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Abstract

In this paper, we derive a Capital Asset Pricing Model (CAPM) formula from a general equilibrium model that allows agents to trade non-recourse mortgage debt in addition to risk free debt and risky equity. As in Geanakoplos and Zame (2014), borrowers must post collateral to obtain a mortgage loan, and effective mortgage payments are endogenous because default depends on the collateral value. Our main result is that binding collateral constraints (or, in terms of observables, a positive expected mortgage default rate) amplify the effect of a higher systematic risk on the risk premium of a portfolio with non-recourse mortgage debt.

JEL Classification: D52, D53, G12, R31

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1 Introduction

An important line of research in modern financial theory has involved finding a model for pricing financial assets. The Capital Asset Pricing Model (CAPM) is one of the most famous and important of such models because it provides an intuitive pricing condition based on simple means and covariaces of asset returns. Its roots go back to Markowitz (1952), who provided a mean-variance theory of portfolio choice, and James Tobin (1958), who included risk-free debt into the portfolio theory. William Sharpe (1964), John Lintner (1965), and Jan Mossin (1966) developed an equilibrium version of Markowitz's mean-variance portfolio theory.[1]

The CAPM formula says that the expected risk premium $E(\mathcal{R}_m)$ of a portfolio z^m that generates an income stream m linearly depends on the risk premium $E(\mathcal{R}_\omega)$ of the market portfolio z^ω and a factor $\beta_{\omega,m}$, called the beta coefficient, which captures the volatility, or *systematic risk*, of portfolio z^m in comparison to the market as a whole. Formally, the *classic CAPM formula* is:

$$E(\mathcal{R}_m) = \beta_{\omega,m} \cdot E(\mathcal{R}_\omega)$$

This result rests on two important assumptions. First, securities markets are competitive and there are no information problems. Second, agents are rational (i.e., they have unique, common, and degenerate beliefs on the occurrence of states of nature) and risk-averse. Other more specialized limiting assumptions include absence of transaction costs, taxes, and restrictions on borrowing and short selling.

Although there have been numerous contributions that attempted to relax some of the strong assumptions of the baseline CAPM model in a general equilibrium framework (see Magill and Quinzii 1996 and Hens and Pilgrim 2002), these models still lack some important aspects of financial investments.[2] One such missing aspect is the possibility that agents' portfolios include non-recourse mortgage debt. This is a predominant type of contract in the United States and other countries that allows investors to leverage their asset positions, subject to the premise that a borrower's effective mortgage payment is the minimum between the depreciated value of the collateral and the mortgage promise. Non-recourse mortgage debt is particularly important in

[1] See Allingham (1991) for a set of sufficient conditions under which an equilibrium exists for this model.

[2] For a discussion of the flaws of the baseline CAPM, see the famous seminal work of Black, Jensen, and Scholes (1972), and more recently, Fama and French (2004).

the real estate industry because in most transactions, the real estate investor/household obtains a non-recourse mortgage loan to buy a real estate asset/house.[3] For example, a household may obtain a loan to buy a house and use the house as collateral to secure the mortgage. Another example we have in mind is that of a real estate investor that secures a mortgage loan with one of its properties and uses the loan to buy real estate equity (e.g., equity shares of a shopping mall). These are examples that motivate our departure from the classic CAPM and highlight the importance of including non-recourse mortgage debt among the set of assets available to investors.

The goal of this paper is to derive a CAPM formula from a general equilibrium model that incorporates non-recourse mortgage debt. Our model follows the tradition of the collateral general equilibrium literature introduced by Geanakoplos (1997) and Geanakoplos and Zame (2014). In this class of models, a mortgage debt contract is defined not only by its promise, but also by the type and quantity of collateral that secures the loan. Non-recourse mortgage contracts are not subject to any moral hazard or adverse selection. Roughly speaking, the only incentive to deliver on a promise is that otherwise the collateral can be seized. To deal with default and non-recourse debt, Geanakoplos and Zame (2014) consider an agency (or clearing house in the literature terminology) that collects mortgage payments from borrowers (including the collateral in case of default) and chooses the "effective" mortgage payment to the lenders. The effective mortgage payment is thus endogenous. The agents' choice of loan amount, mortgage price and the Loan-To-Value are also endogenous. Importantly, the Loan-To-Value (LTV) equals the ratio between the mortgage price and the collateral price.

Under the assumption that agents have mean-variace preferences, we derive an explicit CAPM formula for a portfolio that generates an income stream m. This formula has two components. The first one is the term that appears in the classic CAPM formula: the product of the beta coefficient ($\beta_{\omega,m}$) and the expected risk premium of the market portfolio z^{ω} (in our notaiton $\beta_{\omega,m} \cdot E(\mathcal{R}_{\omega})$). The second component, which here we refer to as the *collateral premium*, is new. It consists of the product of the following three elements: (i) a function $f(\mu)$ of the agents' shadow costs associated with their collateral constraints (μ), with $f(\mu) > 0$ if $\mu > 0$, and

[3]When the collateral is a security (e.g., stock or bond), the loan can be seen as a repo contract and the collateral is fungible. Importantly, in a repo transaction, the lender – not the borrower – of money keeps the collateral and can further lend it in repo or sell it in the security market (see Bottazzi, Luque, and Pascoa 2012).

$f(0) = 0$; (ii) an increasing function g of the beta coefficient $\beta_{\omega,m}$; and (iii) the size of the short position on non-recourse mortgage debt (z^m_{d-}) associated to income stream m. Thus, our adjusted CAPM formula is:

$$E(\mathcal{R}_m) = \underbrace{\beta_{\omega,m} \cdot E(\mathcal{R}_\omega)}_{\text{classic CAPM component}} + \underbrace{f(\mu)g(\beta_{\omega,m})z^m_{d-}}_{\text{collateral premium}}$$

For an income stream m that is generated with a portfolio that contains non-recourse mortgage debt (i.e., $z^m_{d-} > 0$), our CAPM formula asserts that binding collateral constraints ($\mu > 0$) amplify the impact of the volatility, or systematic risk, on the risk premium of portfolio z^m. Interestingly, our adjusted CAPM formula coincides with the classic CAPM formula when the income stream m satisfies one of the following conditions: (1) m is generated by a portfolio that contains non-recourse mortgage debt ($z^m_{d-} > 0$), but collateral constraints are non-binding for all agents in the economy ($\mu = 0$), or (2) m is generated by a portfolio that does not include non-recourse mortgage debt ($z^m_{d-} = 0$). Importantly, we are also able to express the shadow price of collateral constraints, μ, as a weighted average of future mortgage default rates.

Our result, that binding collateral constraints (or, in terms of observables, a positive expected mortgage default rate) amplify the effect of systematic risk on the risk premium of a portfolio with non-recourse mortgage debt, may help rationalize the empirical fact that the predicted return for real estate assets using the classic CAPM formula was not aligned with the associated perceived risk in the capital market.[4] In a widely used book on real estate investments, Geltner, Miller, Clayton, and Eichholtz (2007) provide ample discussion of existing evidence on this issue. This sort of negative result contrasts with the popularity of the CAPM in the investment industry during the 1970s and 1980s, and the spirited debate in those years of whether the CAPM could be applied to real estate decision making.[5] Our findings suggest that the collateral premium can account for

[4]This result and more generally the pricing of risk for real estate assets have been the object of an active line of research; see for example Ibbotson, Diermier, and Siegel (1984) on the role of asset illiquidity, Lustig and Van Nieuwerburgh (2005, 2010) on the role of housing wealth and human wealth on risk premia, and Ambrose, Diop, and Yoshida (2016), Diop (2017), and Tuzel (2010) on the role of competition in the real estate market .

