acquisition or increasing gains from a larger exchange), we should modify the domain of the trader's utility function as follows: $u^i(x_0, F[i;\mathbf{I}], x_1, x(1), \ldots, x(\Xi))$. This extension is left for future research.

p. 600). Further research could be done in choosing an equilibrium with specific (normative) properties through the choice of a the measurable selector. This approach could be fruitful in addressing policy considerations in the market microstructure of SXs.

Next, we introduce the equilibrium concept for the process of SX formation (stage 1). Because existence of this equilibrium relies on the Allouch and Wooders (2008) existence theorem, we need to satisfy AW's assumptions, including the "ε_0 coalition formation costs". These "communication costs" (in AW's terminology) consist of small frictions in period 0 and affect not only traders' opportunities to change SX memberships, *but also* opportunities to purchase a different commodity amount. The communication costs for a SX with size $|S|$ are given by $c(\varepsilon_0) = \varepsilon_0|S|$, $\varepsilon_0 > 0$.

In the equilibrium Definition 1 below, we denote by $\tilde{F}_{F(N)}(\mathbf{I})$ an off-equilibrium deviation by the set of traders N. Roughly speaking, a deviation by a set of traders $N \subset \mathbf{I}$ from a SX structure $F(\mathbf{I})$ determines a SX structure that includes $F(N) = \{S'_k\}_{k=1}^{K'}$ and also those SXs in $F(\mathbf{I}) = \{S_k\}_{k=1}^{K}$, but without those traders belonging to N. Formally, we write $\tilde{F}_{F(N)}(\mathbf{I}) = F(N) \cup \{S_k \cap (\mathbf{I} \backslash N)\}_{k=1}^{K}$. Note that the set of possible deviations is "large" in the sense that we allow traders in N to form any type of SX structure among themselves.

Definition 2 ($c(\varepsilon_0)$*-equilibrium of the SX formation*): A $c(\varepsilon_0)$-SX structure equilibrium for period 0 is an ordered triple $((x_0^I, F(\mathbf{I})), \Pi)$ that consists of a consumption vector $x_0^I \equiv (x_0^i : i \in \mathbf{I})$, a SX structure $F(\mathbf{I})$, and a participation price system $\Pi = \{\pi^i(S) \in \mathbb{R} : i \in S \text{ and } S \subseteq \mathbf{I}\}$ such that,

(D2.i) $\sum_{i\in\mathbf{I}}(x_0^i - \omega_0^i) + \sum_{S_k \in F(\mathbf{I})} z(S_k) \leq 0$.

(D2.ii) For each $S \subset \mathbf{I}$, profits are non-positive, i.e., $\sum_{i\in S} \pi^i(S) - z(S) \leq 0$.

(D2.iii) For each $i \in \mathbf{I}$, any $N \subset \mathbf{I}$ with $i \in N$, and any SX structure deviation $\tilde{F}_{F(N)}(\mathbf{I})$, if $V^i(y_0^i, \tilde{F}_{F(N)}[i;\mathbf{I}]) > V^i(x_0^i, F[i;\mathbf{I}])$, then $y_0^i - \omega_0^i + \pi^i(\tilde{F}_{F(N)}[i;\mathbf{I}]) > -\varepsilon_0$.

(D2.iv) $-\varepsilon_0 N \leq \sum_{S_k} z(S_k) + \sum_{S_k\in F(\mathbf{I})} \sum_{i\in S_k} \pi^i(S_k) \leq 0$

Remarks on Definition 2:

1. Equilibrium condition (D2.i) is a feasibility condition that requires traders' period 0 good endowment to cover (a) the purchase of the aggregate consumption good in period 0 and (b) the SX formation costs corresponding to structure $F(\mathbf{I})$.
2. Equilibrium condition (D2.ii) says that no SX can have positive profits. Condition (D2.iv) limits the extent to which these profits can be negative (aggregate losses are bounded). In particular, condition (D2.iv) says that SXs cannot be significantly far, in aggregate, inside their budget sets. Note that this condition is similar to AW's condition (iv) of the definition of a $c(\varepsilon_0)$-equilibrium.
3. Our equilibrium condition (D2.iii) is analogous to AW's $c(\varepsilon_0)$-equilibrium condition (iii) and requires the maximization of utility, given the communication costs and the period 0 budget constraint (more discussion on this below). This condition applies not only to alternative off-equilibrium SXs, but also to alternative off-equilibrium commodity consumption: It is a joint individual optimization condition. Roughly speaking, communication costs affect both the opportunities to

change SX memberships and the opportunities to purchase a different amount of the consumption good.

It is important to notice that in the definition of $c(\varepsilon_0)$-equilibrium, the budget constraint does not need to be balanced for all consumers [condition (D2.iii)]. However, in equilibrium, most consumers cannot be very far outside of their budget sets (see Allouch and Wooders 2008, p. 253). This weaker condition of the price-taking equilibrium notion is needed to establish the equivalence between a price-taking equilibrium and an approximate core (namely, the AW's ε_0-core). The existence of a price-taking equilibrium in AW follows from the non-emptiness of the core. However, in a setting with local public goods and increasing returns from larger coalitions, non-emptiness of the core requires communication costs. Next, we briefly explain the subtleties of AW's concept of the core with communication costs and its relation with previous works.

AW's require quasi-concavity of utility functions over private commodities, but they do not make such an assumption for club memberships. But then, Shapley and Shubik (1966) and Wooders (1978, 1983) point out, the core may be empty under general conditions. However, certain quasi-cores are non-empty when the number of individuals is large. This is the case for the AW's ε_0-core, which is based on Wooders' (1983) important result that cooperative games with many players have non-empty approximate cores in economies with local public goods; this result is the basis for a number of influential works in the literature of club formation. The non-emptiness of approximate cores of games with many players has been used to establish the existence of a price-taking equilibrium through decentralization of outcomes in approximate cores of economies in a variety of settings, including economies with coalition formation costs such as in AW (see Allouch and Wooders 2008, pp. 260–261). In view of the mild assumptions required to obtain non-emptiness of approximated cores of large games, numerous models have adopted this approach (such as, with indivisibilities, non-convexities, coalition production, or multiple-sided matching models). So far, an application to financial markets and the introduction of a pre-trading stage have not been predicted to our knowledge.

4. Any deviation by a subset of traders N incorporates the new risk-sharing possibilities that these traders have in the new candidate SXs; therefore, the new SXs may have different membership prices. By the same argument as above, the trading equilibrium selection implies that the function $V^i(y_0^i, \tilde{F}_{F(N)}[i; \mathbf{I}])$ in condition (D2.iii) is also well defined.

5. Depending on the composition of the set of traders, it may be that some traders cannot be accommodated in their preferred SXs. Allouch and Wooders (2008) show that when the economy is large, only a small percentage of (leftover) consumers (traders in our terminology) cannot be accommodated in their preferred SXs. The modified notion of equilibrium with remainders (ε_1-remainder $c(\varepsilon_0)$-equilibrium in AW's terminology) assumes that all consumers in the economy are competitive or almost competitive except for perhaps a small percentage of "leftover" consumers. Here, for simplicity of exposition, we ignore those "remainders" and just refer to the equilibrium in the second stage as a $c(\varepsilon_0)$-equilibrium for the SX formation process. As argued by Allouch and Wooders (2008), when the economy is large, these leftovers

constitute only a small proportion of the total population, and a solution that ignores this small exceptional set of remainders provides a reasonable approximation of an exact solution.