[5]For an early study on this issue, see Ross and Zisler (1991), who find that the equity real estate investment trust (REIT) index is far more volatile than other types of real estate indexes. They also indicate that real estate risk lies plausibly midway between that of stocks and bonds, in the 9 percent to 13 percent range. See also Titman and Warga (1986) for an analysis of REITs risk-adjusted performance using both single index (i.e., CAPM) and multiple index (i.e., APT) models.

a significant fraction of the risk premium of a portfolio (or asset) when collateral constraints are binding. Even if the collateral borrowing constraint friction is small at the individual level (say, μ^i small for agent i), it can generate a substantial deviation from the predicted CAPM portfolio risk premium at the aggregate level. In this sense, our result resembles Lustig and Van Nieuwerburgh's (2010) finding that frictions of collateral constraints generate substantial deviations of perfect risk-sharing at the regional level.[6]

1.1 Further relationship with the literature

The seminal works of Geanakoplos (1997) and Geanakoplos and Zame (2014) have inspired many scholars, and significant contributions have followed – see, for example, Araujo, Pascoa, and Torres-Martinez (2002), Kubler and Schmedders, and Fostel and Geanakoplos (2015), and references therein. However, to our knowledge there is no paper that derives an explicit CAPM formula for this class of models. The only exception is Fostel and Geanakoplos (2015), but their contribution rests on a numerical example for the particular case of a binominal economy without an explicit CAPM formula for a portfolio risk premium.

Another important work that relates to our paper is Garleanu and Pedersen (2011). These authors derive a CAPM formula from an equilibrium model with a funding constraint ("margins constraint" in their terminology) that requires *both* short and long investors to pay a margin (cost) proportional to their positions. Their CAPM formula departs from the classic CAPM formula in that there is an additional term that captures the funding friction specific to the margin. However, there are fundamental differences between our model – based on Geanakoplos and Zame (2014) – and Garleanu and Pedersen's (2011) model. First, we consider the specific case of non-recourse mortgages, where only the borrower (short) consitutes collateral, whereas they consider the case of asset trading subject to exogenous margin requirements on both long and short positions. Secondly, and related to the first difference, we specifically allow for strategic default, which in turn endogenously determines the effective mortgage payment, whereas in their model there is no default and asset payoffs are exogenously given. Thirdly, in our model the LTV

[6]Lustig and Van Nieuwerburgh (2010) propose a dynamic general equilibrium model with limited commitment and default resulting in the loss of housing collateral that rationalizes the fact that the consumption-to-income dispersion ratio co-moves positively with housing collateral scarcity.

is equal to the mortgage price and thus endogenous, while in their model the margin is exogenous. Fourthly, in our model an agent's optimization problem is subject to both a budget constraint and the constraint to put up collateral (collateral constraint), whereas in their model there is only one constraint – the margin constraint, which embeds both the agent's wealth and the margins requirements, and thus it is not possible to isolate the impact of the collateral component from other changes in the budget constraint of an investor.

The remainder of this paper is as follows. In Section 2, we present the general equilibrium model with non-recourse debt. In Section 3, we present our pricing results derived from a CAPM economy. In Section 4, we conclude. For the sake of presentation, we leave the proofs of our main results for the Appendix.

2 General equilibrium with non-recourse mortgage debt

Let us consider an economy with one consumption good, I agents, two dates, $t = 0, 1$, and a finite set $\mathbf{S} = \{1, ..., S\}$ of states of nature at date $t = 1$. Denote the corresponding vector of probabilities for each state by $\rho = (\rho_1, ..., \rho_S)$, where ρ_s is the probability associated to state $s \in \mathbf{S}$. In other words, $\rho \in \mathbb{R}^S$ is a probability measure in the space $(S, \mathcal{P}, \rho)$, where $\mathcal{P}$ is the sigma-algebra that contains all subsets of S and $\rho(\tau) = \sum_{s \in \tau} \rho_s$ is the probability of an event $\tau \in \mathcal{P}$. We assume that these probabilities are common across all agents of the economy.

Agent $i = 1, ..., I$ is endowed with a vector of initial endowments $\omega^i = (\omega_0^i, \omega_{\mathbf{1}}^i)$, where $\omega_0^i \in \mathbb{R}_+$ is the endowments in units of the consumption good at date 0 and $\omega_{\mathbf{1}}^i = (\omega_1^i, ..., \omega_S^i) \in \mathbb{R}_+^S$ is the vector of endowments (in units of the consumption good) at each state of date 1. Agent i has preferences represented by a utility function $u^i : X^i \to \mathbb{R}$, where $X^i \subset \mathbb{R}^{S+1}$ denotes the agent i's consumption space. We denote the agent i's consumptions at date 0 and state s of date 1 by x_0^i and x_s^i, respectively. The vector $x_{\mathbf{1}}^i = (x_s^i)_{s=1}^S$ represents the agent i's consumption bundle at date 1, and $x^i = (x_0^i, x_{\mathbf{1}}^i)$ represents the agent i's consumption bundle at both dates 0 and 1. We denote by X^I the consumption space for all agents in the economy.

The consumption good is durable and gets depreciated at a rate $1 - D_s$ at state s of date $t = 1$. Thus, if an agent $i = 1, ..., I$ consumes h units of the good at $t = 0$, then at state s of date $t = 1$ this agent is endowed with $\omega_s^i + D_s h$ units of the consumption good.

Agents are allowed to trade $J+1$ assets. The first J assets can be stocks, bonds, real estate equity, etc. Agents own these assets to different degrees. We write agent i's vector of asset shares of ownership in these J assets by $\delta^i \in \mathbb{R}^J_{++}$ and let $\delta = \sum_i \delta^i$. The payoff vector of asset $j = 1, ..., J$ is denoted by $Y^j = (Y^j_1, ..., Y^j_s, ..., Y^j_S) \in \mathbb{R}^S_+$, where Y^j_s denotes the return of asset j at state s. Among the first J assets, there is a risk-free asset, say $j = 1$, with payoff $Y^1 = \bar{R} \equiv 1 + \bar{r}$ at every state of nature at the second date. We denote the $S \times J$ matrix of returns y^j by

$$Y = \begin{pmatrix} Y^1_1 & \cdots & Y^j_1 & \cdots & Y^J_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y^1_s & \cdots & Y^j_s & \cdots & Y^J_s \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y^1_S & \cdots & Y^j_S & \cdots & Y^J_S \\ & & & & \end{pmatrix}$$

We say that the market generated by the first J assets is complete if the dimension of the columns of matrix Y is equal to S, and Y has a full column rank.

Each agent i chooses a portfolio $\theta^i \in \mathbb{R}^J_+$. We denote the agent i's positions on the first J assets ($j = 1, ..., J$) by $z^i_a = (\theta^i - \delta^i) \in \mathbb{R}^J_+$, and the corresponding market price vector by $q_a \in \mathbb{R}^J$. For simplicity, we do not allow for short sales, and thus we impose a constraint that requires $z^i_a \geq \delta^i \in \mathbb{R}^J_+$.[7]

In addition to the first J assets, we also consider an additional asset, denoted by d, that is non-recourse mortgage debt and promises 1 unit of the consumption good in each state of nature. However, as we explain below, the effective delivery can be smaller than 1 if the value of the depreciated collateral is smaller than the promise.

We denote by z^i_{d+} and z^i_{d-} the agent i's mortgage purchase (lending) and sale (borrowing) of mortgage debt, respectively, and the mortgage market price by q_d. We assume that there is an upper bound κ for mortgage short sales. Thus, the set of feasible financial transactions is

$$Z = (\mathbb{R}^J_+ - \delta^i) \times \mathbb{R}_+ \times [0, \kappa]$$

[7]Short sales could be introduced in this model by allowing agents to borrow equity shares in the repo market. See Bottazzi, Luque, and Pascoa (2012) for an equilibrium model of repo markets in which exogenous bounds on short sales are not required for equilibrium existence.

where the first space $((\mathbb{R}^J_+ - \delta^i))$ corresponds to the trading of the first J assets, the second space $(\mathbb{R}_+)$ corresponds to the non-recourse mortgage purchase, and the third space $([0, \kappa])$ corresponds to the non-recourse mortgage sale. Notice that for the third component we rule out the possibility that an agent sells more of these assets than the amount it owns (δ^i). We denote an agent i's portfolio by $z^i = (z^i_a, z^i_{d+}, z^i_{d-}) \in Z$.

Borrowing in the non-recourse mortgage debt market requires borrower i to put up an amount of collateral $C^i \in \mathbb{R}_+$ at date $t = 0$ per unit of mortgage sold. We allow collateral requirements to be personalized to capture the idea that borrowers are heterogenous, with possibly different FICO scores and access to the mortgage market. Because the nature of the mortgage is non-recourse and the collateral gets depreciated from date 0 to date 1, the borrower's *effective mortgage payment* is

$$Q^i_s = \min\left\{1\,,\ D_s C^i\right\}$$

at state s of date $t = 1$ for each unit of mortgage sold at $t = 0$. In other words, at state s of $t = 1$, a borrower's mortgage obligation is fully satisfied by paying the minimum between the mortgage promise and the depreciated value of collateral.

As in Geanakoplos and Zame (2014), we consider that there is an agency (or clearing house) that manages the mortgage payments and mortgage returns. The agency determines the mortgage rate of return $\phi_1 \in [0, 1)^S$ paid to the lenders. In particular, given the borrowers' effective mortgage payments, the agency delivers the following return for each unit of mortgage purchased:

$$\Phi_s = \begin{cases} \dfrac{\sum_{i=1}^{I} \bar{z}^i_{d-} \wedge D_s C^i z^i_{d-}}{\sum_{i=1}^{I} \bar{z}^i_{d-}} & \text{if } \sum_{i=1}^{I} \bar{z}^i_{d-} > 0 \\ 1 & \text{if } \sum_{i=1}^{I} \bar{z}^i_{d-} = 0 \end{cases}$$

We normalize the price of the consumption good to 1 and thus we can write the agent i's budget set as follows:

$$\mathbb{B}^i(q, Q^i, \Phi) = \left\{ (x_0^i, (x_s^i)_{s=1}^S) \in X^i \; : \begin{array}{l} x_0^i + C^i z_{d-}^i \; - \omega_0^i \leq -q_d z_{d+}^i + q_d z_{d-}^i - q_a z_a^i \\ x_s^i - \omega_s^i - Y\delta^i \leq \Phi_s z_{d+}^i - Q_s^i z_{d-}^i + D_s x_0^i + D_s C^i z_{d-}^i + Y z_a^i \\ (z_{d+}^i, z_{d-}^i, z_a^i) \in Z \\ \quad x_0^i \geq 0 \end{array} \right\}$$

We refer to the last constraint in $\mathbb{B}^i(q, Q^i, \Phi)$, $x_0^i \geq 0$, as the *collateral constraint*.[8] It requires that the borrower posts the required collateral $C^i z_{d-}^i$ when selling an amount z_{d-}^i of non-recourse mortgage debt. We denote the agent i's shadow value associated with this collateral constraint by μ^i. When $\mu^i > 0$, agent i's collateral constraint is binding. The economic interpretation is that agent i assigns a positive shadow cost (in terms of utility) associated with the requirement of constituting collateral in order to get a loan.

Our economy satisfies the hypothesis of the CAPM model. In particular, we assume that (i) there is a probability $\rho = (\rho_1, ..., \rho_S)$ for the states at date 1, with $\rho_s > 0$ for all $s = 1, ..., S$; (ii) the market subspace of returns generated by the columns of matrix Y includes a risk-free asset, i.e., $\mathbf{1} \equiv (1, ..., 1) \in \langle Y \rangle$; (iii) date $t = 1$ endowments are generated by matrix Y, that is, $\omega_{\mathbf{1}}^i \in \langle Y \rangle$, $i = 1, ..., I$; and (iv) $u^i : \mathbb{R}^{S+1} \to \mathbb{R}$ is a mean-variance utility function for all $i = 1, ..., I$. For tractability, we consider that agents' CAPM utility function is additively separable and quadratic, with the following form:[9]

$$u^i(x) = \alpha_0^i x_0^i - \frac{1}{2} \sum_{s=1}^{S} \rho_s (\alpha_1^i - x_s^i)^2 \,, \quad i = 1, ..., I,$$

where $\alpha_0 = \sum_i \alpha_0^i \in \mathbb{R}_{++}$ is the aggregate *impatience* parameter for consumption at date $t = 0$, and $\alpha_1 = \sum_i \alpha_1^i \in \mathbb{R}_{++}$ is the aggregate *risk tolerance* parameter for consumption at date $t = 1$. In order to guarantee that $u^i(x)$ is strictly increasing on the set of feasible state-contingent

[8] Alternatively, we could have defined $\tilde{x}_0^i = x_0^i + C^i z_{d-}^i$ and imposed the following constraint: $\tilde{x}_0^i \geq C^i z_{d-}^i$.

[9] We do not consider a penalty on negative consumption plans as in Eichberger, Rheinberger and Summer (2014). Instead, we use the fact that the borrower must constitute collateral in order to obtain a mortgage loan. See Eichberger, Rheinberger, and Summer (2014) and Magill and Quinzii (2000) for an interpretation of negative consumption plans usually admitted in the CAPM.

consumptions, we assume that, for each agent $i = 1, ..., I$,

$$\alpha_1 \mathbf{1} - (\omega_{\mathbf{1}} + Y\delta + K) \in \mathbb{R}^S_{++} \tag{1}$$

where $\omega_{\mathbf{1}} = \sum_i \omega^i_{\mathbf{1}}$ are the aggregate endowments at date 1, and $K = |I|\kappa$ is the upper bound on mortgage debt. Condition (1) requires α^i_1 to be sufficiently large in order to avoid local satiation.[10] The vector of aggregate marginal utilities is:

$$\begin{aligned}\bar{\gamma} = \sum_{i=1}^{I} \nabla u^i(\bar{x}^i) &= \sum_{i=1}^{I} \left(\alpha^i_0 \ , \ \left(\alpha^i_1 - \bar{x}^i_s\right)_{s=1}^{S} \right) \\ &= \left(\sum_{i=1}^{I} \alpha^i_0 \ , \ \sum_{i=1}^{I} \left(\alpha^i_1 - \bar{x}^i_s\right)_{s=1}^{S} \right)\end{aligned}$$

where $\sum\limits_{i=1}^{I} \alpha^i_0 = \alpha_0$, and $\sum\limits_{i=1}^{I} \alpha^i_1 = \alpha_1$.

Definition 1: $(\bar{x}, \bar{z}_{d+}, \bar{z}_{d-}, \bar{z}_a, \bar{q}, \bar{\Phi}) \in X^I \times Z^I \times \mathbb{R}^{J+1}_+ \times (0,1]^S$ *is an equilibrium for our economy if*:

- $\bar{x} \in \max \left\{ u^i(x^i_0 + C^i z^i_{d-} \ , \ x^i_1) : x^i \in \mathbb{B}^i(\bar{q}, \bar{Q}^i, \bar{\Phi}_1) \right\}$, for all $i = 1, ..., I$,
- $\sum\limits_{i=1}^{I} \bar{z}^i_{d+} - \sum\limits_{i=1}^{I} \bar{z}^i_{d-} = 0$ *and* $\sum\limits_{i=1}^{I} \bar{z}^i_a = 0$,
- $\sum\limits_{i=1}^{I} (\bar{x}^i_0 + C^i \bar{z}^i_{d-}) = \sum\limits_{i=1}^{I} \omega^i_0$,
- $\sum\limits_{i=1}^{I} \bar{x}^i_s = \sum\limits_{i=1}^{I} (\omega^i_s + D_s \omega^i_0 + Y_s \delta^i), \ \ \forall\, s = 1, ..., S,$
- $\bar{\Phi}_s = \begin{cases} \dfrac{\sum\limits_{i=1}^{I} \bar{z}^i_{d-} \wedge D_s C^i z^i_{d-}}{\sum\limits_{i=1}^{I} \bar{z}^i_{d-}} & \text{se } \sum\limits_{i=1}^{I} \bar{z}^i_{d-} > 0 \\ 1 & \text{se } \sum\limits_{i=1}^{I} \bar{z}^i_{d-} = 0 \end{cases}.$

For a proof of existence of a collateral equilibrium, see Geanakoplos and Zame (2014).