Our definition of equilibrium for the securities exchange economy is:

Definition 3 ($c(\varepsilon_0)$-*equilibrium of the SX economy*): A vector $(x_0^I, F(\mathbf{I}), \Pi, (x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, q(S_1), \ldots, q(S_K)(F(\mathbf{I})))$ constitutes a $c(\varepsilon_0)$-equilibrium for our SX economy if

(D3.i) $(x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, q(S_1), \ldots, q(S_K)(F(\mathbf{I})))$ is a securities trading equilibrium for $F(\mathbf{I})$.

(D3.ii) Given the securities trading equilibrium $(x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, q(S_1), \ldots, q(S_K)(F(\mathbf{I})))$, the vector $(x_0^I, F(\mathbf{I}), \Pi)$ is a $c(\varepsilon_0)$ -equilibrium for the SX formation process.

4 Existence of a $c(\varepsilon_0)$-equilibrium

A variety of assumptions are needed to guarantee existence of equilibrium. We provide formal arguments showing how our assumptions imply AW's (2008) assumptions. We also explain the main subtleties of the existence proof. We start by explaining the necessity of considering a compact metric set of attributes in our model where the AW's continuity conditions are satisfied.

4.1 Compact metric set of attributes

AW's (2008) approximate price-taking equilibrium existence result relies on the notion of an "approximate Edgeworth equilibrium" (roughly speaking, a state of the economy whose replicas are in the approximate cores of the corresponding replica economies). AW proves that, for a given approximate Edgeworth equilibrium state of the economy, there is a price system that decentralizes that equilibrium state and also that the set of approximate Edgeworth equilibria is non-empty. Both results imply the existence of an approximate price-taking equilibrium.

New difficulties arise with adoption of an Edgeworth equilibrium approach in the context of club goods. We mention here two important ones. First, the core of the replica economy cannot be embedded in a finite dimensional space because new commodities (clubs) emerge when one replicates the set of consumers. Second, quasi-concavity of the utility function in a club goods economy is not required, as that assumption would be too restrictive. But without quasi-concavity, one cannot use previous approaches that rely on the equal treatment property. AW overcomes this and other difficulties. One critical step in AW to prove that the set of Edgeworth equilibria is non-empty is to extend the results in replica economies with a finite set of attribute types to a compact metric set of attributes. This is because AW needs to consider small perturbations of the attributes (i.e., small perturbations in a set of economies). Along this extension, AW makes specific assumptions that guarantee that a consumer's payoff varies continuously when attributes (economies) are smoothly perturbed. Another important assumption in AW is "desirability of wealth" (private goods can compensate for memberships in large clubs), which ensures per capita boundedness, an important condition to apply previous results in the literature.

We apply the price-taking equilibrium existence result of Allouch and Wooders (2008) in the first stage of our model, when traders sort themselves into different SXs. To this end, we also have to consider a compact set of attributes (economies) and write assumptions that guarantee that the AW's assumptions are also satisfied in our economy. Some of the difficulties in our model come from the fact that in AW's (2008) setting, the utility that agents achieve from the consumption of private goods and club goods is a fundamental. In our case, however, the utility that a trader achieves in a SX (club) is endogenous because it depends on the equilibrium attained in his SX, which is the outcome of the second stage in our model.

For our economy, $\mathcal{E} = \{(u^i, \omega^i, e^i) : i \in \mathbf{I}\}$, the traders' attributes that matter are utilities and endowments. In order to obtain a compact set of attributes—a compact set of economies—endowed with a metric that satisfies the continuity conditions of Allouch and Wooders (2008), we need to apply the transversality theorem in finite dimension in the same fashion as in Geanakoplos and Polemarchakis (1986). Thus, we consider a set of parameterized utility functions $\mathcal{U}$, with the set of parameters defined in a finite dimensional space, endowed with the induced Euclidean metric.[8] Furthermore, we assume that the functions in $\mathcal{U}$ satisfy Assumption (**A1**). Lastly, let $\mathbf{W} = \mathbb{R}^L_{++} \times \mathbb{R}^J_+$ denote the finite dimensional set of endowments where we fix the Euclidean metric.

To write our assumptions using the same notation style as in Allouch and Wooders (2008), let $(\boldsymbol{\Theta}, d)$ denote the set of attributes (economies) endowed with metric d, that is, $\boldsymbol{\Theta} = \mathcal{U} \times \mathbf{W}$, and d is the product metric. The attribute function $\alpha : \mathbf{I} \to \boldsymbol{\Theta}$ associates attributes to each trader, $\alpha(i) \in \boldsymbol{\Theta}$. These attributes consist of a utility function $u^{\alpha(i)}$ and endowments $\left(\omega^{\alpha(i)}, e^{\alpha(i)}\right)$ such that $\left(u^{\alpha(i)}, \omega^{\alpha(i)}, e^{\alpha(i)}\right) \in \boldsymbol{\Theta}$. The pair $(\mathbf{I}, \alpha)$ designates an economy where a trader's attributes are defined by $\alpha(i)$. Henceforth, we use the notation $(\mathbf{I}, \alpha)$ and $(\mathbf{I}, \beta)$, or $\alpha(i)$ and $\beta(i)$, when we want to distinguish between two different economies or between two different types of trader i in these economies, respectively. Otherwise, we use the previous notation (u^i, ω^i, e^i); in this case, that means that the property holds for trader i whatever the economy in Θ.

4.2 Additional assumptions

To prove the existence of an equilibrium SX structure, and in particular a $c(\varepsilon_0)$-equilibrium of the SX formation, we need a set of assumptions on the attributes of traders. In assumptions (A2.i) and (A2.iii) below, we will write $\tau > 0$ to denote the uniform lower bound on period 0 endowments of the consumption good.

(A2.i) *For all* $i \in \mathbf{I}$, $\omega_0^i > \tau$ *with* $\tau > 0$, *and given* $\varepsilon > 0$, *there exists* $\lambda > 0$ *such that for any set* $\mathbf{I}$ *and pair of economies* $(\mathbf{I}, \alpha)$ *and* $(\mathbf{I}, \beta)$, *if* $d(\alpha(i), \beta(i)) \leq \lambda$ *for any* i, *then* $\omega_0^{\alpha(i)} \leq \omega_0^{\beta(i)} + \varepsilon$. *Also,* $u^i(\cdot, x_1, x(1), \ldots, x(\Xi))$ *is continuous, increasing, and strictly quasi-concave.*

[8] An example is a set of parameterized Cobb–Douglas utility functions, where the set of parameters is defined in a finite dimensional space.