[10] Local satiation occurs when: $\forall x \in X$ and $\varepsilon \in \mathbb{R}_+, \exists y \in X : |x - y| < \varepsilon$ and $y \succ x$.

3 Adjusted CAPM with non-recourse mortgage debt

For our discussion below, we use the probability-induced inner products $\langle x^i_{\mathbf{1}}, z^i_{\mathbf{1}} \rangle = \sum_{s=1}^{S} \rho_S x^i_s z^i_s$, $\forall x^i_{\mathbf{1}}, z^i_{\mathbf{1}} \in \mathbb{R}^S$, and represent the mean, variance, and covariance functions in the standard way: $E(m) = \sum_{s=1}^{S} \rho_s m_s$ for the mean of vector $m \in \mathbb{R}^S$; $var(m) = \sum_{s=1}^{S} \rho_s \left(m_s - E(m)\right)^2$ for the variance of vector $m \in \mathbb{R}^S$; and $cov(m, m') = \sum_{s=1}^{S} \rho_s \left(m_s - E(m)\right)\left(m'_s - E(m')\right)$ for the covariance between vectors $m \in \mathbb{R}^S$ and $m' \in \mathbb{R}^S$.

3.1 Equilibrium pricing of an income stream

The following proposition provides a pricing formula for an income stream m generated by a feasible portfolio z^m for an agent i.

Proposition 1: *Let z^m_{d-} denote the short position on non-recourse mortgage debt required to generate income stream m generated by a linear combination of the existing $J+1$ assets. If $(\bar{x}, \bar{z}_{d+}, \bar{z}_{d-}, \bar{z}_a, \bar{q}, \bar{\Phi})$ is an equilibrium for our economy with non-recourse mortgage debt, then:*

- $\bar{\gamma} = (\bar{\gamma}_0, \bar{\gamma}_1)$ *satisfies the following condition:* $\bar{\gamma} = (\alpha_0\,, \mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}})$.
- *If $c(m)$ is the equilibrium price of an income stream m generated by a linear combination of the existing $J+1$ assets, then $c(m)$ satisfies the following condition:*[11]

$$c(m) = \frac{1}{\tilde{\alpha}_0 + \mu}\left(E\left(\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}\right)E(m) - cov(\tilde{\omega}_{\mathbf{1}}, m) - z^m_{d-}\overline{\mu C}\right) \tag{2}$$

where $\mu = \sum_{i=1}^{I} \mu^i$, $\overline{\mu C} = \sum_{i=1}^{I} \mu^i C^i$, $\tilde{\omega}_{\mathbf{1}} = (\tilde{\omega}_s)_{s=1}^{S} = (\omega_s + D_s\omega_0 + Y_s\delta)_{s=1}^{S}$, $\omega_0 = \sum_{i=1}^{I} \omega^i_0\,, \omega_s = \sum_{i=1}^{I} \omega^i_s\,,$ *and* $\tilde{\alpha}_0 = \alpha_0 + \sum_{s=1}^{S} D_s\left(\alpha_1 - D_s\omega_0 - \omega_s - Y_s\delta\right)$.

Before discussing pricing equation (2), we make four remarks regarding variables $\bar{\gamma}$ and $\overline{\mu C}$, and parameters $\tilde{\alpha}_0$ and $\tilde{\omega}_{\mathbf{1}}$.

[11] For simplicity, we present the pricing equation of an income stream generated by an equilibrium portfolio $\bar{z}$ with non-binding short-selling constraints on the first J assets. For the last asset (non-recourse mortgage debt), notice that Khun-Tucker optimality implies $\sigma^i_{b+}\bar{z}^i_{b+} = 0$ and $\sigma^i_{b-}\bar{z}^i_{b-} = 0$, where σ^i_{b+} and σ^i_{b-} are the shadow values corresponding to sign constraints $\bar{z}^i_{b+} \geq 0$ and $\bar{z}^i_{b-} \geq 0$, respectively. If m is an income stream generated by an out-of-equilibrium portfolio z, then we would modify the last term of equation (2) as follows: $(\sigma^i_{b+}z^i_{b+} + \sigma^i_{b-}z^i_{b-} - \sum_{i=i}^{I} z^i_{b-}\mu^i C^i)/(\tilde{\alpha}_0 + \mu)$.

Remark 1: *The vector $\bar{\gamma} = (\bar{\gamma}_0, \bar{\gamma}_1)$ is the equilibrium marginal evaluation of income streams that can be generated with the financial structure given for this economy. The first item in Proposition 1 follows because* $\bar{\gamma} = \left(\sum_{i=1}^{I} \alpha_0^i \ , \ \sum_{i=1}^{I} \ (\alpha_1^i - \bar{x}_s^i)_{s=1}^S \right) = \left(\alpha_0 \ , \mathbf{1}\alpha_1 - (\omega_s + D_s\omega_0 + Y_s\delta)_{s=1}^S \right) = (\alpha_0 \ , \mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}})$.

Remark 2: *Recall that μ^i is the agent i's shadow cost associated with the collateral constraint and, therefore, $\overline{\mu C} = \sum_{i=1}^{I} \mu^i C^i$ stands for the aggregated shadow costs associated with the collateral constraints weighted by the corresponding collateral requirements.*

Remark 3: *Because the consumption good is durable, the impatience parameter α_0 must be adjusted with the term*

$$\sum_{s=1}^{S} D_s \left(\alpha_1 - D_s\omega_0 - \omega_s - Y_s\delta \right)$$

We denote the resulting new impatience parameter by $\tilde{\alpha}_0$. This adjustment captures the fact that the durable consumption good, which serves as collateral for non-recourse mortgages, increases utility. This is a unique feature of this model that is absent in other CAPM models without collateral constraints (see, for example, Eichberger, Rheinberger, and Summer 2014). If the consumption good that serves as collateral were perishable (i.e., $D_s = 0$, for all $s \in S$), then $\tilde{\alpha}_0 = \alpha_0$.

Remark 4: *In our CAPM pricing formula for an income stream with non-recourse mortgage risk, the accumulated risky returns have to be adjusted with the expected non-recourse mortgage payments deficit and the amount of good depreciated from the first date, i.e.,*

$$\tilde{\omega}_{\mathbf{1}} = (\tilde{\omega}_s)_{s=1}^S = (\omega_s + D_s\omega_0 + Y_s\delta)_{s=1}^S.$$

Equation (2) is the pricing formula for an income stream m generated by a linear combination of the existing $J + 1$ assets, given an equilibrium $(\bar{x}, \bar{z}_{d+}, \bar{z}_{d-}, \bar{z}_a, \ \bar{q}, \bar{\Phi})$. Pricing equation (2) coincides with the *classic CAPM* pricing formula of an income stream m,

$$c(m) = \frac{1}{\alpha_0} \left(E\left(\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}} \right) E(m) - cov(\tilde{\omega}_{\mathbf{1}}, m) \right), \tag{3}$$

when the good is perishable $(1 - D_s = 1)$ and the collateral constraints are non-binding $(\mu^i =$

0 for all i) and/or income stream is not generated with a portfolio that contains non-recourse mortgage debt ($z^m_{d-} = 0$).

If the income stream m is generated with a portfolio that includes non-recourse mortgage debt (i.e., $z^m_{d-} > 0$), collateral requirements are positive (i.e., $C^i > 0$), and there is at least one agent with its collateral constraint binding (i.e., $\mu^i > 0$ for at least one i), then the term $(z^m_{d-}\overline{\mu C})$ in the pricing equation (2) is positive, which in turn reduces the market price $c(m)$ of income stream m below what would be predicted in absence of non-recourse mortgage debt. Roughly speaking, an agent's disutility of constituting collateral (above what would be its optimal consumption of the good in absence of the collateral constraint) has a negative impact on the price of any income stream generated by the financial market.

Finally, we note that an important covenant in a mortgage contract is whether the mortgage is subject to recourse or non-recourse law. In the former, in case of default the lender seizes the collateral and all the borrower's additional resources until the debt obligation has been fully met (consumption can be negative in equilibrium at the state(s) when the agent defaults). In the latter, the borrower is not obliged to honor the initial mortgage promise when the value of the collateral is smaller than that promise. In that case, the borrower (e.g., household) defaults and delivers the collateral (e.g., house) to the lender. In this paper we have examined the case of non-recourse mortgage debt. Another strand of the general equilibrium literature considers *recourse* debt instead (see Quintin 2013 and Eichberger, Rheinberger, and Summer 2014).[12]

Although an economy with recourse debt is distant from the goal in our paper, it is worth mentioning that there are fundamental differences between the two models (recourse versus non-recourse). First, and most importantly, the shadow value associated with collateral constraints is specific to economies with non-recourse mortgage debt, and is absent if we consider recourse debt instead. Second, in addition to the second date endowments and equity returns, the market portfolio $z^{\tilde{\omega}}$ has to be adjusted with the amount of good depreciated from the first date. There are also differences regarding the impatience parameter ($\tilde{\alpha}_0$) because in our model the equilibrium consumption of one unit of the consumption good at date $t = 0$ influences not only the agent's

[12]Even in the midst of the most severe housing downturn on record, most households in the United States with negative equity chose to continue meeting their financial obligations. This suggests that residential borrowers do incur costs beyond their collateral, in which case the issues formalized in Quintin (2013) become relevant, as shown in Quintin (2012).

utility at date $t = 0$, but also the agent's utility at date $t = 1$. This is because the durable consumption good purchased at date $t = 0$ increases the agent's wealth at date $t = 1$.

3.2 Risk premium

The pricing formula in Proposition 1 allows us to derive an adjusted CAPM formula for the risk premium of a portfolio z^m that generates an income stream m. Before presenting our next proposition, we need to introduce some useful notation.

We define the return per unit of income invested associated with portfolio z^m as $R_m = m/c(m)$, $\forall m \in \langle Y \rangle$, if the price of an income stream m is positive ($c(m) > 0$). Similarly, we define the return of the market portfolio $z^{\tilde{\omega}_1}$ as $R_{\tilde{\omega}_1} = \tilde{\omega}_1 / c_{\bar{q}}(\tilde{\omega}_1)$ when $c_{\bar{q}}(\tilde{\omega}_1) > 0$, where $\tilde{\omega}_1$ is the accumulated revenue generated by the market portfolio $z^{\tilde{\omega}_1}$. The *excess return* of portfolio z^m is the random variable $\mathcal{R} : S \to \mathbb{R}$ defined by $\mathcal{R}_m = R_m - \bar{R}$, where $\bar{R} = (1 + \bar{r})$, where $\bar{r}$ is the risk-free rate. The expected value of $\mathcal{R}_m$, defined by $E(\mathcal{R}_m) = E(R_m) - \bar{R}$, is called the *risk-premium* of portfolio z^m. Similarly, we define the *risk-premium of the market portfolio* as $E(\mathcal{R}_{\tilde{\omega}_1}) = E(R_{\tilde{\omega}_1}) - \bar{R}$.

Proposition 2: *If $(\bar{x}, \bar{z}_{d+}, \bar{z}_{d-}, \bar{z}_a, \bar{q}, \bar{\Phi})$ is an equilibrium for our economy with non-recourse mortgage debt, then the equilibrium risk premium of a portfolio z^m with associated income stream m generated by the matrix of asset returns $\langle Y, \Phi \rangle$ is*

$$E(\mathcal{R}_m) = \underbrace{\beta_{\tilde{\omega}_1 m} E(\mathcal{R}_{\tilde{\omega}_1})}_{\text{classic CAPM risk premium component}} + \underbrace{f(\mu) g(\beta_{\tilde{\omega}_1 m}) z^m_{d-}}_{\text{collateral premium}} \tag{4}$$

where

$$\beta_{\tilde{\omega}_1 m} = \frac{cov(R_{\tilde{\omega}_1}, R_m)}{var(R_{\tilde{\omega}_1})}$$

is the beta coefficient associated with portfolio z^m,

$$f(\mu) = \frac{\bar{R} z^m_{d-} \overline{\mu C}}{\tilde{\alpha}_0 + \mu + z^m_{d-} \overline{\mu C}}$$

and $g(\beta_{\tilde{\omega}_1 m})$ is an increasing function of $\beta_{\tilde{\omega}_1 m}$.

The first term on the right hand side of equation (4) is the same as in the classic CAPM. This

classic CAPM risk premium component is the product of two terms: the portfolio's $\beta_{\tilde{\omega}_1 m}$, and the expected risk premium $E(\mathcal{R}_{\tilde{\omega}_1})$ investors require from the market portfolio. The β-component is often called the *quantity of risk*. It measures the volatility, or *systematic risk*, of portfolio z^m in comparison to the market as a whole. Roughly speaking, $\beta_{\tilde{\omega}_1 m}$ is the sensitivity of the expected risk premium of income stream m to the expected risk premium of the market income stream $\tilde{\omega}_1$. A $\beta_{\tilde{\omega}_1 m}$ below 1 is specific to an income stream m with lower volatility than the market, or a volatile income stream whose price movements are not highly correlated with the market. A $\beta_{\tilde{\omega}_1 m}$ greater than 1 generally means that portfolio z^m both is volatile and has a high correlation with the market portfolio.[13]

The second term of equation (4) is new and is specific to our economy with non-recourse mortgage debt. We refer to this second component as the *collateral premium*. This additional term is zero when either (1) the income stream is generated by a portfolio that does not include a short position on non-recourse mortgage debt (i.e., $z^m_{d-} = 0$), or (2) if $z^m_{d-} > 0$, either (2.1) collateral constraints are non-binding for all agents (i.e., $\mu^i = 0$, so $f(\mu) = 0$), or (2.2) portfolio z^m coincides with the market portfolio[14]. When the collateral premium is null, equation (4) coincides with the classic CAPM formula: $E(\mathcal{R}_m) = \beta_{\tilde{\omega}_1 m} E(\mathcal{R}_{\tilde{\omega}_1})$.

When $z^m_{d-} > 0$, $\mu^i C^i > 0$ for some agents, and $m \neq \tilde{\omega}_1$, the collateral premium is non-zero and we can conclude the following:

Corollary 1: *Binding collateral constraints amplify the impact that systematic risk has on the risk premium of a portfolio z^m that includes non-recourse mortgage debt.*

3.3 Shadow cost of collateral constraints

Finally, we are able to express the agent i's shadow cost μ^i as a function of parameters of our economy.