Assumption (A2.i) requires period 0 endowments to be uniformly bounded away from 0, and, for near economies, that traders' endowments not differ significantly. The first part of (A2.i) appears in AW's condition (g), "continuity with respect to attributes 2". Assumptions in the second part of (A2.i) are standard and imply AW's assumptions (a) monotonicity, (b) continuity, and (c) quasi-concavity on $V^i(\cdot, F[i; \mathbf{I}])$.

(A2.ii) *Given* $\varepsilon > 0$, *there exists* $\lambda > 0$ *such that, for any SX S and any attribute functions* α *and* β, *if* $d(\alpha(i), \beta(i)) \leq \lambda$ *for every* $i \in S$, *then* $z^{\alpha}_{S^{\alpha}} \leq z^{\beta}_{S^{\beta}} + \varepsilon$.

Assumption (A2.ii) imposes a continuity condition on z with respect to attributes, also needed for our existence result of a $c(\varepsilon_0)$-equilibrium of the process of SX formation. There, S^{α} and S^{β} are two SXs comprising the same traders, but with attributes α and β, respectively. AW's assumption (h), namely "continuity with respect to attributes 3", is precisely our assumption (A2.ii).

The remaining assumptions differ slightly from AW, since the club good associated with each SX, interpreted here as the securities trading facility, is endogenous in our model.

(A2.iii) *If* $\tau > 0$ *is the uniform lower bound on period 0 endowments of the consumption good and* $u^i(\omega^i_0 - \tau, \omega^i_1, \omega^i(1), \ldots, \omega^i(\Xi)) < u^i(x_0, x_1, x(1), \ldots, x(\Xi))$, *then* $x_0 > 0$.

Assumption (A2.iii) guarantees that period 0 endowments are desirable and implies AW's condition (d) "desirability of endowment". Notice that their assumption can be rewritten in our notation as follows: If $V^i(\omega^i_0 - \tau, \{i\}) < V^i(x^i_0, F[i; \mathbf{I}])$, then $x^i_0 > 0$. Since $V^i(\omega^i_0 - \tau, \{i\}) = u^i(\omega^i_0 - \tau, \omega^i_1, \omega^i(1), \ldots, \omega^i(\Xi))$, and $V^i(x^i_0, F[i; \mathbf{I}]) = u^i(x^i_0, \tilde{x}^i_1, \tilde{x}^i(1), \ldots, \tilde{x}^i(\Xi))$. Thus, we conclude that Assumption (A2.iii) implies AW's condition (d).

(A2.iv) *Given any* $\varepsilon > 0$, *there exist* $\rho_{\varepsilon} > 0$, *such that, for any economy* $(\mathbf{I}, \alpha)$ *and any* $i \in \mathbf{I}$, $u^i(x^i_0, x_1, x(1), \ldots, x(\Xi)) + \rho_{\varepsilon} < u^i(x^i_0 + \varepsilon, x_1, x(1), \ldots, x(\Xi))$, *for any* x^i_0 *and* $(x_1, x(1), \ldots, x(\Xi))$.

Assumption (A2.iv) says that the good is valuable in period 0. In particular, each trader's "marginal" utility of consumption in period 0 has a uniformly strictly positive lower bound. Assumption (A2.iv) implies AW's condition (e) "private goods are valuable", which in terms of our notation can be written as follows: Given any attribute θ and any $\varepsilon > 0$, there is $\rho^{\theta}_{\varepsilon} > 0$ such that, for all $i \in \mathbf{I}$ with $\alpha(i) = \theta$ and all $x^i_0 \in \mathbb{R}_+$, $V^i(x^i_0, F[i; \mathbf{I}]) + \rho^{\theta}_{\varepsilon} < V^i(x^i_0 + \varepsilon, F[i; \mathbf{I}])$ holds. It stands to reason that this assumption follows immediately from our assumption (A2.iv).

Assumption (A2.iv) is also necessary to prove AW's assumption (f) "continuity with respect to attributes". In terms of our notation, this assumption says that given $\varepsilon > 0$, there exist $\gamma > 0$, such that, for any set $\mathbf{I}$ and pair of economies $(\mathbf{I}, \alpha)$ and $(\mathbf{I}, \beta)$, if $d(\alpha(i), \beta(i)) \leq \gamma$, then $V^{\alpha(i)}(x^i_0, F[i; \mathbf{I}^{\alpha}]) < V^{\beta(i)}(x^i_0 + \varepsilon, F[i; \mathbf{I}^{\beta}])$, for any i and any $x^i_0 \in \mathbb{R}_+$. As AW explains, this condition ensures that if the attributes of consumers in a club were slightly perturbed, then a small increase in their private goods allocations would compensate for any loss in utility due to the perturbation.

To prove that AW's assumption (f) holds in our SX economy, we need to introduce one assumption that, roughly speaking, requires continuity of the utility function when we "slightly" change the traders' attributes in the economy.

(A2.v) *Given any* $\delta > 0$, *there exist* $\lambda > 0$, *such that, for any pair of economies* $(\mathbf{I}, \alpha)$ *and* $(\mathbf{I}, \beta)$, *if* $d(\alpha(i), \beta(i)) \leq \lambda$ *for all* $i \in \mathbf{I}$ *then, for any* x_0^i *and* $(x_1, x(1), \ldots, x(\Xi))$, $\left|u^{\alpha(i)}(x_0^i, x_1, x(1), \ldots, x(\Xi)) - u^{\beta(i)}(x_0^i, x_1, x(1), \ldots, x(\Xi))\right| < \delta$.

Proposition 1 below follows from Assumption (A2.v) and an important intermediate result that asserts that the securities trading equilibrium $\tilde{x}(F[i; \mathbf{I}])$ is continuous in traders' attributes (endowments and utilities). To prove this intermediate result, we apply the transversality theorem. For this, we consider the concept of a generic subset of the set (Θ, d) of attributes. By a generic set, we mean an open and dense set where the complement has zero measure—see Geanakoplos and Polemarchakis (1986). We explicitly refer to the next result as a proposition, as it stands as a contribution to the theory of clubs. The proof is long and therefore is left for the "Appendix."

Proposition 1 *Assume that (A2.v) holds. Then, there are a generic set of economies* $\mathcal{U}' \times \mathbf{W}'$, *for which, given* $\lambda > 0$, *there is a* $\gamma > 0$ *such that, for any pair of economies* $(\mathbf{I}, \alpha)$ *and* $(\mathbf{I}, \beta)$, *if* $d(\alpha(i), \beta(i)) \leq \gamma$ *for any* i, *then* $|V^{\alpha(i)}(x_0, F[i; \mathbf{I}^\alpha]) - V^{\beta(i)}\left(x_0, F[i; \mathbf{I}^\beta]\right)| < \lambda$.

Claim 1 *Proposition 1 and Assumption (A2.iv) imply AW's Assumption (f).*

Proof Proposition 1 asserts that, for each SX structure $F(\mathbf{I})$, there is a generic set $\left(\mathcal{U}' \times \mathbf{W}'\right)(F(\mathbf{I}))$ where the securities trading equilibrium is continuous in traders' attributes. Now, the finite intersection $\mathcal{U}'' \times \mathbf{W}'' \equiv \bigcap_{F(\mathbf{I}) \in \mathbf{F}(\mathbf{I})} \left(\mathcal{U}' \times \mathbf{W}'\right)(F(\mathbf{I}))$ is a generic set, where the securities trading equilibrium is continuous in traders' attributes for every SX structure. Given an economy $(\mathbf{I}, \alpha)$ belonging to the generic set $\mathcal{U}'' \times \mathbf{W}''$, we can find a compact subset of economies $\Theta' \subset \Theta$ that contains $(\mathbf{I}, \alpha)$ (since $\mathcal{U}'' \times \mathbf{W}''$ is open) where Proposition 1 holds. We now prove that assumption (f) of AW holds in this compact set of economies for $x_0^i \leq \sum \omega_0^i + \varepsilon$, with $\varepsilon > 0$.[9] Our assumption (A2.iv) implies that given $\varepsilon > 0$, there exist ρ_ε such that, $u^{\alpha(i)}(x_0^i + \varepsilon, x_1, x(1), \ldots, x(\Xi)) - u^{\alpha(i)}(x_0^i, x_1, x(1), \ldots, x(\Xi)) > \rho_\varepsilon$. On the other hand, given ρ_ε, Proposition 1 implies that there exists $\gamma > 0$ such that, if $d(\alpha(i), \beta(i)) \leq \gamma$, then $|u_1^{\beta(i)}(x_0^i, \tilde{x}(F[i; \mathbf{I}])) - u_1^{\alpha(i)}\left(x_0^i, \tilde{x}(F[i; \mathbf{I}])\right)| < \rho_\varepsilon$. Therefore, $V^{\beta(i)}(x_0^i + \varepsilon, F[i; \mathbf{I}^\beta]) - V^{\alpha(i)}(x_0^i, F[i; \mathbf{I}^\alpha]) = u^{\beta(i)}(x_0^i + \varepsilon, \tilde{x}(F[i; \mathbf{I}^\beta])) - u^{\alpha(i)}(x_0^i, \tilde{x}(F[i; \mathbf{I}^\alpha])) = u^{\beta(i)}(x_0^i + \varepsilon, \tilde{x}(F[i; \mathbf{I}^\beta])) - u^{\beta(i)}(x_0^i, \tilde{x}(F[i; \mathbf{I}^\beta])) + u^{\beta(i)}(x_0^i, \tilde{x}(F[i; \mathbf{I}^\beta])) - u^{\alpha(i)}(x_0^i, \tilde{x}(F[i; \mathbf{I}^\alpha])) > 0$, that is, if $d(\alpha(i), \beta(i)) \leq \gamma$, $V^{\beta(i)}(x_0^i + \varepsilon, F[i; \mathbf{I}^\beta]) - V^{\alpha(i)}(x_0^i, F[i; \mathbf{I}^\alpha]) > 0$. □

Certainly, we want our model to allow for the formation of a large unique SX in equilibrium.[10] We do not want to impose any assumption that exogenously bounds SX sizes.[11] For this, we need to show that AW's assumption "desirability of wealth"

[9] Observe that AW (2008, pp. 271–272) only requires assumption (f) to be satisfied for a consumption x_0^i bounded above by the aggregate endowments plus some $\varepsilon > 0$.

[10] Anecdotal evidence shows that this was a common belief until 2006. See *The Economist*, March 25, 2006 (http://www.economist.com/node/6978712): "*Liquidity and technology will inevitably make trading a natural monopoly*".

[11] Cole and Prescott (1997), Ellickson et al. (2001), Luque (2013), and Wooders (1980,1997) require bounded club sizes. In contrast, Wooders (1989) and Allouch and Wooders (2008) permit both

is satisfied. This assumption says that there is $x_0^* \in \mathbb{R}_+$ and an integer η such that for any economy $(\mathbf{I}, \alpha)$ and any $i \in \mathbf{I}$, there is a coalition $S \in \mathbf{I}$ with $|S| \leq \eta$ and a club structure $F(S)$ satisfying $V^i(x_0^i + x_0^*, F[i; S]) \geq V^i(x_0^i, F[i; \mathbf{I}])$, for any $F(\mathbf{I})$ and any $x_0^i \in \mathbb{R}_+$. Another assumption is needed to guarantee equilibrium existence when traders may experience ever-increasing gains from larger SXs, while at the same time, small SXs are allowed.

(A3) *There is* $x_0^* \in \mathbb{R}_+$ *such that for any economy* $(\mathbf{I}, \alpha)$, *any trader* $i \in \mathbf{I}$, *and any* $x_0^i \in \mathbb{R}_+$, *we have* $u^i(x_0^i + x_0^*, \omega_1^i, \omega^i(1), \ldots, \omega^i(\Xi)) \geq u^i(x_0^i, \sum_i(\omega_1^i,\ \omega^i(1), \ldots, \omega^i(\Xi)))$.

Assumption (A3) says that, even in the worst-case scenario where trader i cannot diversify risk in any SX and, as a consequence, consumes his endowment $(\omega_1^i, \omega^i(1), \ldots, \omega^i(\Xi))$, the trader prefers to consume a very large amount of the private good in period 0, $x_0^i + x_0^*$, rather than consume the aggregate endowments in periods 1 and 2. Notice that the consumption $x_0^i + x_0^*$ can be very great and even unfeasible for the trader.

Claim 2 *Assumption (A3) implies AW's "desirability of wealth" assumption.*

Proof Notice that by $V^i(x_0^i + x_0^*, \{i\}) = u^i(x_0^i + x_0^*, \omega_1^i, \omega^i(1), \ldots, \omega^i(\Xi))$. By (A3), $u^i(x_0^i + x_0^*, \omega_1^i, \omega^i(1), \ldots, \omega^i(\Xi)) \geq u^i(x_0^i, \sum_i(\omega_1^i, \omega^i(1), \ldots, \omega^i(\Xi)))$ and $u^i(x_0^i, \sum_i(\omega_1^i, \omega^i(1), \ldots, \omega^i(\Xi)) \geq V^i(x_0^i, F[i; \mathbf{I}])$. Then, transitivity implies that $V^i(x_0^i + x_0^*, \{i\}) > V^i(x_0^i, F[i; \mathbf{I}])$, which satisfies the "desirability of wealth" assumption for a SX $S = \{i\}$ (and therefore, for $\eta = 1$ in AW's terminology). □

We have now shown that all assumptions required in Theorem 2 of Allouch and Wooders (2008) are satisfied (namely, AW's assumptions (a)–(h) and "desirability of wealth").[12] We are now ready to state Lemma 3, which is needed for the proof of Theorem 1 below:

Lemma 3 *Assume that A2 and A3 hold. Then, there exists a generic set of SX economies for which there is a* $c(\varepsilon_0)$*-equilibrium with possibly ever-increasing gains from larger SXs.*

4.3 The existence result

In our equilibrium existence result, we assume that assumptions (A1), (A2), and (A3) hold.

Footnote 11 continued
unbounded club sizes and ever-increasing gains from larger clubs. The main difference between the two cases is the extent to which approximation or coalition formation costs are required to obtain existence of equilibrium. Konishi et al. (1998) allow for an arbitrary number of jurisdictions, but take as given the size of the population, rather than looking at "large" economies. See Luque (2014) for a review of the different approaches to the presence of equilibrium in local public good economies.