[13]Notice that $\beta_{\tilde{\omega}_1 m} > 1$ ($\beta_{\tilde{\omega}_1 m} < 1$) implies that $var(m)/var(\tilde{\omega}_1) > 1$ ($var(m)/var(\tilde{\omega}_1) < 1$, repectively).

[14]If $z^m = z^{\tilde{\omega}_1}$, then $c(m) = c(\tilde{\omega}_1)$ and $\beta_{\tilde{\omega}_1 m} = 1$, so $g(\beta_{\tilde{\omega}_1 m}) = 0$ (see the proof of Proposition 2 in the Appendix).

Proposition 3: *If* $(\bar{x}, \bar{z}_{d+}, \bar{z}_{d-}, \bar{z}_a, \bar{q}, \bar{\Phi})$ *is an equilibrium for our economy with non-recourse mortgage debt, then the agent* i*'s shadow cost of its collateral constraint is equal to*

$$\mu^i = \frac{1}{C^i}\left(\underbrace{\sum_{s=1}^{S} \pi_s^i(\Phi_s - Q_s^i)}_{\text{expected mortgage default rate}} + \sigma_{d+}^i + \sigma_{d-}^i\right) \tag{5}$$

where σ_{d+}^i *and* σ_{d-}^i *are the agent* i*'s shadow values corresponding to the sign constraints* $z_{d+}^m \geq 0$ *and* $z_{d-}^m \geq 0$*, and* π_s^i *is the agent* i*'s shadow value of its budget constraint at state* s*.*

In equation (5), $\Phi_s - Q_s^i$ is the difference between the effective mortgage rate paid to the lender and the mortgage payment that borrower i delivers (the minimum between the mortgage promise and the value of the depreciated collateral).[15] Thus, the formula in Proposition 3 expresses agent i's shadow cost of its collateral constraint, μ^i, as a weighted average of future default rates.

The following result follows from Propositions 2 and 3.

Corollary 2: *A positive expected default rate for non-recourse mortgages amplifies the impact that a higher systematic risk has on the risk premium of a portfolio that includes non-recourse mortgage debt.*

4 Conclusions

This paper considers Geanakoplos and Zame's (2014) general equilibrium model with collateral constraints and non-recourse debt and derives explicit pricing conditions for a CAPM economy. Our analysis reveals that there are important differences between our pricing conditions and those derived from the classic CAPM economy without default risk.

First, in the classic CAPM pricing formula of an income stream without credit risk, the price of a portfolio is a decreasing linear function that depends on the covariance between the income stream associated with the portfolio and the aggregated risky returns ω_1 (if ω_1 belongs to the space generated by Y, then it is considered the market portfolio). When we consider non-recourse credit

[15] In the model, we consider the case of personalized collateral (collateral coefficients differ across agents). If we had considered non-personalized collateral (i.e., $C^i = C$, for all i), then $Q_s^i = \Phi_s$ for all i, and, therefore, $\mu^i = \left(\sigma_{d+}^i + \sigma_{d-}^i\right)/C$.

risk, the price of a portfolio has an additional term on the right hand side of the classic pricing component. This new term is an increasing function of the shadow values of collateral constraints, but is preceded by a negative sign in the pricing equation. Thus, we conclude that the more binding are the collateral constraints, the lower is the value of an income stream in equilibrium. This negative impact on the price of an income stream is due to the disutility of constituting additional units of collateral above what would be the optimal consumption in absence of the (binding) collateral constraints. Moreover, in our CAPM pricing formula for an income stream with non-recourse mortgage risk, the accumulated risky returns have to be adjusted with the expected non-recourse mortgage payments deficit and the amount of good depreciated from the first date.

Second, in the classic CAPM formula, the expected risk premium of a portfolio equals the product of a beta coefficient (a proxy for systematic risk) and the market risk premium. We show that when a portfolio is built with a short position on non-recourse mortgage debt, a new component enters into this CAPM equation. We call this component the collateral premium, and show it is an increasing function of both the shadow value of collateral constraints and the beta coefficient. Moreover, because the shadow value of collateral constraints can be written as the weighted average of the future default rates, we can conclude the following: an economy with a positive expected default rate on non-recourse mortgages amplifies the positive impact that a higher systematic risk has on the risk premium of a portfolio built with non-recourse mortgage debt.

This result may explain the failure of the classic CAPM formula for real estate investments, which in general have a strong component of non-recourse mortgage debt. Our formula also applies to any financial portfolio that is built using non-recourse mortgage debt. Of course, empirical work is needed to test our pricing formula. We leave this for future research. Usual precautions should be taken into account when running a multi-period regression because the expected mortgage default rate in the economy might fluctuate over time.

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5 Appendix

This Appendix contains the proofs of Propositions 1, 2 and 3.

Proof of Proposition 1:

Consider the following optimization problem:

$$\min -u(x_0^i + C^i z_{d-}^i \ , \ x_1^i) \ \ \textit{subject to:} \begin{cases} x \in \mathbb{B}^i(q, Q, \Phi) \\ x_0^i \geq 0 \\ z_{d+}^i \geq 0 \\ z_{d-}^i \geq 0 \end{cases}$$

That is, $\min -u(x_0^i + C^i z_{d-}^i \ , \ x_1^i)$ subject to:

$$\begin{cases} x_0^i \leq \omega_0^i - q_d z_{d+}^i + (q_d - C^i) z_{d-}^i - q_a z_a^i \\ x_s^i \leq \omega_s^i + Y_s \delta_s^i + \Phi_s z_{d+}^i - (D_s C^i - Q_s^i) z_{d-}^i + D_s x_0^i + Y_s z_a^i \\ x_0^i \geq 0 \\ z_{d+}^i \geq 0 \\ z_{d-}^i \geq 0 \end{cases}$$

Thus, the Lagrangian for this problem is as follows:

$$\begin{aligned} L^i(x^i, z^i, \pi^i, \mu^i, \sigma_{d+}^i, \sigma_{d-}^i) = & -u^i(x_0^i + C^i z_{d-}^i \ , \ x_1^i) + \pi_0^i(x_0^i - \omega_0^i + q_d z_{d+}^i - (q_d - C^i) z_{d-}^i + q_a z_a^i) + \\ & + \sum_{s=1}^{S} \pi_s^i \left[x_s^i - D_s x_0^i - \omega_s^i - Y \delta^i - \Phi_s z_{d+}^i - (D_s C^i - Q_s^i) z_{d-}^i - Y_s z_a^i \right] \\ & - \mu^i x_0^i - \sigma_{d+}^i z_{d+}^i - \sigma_{d-}^i z_{d-}^i \end{aligned} \tag{6}$$

where $x^i = (x_0^i + C^i z_{d-}^i \ , \ (x_s^i)_{s=1}^S)$, $\pi^i \in \mathbb{R}^{S+1}$, $\mu^i \geq 0$, $\sigma_{d+}^i \geq 0$ and $\sigma_{d-}^i \geq 0$ are the Lagrangian multipliers.

Using the Kuhn-Tucker conditions in the Lagrangian above which are necessary at equili-

brum, we have:

$$\frac{\partial L^i}{\partial x_0^i} = -\frac{\partial u^i}{\partial x_0^i}(\bar{x}^i) + \pi_0^i - \sum_{s=1}^{S} D_s \pi_s^i - \mu^i = 0 \Leftrightarrow \frac{\partial u^i}{\partial x_0^i}(\bar{x}^i) = \pi_0^i - \sum_{s=1}^{S} D_s \pi_s^i - \mu^i \ , \forall \ s = 1, ..., S \tag{7}$$

$$\frac{\partial L^i}{\partial x_s^i} = -\frac{\partial u^i}{\partial x_s^i}(\bar{x}^i) + \pi_s^i = 0 \Leftrightarrow \frac{\partial u^i}{\partial x_s^i}(\bar{x}^i) = \pi_s^i \ , \forall \ s = 1, ..., S \tag{8}$$

$$\frac{\partial L^i}{\partial z_{d+}^i} = q_d \pi_0^i - \sum_{s=1}^{S} \pi_s^i \Phi_s - \sigma_{d+}^i = 0 \Leftrightarrow \sum_{s=1}^{S} \pi_s^i \Phi_s - q_d \pi_0^i = -\sigma_{d+}^i \tag{9}$$

$$\frac{\partial L^i}{\partial z_{d-}^i} = -\frac{\partial u^i}{\partial x_0^i}(\bar{x}^i)\, C^i - \pi_0^i (q_d - C^i) - \sum_{s=1}^{S} \pi_s^i (D_s C^i - Q_s^i) - \sigma_{d-}^i = 0 \Leftrightarrow$$

$$\Leftrightarrow -\left(\pi_0^i - \sum_{s=1}^{S} D_s \pi_s^i - \mu^i\right) C^i - \pi_0^i (q_d - C^i) - \sum_{s=1}^{S} \pi_s^i (D_s C^i - Q_s^i) - \sigma_{d-}^i = 0$$

$$\Leftrightarrow \pi_0^i q_d - \sum_{s=1}^{S} \pi_s^i Q_s^i = -\sigma_{d-}^i + C^i \mu^i \tag{10}$$

$$\frac{\partial L^i}{\partial z_a^i} = q_a \pi_0^i - \sum_{s=1}^{S} \pi_s^i Y_s = 0 \Leftrightarrow -q_a \pi_0^i + \sum_{s=1}^{S} \pi_s^i Y_s = 0 \tag{11}$$

$$\frac{\partial L^i}{\partial \pi_0^i} = x_0^i - w_0^i + q_d z_{d+}^i - (q_d - C^i) z_{d-}^i + q_a z_a^i = 0 \tag{12}$$

$$\frac{\partial L^i}{\partial \pi_s^i} = x_s^i - D_s x_0^i - w_s^i - Y_s \delta^i - \Phi_s z_{d+}^i - (D_s C^i - Q_s^i) z_{d-}^i - Y_s z_a^i = 0 \ , \forall \ s = 1, ..., S \tag{13}$$

Substituting 8 into 7, we get:

$$\frac{\partial u^i}{\partial x_0^i}(\bar{x}^i) = \pi_0^i - \sum_{s=1}^{S} D_s \frac{\partial u^i}{\partial x_s^i}(\bar{x}^i) - \mu^i \Leftrightarrow \pi_0^i = \frac{\partial u^i}{\partial x_0^i}(\bar{x}^i) + \sum_{s=1}^{S} D_s \frac{\partial u^i}{\partial x_s^i}(\bar{x}^i) + \mu^i.$$

From the previous equality and 8, we get:

$$\pi^i = \left(\pi_0^i, \left(\pi_s^i\right)_{s=1}^S\right) = \left(\frac{\partial u^i}{\partial x_0^i}(\bar{x}^i) + \sum_{s=1}^{S} D_s \frac{\partial u^i}{\partial x_s^i}(\bar{x}^i) + \mu^i \ , \ \left(\frac{\partial u^i}{\partial x_s^i}(\bar{x}^i)\right)_{s=1}^S\right).$$

On the other hand, from 11 we get:

$$\left\langle \left(\pi_0^i, \left(\pi_s^i\right)_{s=1}^S\right), \left(-q_a \, , (Y_s)_{s=1}^S\right) \right\rangle = 0,$$

from 9 we get:

$$\left\langle (\pi_0^i, \left(\pi_s^i\right)_{s=1}^S \right), \left(-q_d\,, \left(\Phi_s\right)_{s=1}^S\right)\right\rangle = -\sigma_{d+}^i,$$

and from 10 we get:

$$\left\langle \left(\pi_0^i, \left(\pi_s^i\right)_{s=1}^S \right), \left(q_d\,, \left(-Q_s^i\right)_{s=1}^S\right)\right\rangle = -\sigma_{d-}^i + C^i\mu^i.$$

Therefore, if we consider an income stream m which is attainable in the market by at least an agent i through a portfolio $z^m \in Z$ we conclude that:

$$\left\langle \pi^i, \tau \right\rangle = -\sigma_{d+}^i z_{d+}^m + (C^i\mu^i - \sigma_{d-}^i) z_{d-}^m,$$

where $\tau = (-c(m), m) \in \mathbb{R}^{S+1}$ and $\tau = Tz^m$ with $T = \begin{pmatrix} -q_d & q_d & -q_a \\ R_s & -Q_s^i & Y_s \\ & & \end{pmatrix}$.

By summing across the agents, we have:

$$\left\langle \pi^i, \tau \right\rangle = -\sigma_{d+}^i z_{d+}^m + (C\mu^i - \sigma_{d-}^i) z_{d-}^m \Leftrightarrow \tag{14}$$

$$\Leftrightarrow \left\langle \left(\frac{\partial u^i}{\partial x_0^i}(\bar{x}^i) + \sum_{s=1}^S D_s \frac{\partial u^i}{\partial x_s^i}(\bar{x}^i) + \mu\ ,\ \left(\frac{\partial u^i}{\partial x_s^i}(\bar{x}^i)\right)_{s=1}^S \right), (-c(m), m) \right\rangle = -\sigma_{d+}^i z_{d+}^m + (C^i\mu^i - \sigma_{d-}^i) z_{d-}^m$$

$$\Leftrightarrow \left\langle (\tilde{\alpha}_0 + \mu\,, \mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}})\ , (-c(m), m) \right\rangle = -\sigma_{d+} z_{d+}^m + (\overline{C\mu} - \sigma_{d-}) z_{d-}^m, \tag{15}$$

where $\tilde{\alpha}_0 = \alpha_0 + \sum_{s=1}^S D_s\left(\alpha_1 - D_s\omega_0 - \omega_s - Y_s\delta\right)$, $\sigma_{d+} = \sum_{i=1}^I \sigma_{d+}^i$, $\sigma_{d-} = \sum_{i=1}^I \sigma_{d-}^i$ and $\overline{C\mu} = \sum_{i=1}^I C^i\mu^i$.

Multiplying 15 by $\frac{1}{\tilde{\alpha}_0+\mu}$ we get:

$$\langle 1, -c(m)\rangle + \frac{1}{\tilde{\alpha}_0 + \mu}\left\langle \mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}\,, m\right\rangle = \frac{-\sigma_{d+} z_{d+}^m + (\overline{C\mu} - \sigma_{d-}) z_{d-}^m}{\tilde{\alpha}_0 + \mu}.$$

Thus:

$$c(m) = E\left(\frac{\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}}{\tilde{\alpha}_0 + \mu}\right) E(m) - \frac{1}{\tilde{\alpha}_0 + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) + \frac{\sigma_{d+} z_{d+}^m - (\overline{C\mu} - \sigma_{d-}) z_{d-}^m}{\tilde{\alpha}_0 + \mu}.$$

Proof of Proposition 2:

From Proposition 1 we know that,

$$c(m) = E\left(\frac{\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}}{\tilde{\alpha}_0 + \mu}\right) E(m) - \frac{1}{\tilde{\alpha}_0 + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) + \frac{\sigma_{d+} z_{d+}^m - (\overline{C\mu} - \sigma_{d-}) z_{d-}^m}{\tilde{\alpha}_0 + \mu}$$

or

$$c(m) = E\left(\frac{\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}}{\tilde{\alpha}_0 + \mu}\right) E(m) - \frac{1}{\tilde{\alpha}_0 + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) + K$$

with $K = \frac{\sigma_{d+} z_{d+}^m - (\overline{C\mu} - \sigma_{d-}) z_{d-}^m}{\tilde{\alpha}_0 + \mu}$ to simplify.