12 AW (2008, Theorem 2) says that: "*Given communication costs, for all sufficiently large economies the core is non-empty and the set of* $c(\varepsilon_0)$*-equilibrium outcomes* (similar notion to our Definition 2) *is equivalent to the core*".

Theorem 1 *If there are sufficiently many traders with attributes represented in the economy, then there exists a generic set of SX economies for which there is a* $c(\varepsilon_0)$*-equilibrium of the SX economy.*

The proof of Theorem 1 is in the "Appendix," but here are its main steps.

- The proof of existence of a "securities trading equilibrium" (Lemma 1) has some novel aspects. This proof first considers one auxiliary economy for each SX, where traders trade the good and the securities only among themselves, and then shows that the equilibrium for this system of auxiliary economies is a trading equilibrium. Notice that we cannot use the standard approach to equilibrium existence in an economy with incomplete markets and a unique trading platform, where an auctioneer chooses both the commodity price and the security prices in the simplex. In our model with multiple SXs, it may happen that this auctioneer chooses the price of one security equal to 1; therefore, the remaining commodity and security prices would be zero. But then, there would be SXs whose traders face commodity and security prices equal to zero, so their budget constraints would hold with equality, and lower semicontinuity of the budget correspondence would fail.
- The application of AW's existence result to prove the existence of a $c(\varepsilon_0)$-equilibrium for the process of SX formation (Lemma 3) is not immediate because some of their assumptions rely on evaluation of the individual's utility function at the club good, which in our model is endogenous. The trickiest part is to show that AW's condition (f) "continuity with respect to attributes" holds for our economy. This condition ensures that traders who are similar in terms of their attributes have similar utility functions and also that traders who are "close" in attribute set are "crowding substitutes" for each other. Our Proposition 1 addresses this issue. Also, we need to make sure that there is a compact subset of a generic set of traders' attributes where the continuity property on attributes holds. The next two points explain the details:

1. The proof of Proposition 1 makes use of the transversality theorem. There are two subtleties in this proof. The first is that we have to show that the securities trading equilibrium is a continuous differentiable function of commodity and security prices. The second subtlety has to do with equilibrium regularity. Specifically, we have to show that there is a generic set of endowments and utilities such that the set of securities trading equilibria is a continuously differentiable function of the endowment and utility assignment. For that purpose, Geanakoplos and Polemarchakis (1986) offer good guidance. However, our framework is different, so we must adapt their proof to an economy where: (1) the securities market clears for each SX; (2) the trading period accounts for the commodity and securities; and (3) selling a security requires having enough endowment of that security (the box constraint must be satisfied).
2. Finally, recall that AW's result relies on the use of a compact set of traders' attributes where the continuity property on attributes holds. We prove that this continuity property holds in a generic set. A generic set is open and dense; thus, for each economy in the generic set, we can fix a compact set that includes this economy and use this set to extend replica economies. Observe that we do not need

to take any limit as the economy grows—AW needs to consider replica economies only to prove their existence result, and in our model, we verify all AW's assumptions.

5 Further remarks and directions for future research

In this paper, we considered a two-stage equilibrium model where in the first stage, traders formed SXs based on their evaluations of the risk-sharing opportunities in the different SXs, and then, in the second stage (once SXs have been formed), securities were traded in each SX. A two-stage model was needed because we embedded the "securities trading equilibrium" (an endogenous object) in the trader's utility function. This was not the case in Allouch and Wooders (2008), where the local public good entered directly as a primitive in the individual's utility function (e.g., park or swimming pool).

Another important feature of our model is the assumption that a trader cannot trade in more than one SX. In an economy with one private good, this assumption implies that the trader is only affected by the actions of other traders in his SX, a setting compatible with previous club models, where an agent's utility depends on other members of his club, but not on agents outside the club. This limitation in club models prevents us from allowing for multiple memberships in our model, because it would imply that the trader's utility would depend on the whole SX structure through equilibrium security prices.[13]

There are several alternative approaches to the endogenous formation of security exchanges. First, one could model the negotiation of membership fees among participants using a transferable utility game. In that setting, one should require that for every SX that does not form, it is impossible to find an array of fees that covers the SX formation cost (with zero profits) and leaves everybody ex-ante better off. Second, network and search models could also be extended to accommodate a first stage that pins down the network structure (see Hojman and Szeidl 2008 for a first approach to endogenous network formation).

There are other elements of our model that can be developed in future research. First, our model relies on the concept of Walrasian equilibrium for the notion of a "securities trading equilibrium" (once SXs are formed). The Walrasian mechanism could be replaced by other market mechanisms, such as direct search, competitive search, Nash bargaining, or auctions (see Biais 1993 for a model of competition between markets under different trading rules). One could also incorporate information problems (see Pancs 2015 for an efficient dark market environment). For these alternative models, one needs to verify the property that the securities trading equilibrium is continuous in traders' attributes as we do in Proposition 1. This enrichment of the model would widen the notion of a SX.

[13] In Faias and Luque (2015a), we provide an equilibrium existence proof for a general equilibrium economy with *exogenous* exchange structures, cross-listings, and multiple memberships. It remains an open question as to how to modify club models to allow for an agent's utility function to depend on the whole structure of clubs.

One could also incorporate default in our model by allowing traders to choose not only the trading platform, but also a clearing house that covers default in those states of nature in period 2 when the trader does not have enough endowment to pay his debt. This analysis would extend Santos and Scheinkman's (2001) analysis of default with two clearing houses to a model with multiple clearing houses that endogenously emerge in equilibrium.

Finally, several policy questions could be studied using the lens of our model, such as market fragmentation/consolidation and the completeness/incompleteness of the SXs' security structures. In Faias and Luque (2015b), we use a simplified version of the model to revisit important policy questions, such as whether it is beneficial for traders to sort themselves into two SXs with different securities in order to trade and share risk among themselves, and whether bond and equity markets should be separated as they typically are. Among other results, we highlight the importance of a discriminatory SX membership price system for certain types of SX structures to emerge.

Appendix

Proof of Theorem 1 This theorem follows from lemmas 1, 2, and 3. Lemmas 1 and 3 have already been stated and proved. Below we prove Lemma 2 and Proposition 1, the latter used to show that AW's condition (f) "continuity with respect to attributes" holds for our economy □

Lemma 4 *There exists a measurable selection* $\tilde{x}^I(F(\mathbf{I})) = (\tilde{x}^i(F[i;\mathbf{I}]) : i \in \mathbf{I})$ *for the equilibrium correspondence* $E(F(\mathbf{I}))$.

Proof of Lemma 2 The proof follows by the Kuratowski–Ryll–Nardzewski measurable selection theorem (a weak measurable correspondence with non-empty closed values into a separable metrizable space admits a measurable selection).[14] In fact, we have that $\mathbf{F}(\mathbf{I})$ is a finite set, and therefore, the equilibrium correspondence $E(\cdot)$ defined in $\mathbf{F}(\mathbf{I})$ is trivially a weak measurable correspondence (see Aliprantis and Border 2006, p. 600). The correspondence takes values in the positive coordinate subset of a finite dimensional space, and therefore, it follows immediately that it is a separable metrizable space.

The correspondence $E(\cdot)$ takes closed values, i.e., if $(x_1^{I,s}, x^{I,s}(1), \ldots, x^{I,s}(\Xi), y^{I,s}, p^s, q^s)$ is a sequence in $E(F(\mathbf{I}))$ that converges to $(x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, p, q)$, then $(x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, p, q)$ also belongs to $E(F(\mathbf{I}))$. Given an equilibrium sequence, if we consider the budget constraints of each trader and pass to the limit, we obtain that in the limit the budget constraint of each trader is satisfied. The same reasoning allows us to prove that market clearing also holds in the limit.

Finally, it remains to show that in the limit each trader is maximizing his utility. Suppose not, so for a trader i there is another bundle $(\check{x}^i, \check{y}^i)$ that is budget feasible such

[14] Notice that the equilibrium correspondence is defined in the finite set of exchange structures; therefore, a continuous measurable selector is not needed. Continuous selectors are in general used to construct continuous objective functions. Thus, they fit only if the correspondence is defined in a continuum set.

that $u^i(x_0, \check{x}^i) > u^i(x_0, \tilde{x}^i)$. Now, let $(\check{x}^i, \check{y}^i) = (\lambda\tilde{x}^{i,s} + (1-\lambda)\check{x}^i, \lambda y^{i,s} + (1-\lambda)\check{y}^i)$ with $\lambda \in [0, 1]$. Observe that $(\check{x}^i, \check{y}^i)$ is budget feasible for s large enough and for λ close to one. Moreover, by continuity, we have that $u_1^i(\check{x}^i) > u_1^i(\tilde{x}^{i,s})$, for s large enough. Then, the strict quasi-concavity implies that $u^i(x_0, \check{x}^i) = u^i(x_0, \lambda\tilde{x}^{i,s} + (1-\lambda)\check{x}^i) > u_1^i(x_0, \tilde{x}^{i,s})$. This is a contradiction because $((\tilde{x}^{i,s})_{i\in\mathbf{I}}, y^{I,s}, p^s, q^s)$ is a trading equilibrium for the given SX structure $F(\mathbf{I})$. □

An immediate consequence of Lemma 2 is that $V^i(x_0, F[i; \mathbf{I}])$ is well defined.

Proof of Proposition 1 This proof requires the following four steps.

Step 1 $\tilde{x}^i(F[i; \mathbf{I}])$ is a C^1 function in prices q.[15]

The first-order necessary and sufficient conditions for an interior optimum of a trader i for which some of his box constraints are binding are:

$$
\begin{aligned}
&D_1 u^i - \tilde{\beta}_1 = 0\\
&D_\xi u^i - \tilde{\beta}(\xi) = 0, \xi = 1, \ldots, \Xi\\
&-x(\xi) + \omega^i(\xi) + A^i_\xi(e^i + y) = 0, \xi = 1, \ldots, \Xi\\
&\tilde{\beta}^T A^i - \tilde{\beta}_1 q + \tilde{\mu}^i = 0\\
&-x_1 + \omega^i_1 - qy = 0\\
&e^i + y = 0
\end{aligned}
$$

where T refers to the transpose of a matrix. The shadow price vectors for the budget constraints in period 1 and node ξ of period 2 are $\tilde{\beta}_1$ and $\tilde{\beta}(\xi)$, respectively, and the shadow price vector for the box constraints is denoted by $\tilde{\mu}$. The trader i's return matrix is also denoted by $A^i = A(S)$, where $S = F[i, \mathbf{I}]$. The element A^i_ξ denotes the line ξ of the return matrix A^i. The Jacobian matrix with the second-order derivatives with respect to $(x_1, x(\xi), \tilde{\beta}(\xi), y, \tilde{\beta}_1, \tilde{\mu})$, where $x(\xi)$ and $\tilde{\beta}(\xi)$ are generic elements of the corresponding Ξ-vector, is:

$$
\mathbf{J} = \begin{bmatrix}
D_1^2 u^i & 0 & 0 & 0 & -1 & 0\\
0 & D_\xi^2 u^i & -1 & 0 & 0 & 0\\
0 & -1 & 0 & A^i & 0 & 0\\
0 & 0 & A^{i\,T} & 0 & -q & 1\\
-1 & 0 & 0 & -q^T & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

It is easy to see that the matrix $\mathbf{J}$ is non-singular (i.e., invertible). For this, we need to show that $\mathbf{J}z = 0$ implies $z = 0$ for all $z = (\check{x}_1, \check{x}(\xi), \check{\beta}(\xi), \check{y}, \check{\beta}_1, \check{\mu})$. So let z be such that $\mathbf{J}z = 0$. Then, $\check{y} = 0$. We also have that $z^T\mathbf{J}z = 0$, and by using $\check{y} = 0$, the matrix reduces to $\check{x}_1^T(D_1^2 u_1^i)\check{x}_1 + \check{x}(\xi)^T(D_\xi^2 u_1^i)\check{x}(\xi) = 0$. This last equality can be written as

$$
\begin{bmatrix} \check{x}_1^T & \check{x}(\xi)^T \end{bmatrix}
\begin{bmatrix} D_1^2 u^i & 0\\ 0 & D_\xi^2 u^i \end{bmatrix}
\begin{bmatrix} \check{x}_1\\ \check{x}(\xi) \end{bmatrix} = 0
$$

15 Here, q stands for the security price vector in trader i's exchange.

which implies $\check{x}_1 = 0$, $\check{x}(\xi) = 0$ by negative definiteness of D^2u^i. Then, back to $\mathbf{J}z = 0$, we obtain $\check{\beta}_1 = 0$, $\check{\beta}(\xi) = 0$, and $\check{\mu} = 0$. Therefore, by the implicit function theorem, we conclude that the trader's excess demand with binding box constraints is a C^1 function on security prices.

If instead a trader i's box constraints are non-binding, then we have to remove the last first-order condition ($e^i + y = 0$), and also the last column and last row of the previous Jacobian matrix. This modified matrix is also non-singular. To see this, let z be such that $\mathbf{J}z = 0$. Then, $z^T\mathbf{J}z = 0$, and using some of the equations of the system $\mathbf{J}z = 0$, $z^T\mathbf{J}z = 0$ reduces to $\check{x}_1^T(D_1^2u_1^i)\check{x}_1 + \check{x}(\xi)^T(D_\xi^2u_1^i)\check{x}(\xi) = 0$. Notice that this last equality can be written as before, and we again can conclude that $\check{x}_1 = 0$ and $\check{x}(\xi) = 0$ by negative definiteness of $D^2u_1^i$. Then, back to $\mathbf{J}z = 0$, we obtain $\check{y} = 0$ since A^i has full rank. Finally, again with $\mathbf{J}z = 0$, $\check{\beta}(\xi) = 0$ and $\check{\beta}_1 = 0$. Therefore, by the implicit function theorem, we conclude that the trader's excess demand with non-binding box constraints is a C^1 function of security prices.