If we denote by $\bar{r}$ the risk-free rate then for the free risk income stream $\mathbf{1}$ we obtain

$$1 = E\left(\frac{\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}}{\tilde{\alpha}_0 + \mu}\right)(1 + \bar{r}) + K$$

that is

$$E\left(\frac{\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}}{\tilde{\alpha}_0 + \mu}\right) = \frac{1 - K}{1 + \bar{r}}$$

Using this expression for $E\left(\frac{\mathbf{1}\alpha_1 - \tilde{\omega}_{\mathbf{1}}}{\tilde{\alpha}_0 + \mu}\right)$, we obtain for $c(m)$,

$$c(m) = \frac{1 - K}{1 + \bar{r}} E(m) - \frac{1}{\tilde{\alpha}_0 + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) + K \;\Leftrightarrow$$

$$\Leftrightarrow \frac{E(m)}{c(m)} = \frac{1 + \bar{r}}{1 - K}\left[1 + \frac{1}{c(m)} \frac{1}{\tilde{\alpha} + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) - \frac{K}{c(m)}\right] \;\Leftrightarrow$$

$$\Leftrightarrow \frac{E(m)}{c(m)} - (1 + \bar{r}) = \frac{1 + \bar{r}}{1 - K}\left[K + \frac{1}{c(m)} \frac{1}{\tilde{\alpha} + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) - \frac{K}{c(m)}\right]$$

$$\Leftrightarrow \frac{E(m)}{c(m)} - (1 + \bar{r}) = \frac{1}{1 - K} \frac{1}{c(m)} \frac{1 + \bar{r}}{\tilde{\alpha} + \mu} cov(\tilde{\omega}_{\mathbf{1}}, m) + (1 + \bar{r}) \frac{K}{1 - K}\left(1 - \frac{1}{c(m)}\right)$$

When we write this formula for the market portfolio we obtain,

$$\frac{E(\tilde{\omega}_{\mathbf{1}})}{c(\tilde{\omega}_{\mathbf{1}})} - (1+\bar{r}) = \frac{1}{1-K}\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\frac{1+\bar{r}}{\tilde{\alpha}+\mu}var(\tilde{\omega}_{\mathbf{1}}) + (1+\bar{r})\frac{K}{1-K}\left(1-\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\right)$$

$$\Leftrightarrow \frac{1}{1-K}\frac{1+\bar{r}}{\tilde{\alpha}+\mu} = \frac{c(\tilde{\omega}_{\mathbf{1}})}{var(\tilde{\omega}_{\mathbf{1}})}\left[\frac{E(\tilde{\omega}_{\mathbf{1}})}{c(\tilde{\omega}_{\mathbf{1}})} - (1+\bar{r}) - (1+\bar{r})\frac{K}{1-K}\left(1-\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\right)\right]$$

Then,

$$\frac{E(m)}{c(m)} - (1+\bar{r}) = \frac{cov(\tilde{\omega}_{\mathbf{1}},m)}{c(m)}\frac{c(\tilde{\omega}_{\mathbf{1}})}{var(\tilde{\omega}_{\mathbf{1}})}\left[\frac{E(\tilde{\omega}_{\mathbf{1}})}{c(\tilde{\omega}_{\mathbf{1}})} - (1+\bar{r}) - (1+\bar{r})\frac{K}{1-K}\left(1-\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\right)\right] +$$

$$+(1+\bar{r})\frac{K}{1-K}\left(1-\frac{1}{c(m)}\right)$$

$$\Leftrightarrow \frac{E(m)}{c(m)} - (1+\bar{r}) = \frac{cov(\tilde{\omega}_{\mathbf{1}},m)}{c(m)}\frac{c(\tilde{\omega}_{\mathbf{1}})}{var(\tilde{\omega}_{\mathbf{1}})}\left[\frac{E(\tilde{\omega}_{\mathbf{1}})}{c(\tilde{\omega}_{\mathbf{1}})} - (1+\bar{r})\right] +$$

$$(1+\bar{r})\frac{K}{1-K}\left[\left(1-\frac{1}{c(m)}\right) - \left(1-\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\right)\frac{cov(\tilde{\omega}_{\mathbf{1}},m)}{c(m)}\frac{c(\tilde{\omega}_{\mathbf{1}})}{var(\tilde{\omega}_{\mathbf{1}})}\right]$$

Finally,

$$E(\mathcal{R}_m) = \frac{cov(R_{\tilde{\omega}_{\mathbf{1}}},R_m)}{var(R_{\tilde{\omega}_{\mathbf{1}}}))}E(\mathcal{R}_{\tilde{\omega}_{\mathbf{1}}}) + \frac{K\bar{R}}{1-K}\left[\left(1-\frac{1}{c(m)}\right) - \left(1-\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\right)\frac{cov(R_{\tilde{\omega}_{\mathbf{1}}},R_m)}{var(R_{\tilde{\omega}_{\mathbf{1}}}))}\right]$$

If $\sigma_{d+} = 0$ e $\sigma_{d-} = 0$ then,

$$E(\mathcal{R}_m) = \frac{cov(R_{\tilde{\omega}_{\mathbf{1}}},R_m)}{var(R_{\tilde{\omega}_{\mathbf{1}}}))}E(\mathcal{R}_{\tilde{\omega}_{\mathbf{1}}}) - \frac{\overline{C\mu}\bar{R}z_{d-}^m}{\tilde{\alpha}_0+\mu+\overline{C\mu}z_{d-}^m}\left[\left(1-\frac{1}{c(m)}\right) - \left(1-\frac{1}{c(\tilde{\omega}_{\mathbf{1}})}\right)\frac{cov(R_{\tilde{\omega}_{\mathbf{1}}},R_m)}{var(R_{\tilde{\omega}_{\mathbf{1}}}))}\right],$$

or,

$$E(\mathcal{R}_m) = \beta_{\tilde{\omega}_1 m} E(\mathcal{R}_{\tilde{\omega}_1}) + f(\mu) g(\beta_{\tilde{\omega}_1 m}),$$

where

$$\beta_{\tilde{\omega}_1 m} = \frac{cov(R_{\tilde{\omega}_1}, R_m)}{var(R_{\tilde{\omega}_1})}$$

$$f(\mu) = \frac{\bar{R} z_{d-}^m \overline{\mu C}}{\tilde{\alpha}_0 + \mu + z_{d-}^m \overline{\mu C}}$$

and

$$g(\beta_{\tilde{\omega}_1 m}) = -\left[\left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right) \beta_{\tilde{\omega}_1 m}\right].$$

When $z_{d-}^m > 0$, $\mu > 0$, and $m \neq \tilde{\omega}_1$ (so $\beta_{\tilde{\omega}_1 m} \neq \frac{c(\tilde{\omega}_1)}{c(m)} \frac{1-c(m)}{1-c(\tilde{\omega}_1)}$), the collateral premium is non-zero. Also, notice that the derivative of $g(\beta_{\tilde{\omega}_1 m})$ with respect to $\beta_{\tilde{\omega}_1 m}$ is positive when $c(\tilde{\omega}_1) > 1$, which always holds since $c(\tilde{\omega}_1)$ must be greater than the price of the risk-free asset, which in our model is 1.

Proof of Proposition 3:

By 7 we have,

$$\pi_0^i q_d = \sum_{s=1}^{S} \pi_s^i \Phi_s + \sigma_{d+}^i.$$

By 8 we have,

$$C^i \mu^i = \pi_0^i q_d - \sum_{s=1}^{S} \pi_s^i Q_s^i + \sigma_{d-}^i.$$

Then,

$$C^i \mu^i = \sum_{s=1}^{S} \pi_s^i (\Phi_s - Q_s^i) + \sigma_{d+}^i + \sigma_{d-}^i$$

and the result follows.