To see why continuity is preserved whether the box constraint is binding or not, notice that when traders in a given SX want to sell more than what they own of a security, the box constraint becomes binding. This pressure is then translated into a positive shadow price μ that affects the security price. Our continuity result says that this shadow price adjusts to changes in the security price so that the demand function changes in a continuous way. When the box constraint is non-binding, continuity is preserved because demand is a continuous function of the security prices. □

Step 2 For any choice of utilities $U \equiv (u^i)_{i\in\mathbf{I}}$ in $\mathcal{U}$, there is a generic set, $\mathbf{W}(U) \subset \mathbf{W}$, of endowments of periods 1 and 2, such that for every economy $\left(U, (\omega^i, e^i)_{i\in I}\right)$ *with* $(\omega^i, e^i)_{i\in\mathbf{I}} \in \mathbf{W}(U)$, the set of security trading equilibria is a continuous differentiable function of the endowments of periods 1 and 2.

Denote the price domain in period 1 by $\mathbf{M}_1 = \mathbb{R}_+^{\sum_{S\in F(\mathbf{I})} J(S)}$. Also, denote by $f : \mathcal{U} \times \mathbf{W} \times \mathbf{M}_1 \to \mathbb{R}^{\sum_{S\in F(\mathbf{I})} J(S)}$ the aggregate excess demand function of securities, given utilities, commodity and security endowments, and security prices. Let the utilities be fixed at U. We want to show that f restricted to U, denoted by $f_{|U}$, is transverse to 0 in order to apply the implicit function theorem. That is, if for all $(\omega, e, q) \in \mathbf{W} \times \mathbf{M}_1$ with $f_{|U}(\omega, e, q) = 0$, the Jacobian matrix $D_{(\omega,e,q)}f_{|U}$ has full rank. For this, we will show that there is a set of independent vectors of directional derivatives with dimension $\sum_{S\in F(\mathbf{I})} J(S)$. Consider a SX S and choose a trader k that belongs to this SX. We can reduce $\omega^k(\xi)$ by $a_{j(S)}(\xi)$, for all ξ, and increase ω_1^k by $q_{j(S)}$, and the budget constraint still holds. The only effect on trader k's demand is an increase in security $j(S)$ by 1 unit, and the net effect on the aggregate excess demand of security $j(S)$ is now $(0, \ldots, 1, \ldots, 0)$. This proves Step 2. □

Step 3 There is a generic set $\mathcal{U}' \times \mathbf{W}'$, such that for every economy with $(u^i)_{i\in\mathbf{I}} \in \mathcal{U}'$ and $(\omega^i, e^i)_{i\in\mathbf{I}} \in \mathbf{W}'$, the set of security trading equilibria is a continuous differentiable function of both the endowment and the utility assignment.[16]

[16] A set is a continuously differentiable function if all of its elements are continuously differentiable functions.

Fix a trading equilibrium $(x_1^I, x^I(1), \ldots, x^I(\Xi), y^I, q)$ for SX structure $F(\mathbf{I})$. Now, choose a SX $S \in F(\mathbf{I})$ and a trader k in S. Trader k's budget constraints in this SX are $x_1 - \omega_1^k + \sum_{j \in \mathbf{J}(S)} q_j y_j = 0$ and $x(\xi) - \omega^k(\xi) = \sum_{j \in \mathbf{J}} a_j(\xi) e_j^k + \sum_{j \in \mathbf{J}(S)} a_j(\xi) y_j$. Since the payoffs of the securities available in SX S are independent, there is a submatrix with full rank: We consider without loss of generality that the first $J(S)$ rows are independent, and we fix B as the non-singular $J(S) \times J(S)$ submatrix with the first $J(S)$ rows. Then, we can obtain the portfolio vector as a function of the commodity bundle, $y^k = B^{-1}[x^k(\xi) - \omega^k(\xi) - \sum_{j \in \mathbf{J}} a_j(\xi) e_j^k]_{\xi=1,\ldots,J(S)}$, and write the following single budget constraint:

$$x_1^k - \omega_1^k + q(S) \cdot B^{-1} \left[x^k(\xi) - \omega^k(\xi) - \sum_{j \in \mathbf{J}} a_j(\xi) e_j^k \right]_{\xi=1}^{J(S)} = 0, \tag{5}$$

where $q(S)$ is the vector of security prices in S. Consider the following utility function for trader k : $u^k(x_0^k, x_1, x(1), \ldots, x(J(S)), x^k(J(S)+1), \ldots, x^k(\Xi))$, which is a function of $(x_1, x(1), \ldots, x(J(S)))$ and where $(x_0^k, x^k(J(S)+1), \ldots, x^k(\Xi))$ are assumed to be fixed. The previous single budget constraint (5) can be written as follows,

$$x_1 - \omega_1^k + p \cdot \left[x(\xi) - \omega^k(\xi) - \sum_{j \in \mathbf{J}} a_j(\xi) e_j^k \right]_{\xi=1}^{J(S)} = 0, \tag{6}$$

where $p = q(S)B^{-1}$. Now, let $\hat{x} = (\hat{x}(1), \ldots, \hat{x}(J(S)))$ be trader k's commodity demand function using the single budget constraint (6). Following Geanakoplos and Polemarchakis (1986), we can now perturb trader k's utility in such a way that $\hat{x}$ does not change, but $D_p\hat{x}$ is altered by a non-singular matrix. Since $D_{q(S)} y^k = D_{\hat{x}} y^k D_p \hat{x} D_{q(S)} p = B^{-1} D_p \hat{x} [B^{-1}]^T$, where $[B^{-1}]^T$ is the transpose of B^{-1}, and both B^{-1} and $[B^{-1}]^T$ are invertible, we conclude that, by perturbing trader k's utility, it is possible to perturb $D_{q(S)} y^k$ without changing y^k.

Now, let $\tilde{\mathbf{S}} = \{r \in \mathbb{R}^{\sum_{S \in F(\mathbf{I})} J(S)} : \|r\| = 1\}$, which has dimension $\sum_{S \in F(\mathbf{I})} J(S) - 1$, and consider the function

$$g : \boldsymbol{\mathcal{U}} \times \boldsymbol{W} \times \boldsymbol{M}_1 \times \tilde{\mathbf{S}} \rightarrow \mathbb{R}^{\sum_{S \in F(\mathbf{I})} J(S)} \times \mathbb{R}^{\sum_{S \in F(\mathbf{I})} J(S)},$$

given by $g(\nu, r) = (f(\nu), r^T D_q f)$, where $\nu = (U, \omega, e, q)$ and Df_q is the Jacobian of the aggregate demand function of securities with respect to security prices. By the previous argument, it is possible to perturb trader k's utility in SX S_k for every SX $S_k \in F(\mathbf{I})$, and in this way, we perturb $D_q f$ without perturbing f. In Step 3, we proved that f is transverse to zero, so we conclude that g is transverse to zero. By the transversality theorem, $g_{(U,\omega,e)}$ is transverse to zero, for (U, ω, e) chosen in a generic set of $\mathcal{U} \times \mathbf{W}$. Since the domain of $g_{(U,\omega,e)}$ has one dimension lower than its range, we conclude that $g^{-1}_{(U,\omega,e)}(0) = \emptyset$. Thus, $r^T D_q f = 0$ has no solution, which means that $D_q f$ (a function of utilities, endowments, and prices) is invertible, and then strong regularity follows. □

Step 4 There is a generic set of economies for which, given $\lambda > 0$, there is a $\gamma > 0$ such that for any set I and pair of economies (I, α) and (I, β), $d(\alpha(i), \beta(i)) \leq \gamma$ for any i, *then* $|V^{\alpha(i)}(x_0, F[i; \mathbf{I}^\alpha]) - V^{\beta(i)}(x_0, F[i; \mathbf{I}^\beta])| < \lambda$.

The proof of this fourth step follows by Assumption (A2.v) and the continuity of security trading equilibria (Step 3). □

Now, at this part of the proof of Theorem 1, observe that, given a SX structure, the security trading equilibrium may not be unique. This would imply that there is an indirect utility $u^i(x_0, \tilde{x}(F[i; \mathbf{I}]))$ for each equilibrium solution $\tilde{x}(F[i; \mathbf{I}])$, and thus more than one function $V^i(x_0, F[i; \mathbf{I}])$. Existence of an equilibrium for the SX economy would require choosing a measurable selector of the equilibrium correspondence $E(\cdot)$. Then, we have to consider that for each SX structure $F(\mathbf{I})$, the utility u^i is evaluated at the equilibrium selection $\tilde{x}^i(F[i; \mathbf{I}])$. Lemma 2 asserts that this measurable selection exists.

References

Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis, 3rd edn. Springer, Berlin (2006)

Allen, B.: Randomization and the limit points of monopolistic competition. J. Math. Econ. **23**, 205–218 (1994)

Allouch, N., Wooders, M.H.: Price taking equilibrium in economies with multiple memberships in clubs and unbounded club sizes. J. Econ. Theory **140**, 246–278 (2008)

Balasko, Y., Cass, D., Siconolfi, P.: The structure of financial equilibrium with exogenous yields: the case of restricted participation. J. Math. Econ. **19**, 195–216 (1990)

Berliant, M., Page, F.: Income taxes and the provision of public goods: existence of an optimum. Econometrica **69**, 771–784 (2001)

Biais, B.: Price formation and equilibrium liquidity in fragmented and centralized markets. J. Finance **48**, 157–185 (1993)

Biais, B., Glosten, L., Spatt, C.: Market microstructure: a survey of microfoundations, empirical results, and policy implications. J. Financ. Mark. **8**, 217–264 (2005)

Bottazzi, J.-M., Luque, J., Pascoa, M.: Securities market theory: possession, repo and rehypothecation. J. Econ. Theory **147**, 477–1500 (2012)

Cole, H.L., Prescott, E.: Valuation equilibrium with clubs. J. Econ. Theory **74**, 19–39 (1997)

Colliard, J.-E., Foucault, T.: Trading fees and efficiency in limit order markets. Rev. Financ. Stud. **25**, 3389–3421 (2012)

Ellickson, B., Grodal, B., Scotchmer, S., Zame, W.: Clubs and the market: large finite economies. J. Econ. Theory **101**, 40–77 (2001)

Faias, M.: Approximate equilibrium in pure strategies for a two-stage game of asset creation. Decis. Econ. Finance **31**, 117–136 (2008)

Faias, M., Luque, J.: Risk Sharing Across Exchanges with Cross-Listings. Mimeo (2015a)

Faias, M., Luque, J.: Market Transparency and Endogenous Securities Market Fragmentation. Mimeo (2015b)

Faias, M., Moreno, E., Páscoa, M.: Real indeterminacy of equilibria and manipulability. J. Math. Econ. **37**, 325–340 (2002)

Gabszewicz, J., Thisse, J.-F.: Price competition, quality and income disparities. J. Econ. Theory **20**, 340–359 (1979)

Geanakoplos, J., Polemarchakis, H.: Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete. Cowles Foundation Working Paper 652 (1986)

Grant, J.: Derivatives trading platform bypasses intermediary banks. The Financial Times, January 17 (2011)

Hens, T., Pilgrim, B.: General Equilibrium Foundations of Finance: Structure of Incomplete Markets Model. Springer, Berlin (2002)

Hojman, D.A., Szeidl, A.: Core and periphery in networks. J. Econ. Theory **139**, 295–309 (2008)

Konishi, H., Le Breton, M., Weber, S.: Equilibrium in a finite local public goods economy. J. Econ. Theory **79**, 224–244 (1998)

Lo, A.W., Mamaysky, H., Wang, J.: Asset prices and trading volume under fixed transactions costs. J. Polit. Econ. **112**, 1054–1090 (2004)
Luque, J.: Heterogeneous Tiebout communities with private production and anonymous crowding. Reg. Sci. Urban Econ. **43**, 117–123 (2013)
Luque, J.: Wages, local amenities and the rise of the multi-skilled city. Ann. Reg. Sci. **52**, 457–467 (2014)
Madhavan, A.: Market microstructure. J. Financ. Mark. **3**, 205–258 (2000)
Mas-Colell, A., Nachbar, J.H.: On the finiteness of the number of critical equilibria, with an application to random selections. J. Math. Econ. **4**, 397–409 (1991)
Miao, J.: Competitive equilibria of economies with a continuum of consumers and aggregate shocks. J. Econ. Theory **128**, 274–298 (2006)
Pagano, M.: Trading volume and asset liquidity. Q. J. Econ. **104**, 255–274 (1989)
Pagnotta, E., Philippon, T.: Competition on Speed. Mimeo, New York (2012)
Pancs, R.: Efficient dark markets. Econ. Theory **59**, 605–624 (2015)
Santos, T., Scheinkman, J.: Competition among exchanges. Q. J. Econ. **116**, 1027–1061 (2001)
Securities and Exchange Comission.: Concept release on equity market structure 34 (2010)
Shapley, L.S., Shubik, M.: Quasi-cores in a monetary economy with nonconvex preferences. Econometrica **34**, 805–827 (1966)
Simon, L., Zame, W.R.: Discontinuous games and endogenous sharing rules. Econometrica **58**, 861–872 (1990)
Stahn, H.: Monopolistic behaviors and general equilibrium: a generalization of Nikaido's work. J. Math. Econ. **32**, 87–112 (1999)
Wooders, M.H.: Equivalence of Lindahl equilibria with participation prices and the core. Econ. Theory **9**, 115–128 (1997)
Wooders, M.H.: A Tiebout theorem. Math. Soc. Sci. **18**(1989), 33–55 (1989)
Wooders, M.H.: The epsilon core of a large replica game. J. Math. Econ. **11**, 277–300 (1983)
Wooders, M.H.: The Tiebout hypothesis; near optimality in local public good economies. Econometrica **48**, 1467–1485 (1980)
Wooders, M.H.: Equilibria, the core, and jurisdiction structures in economies with a local public good. J. Econ. Theory **18**, 328–348 (1978)