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Leverage, Security Bubbles and the Regulation of Repo Markets

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Regulating the repo market: implications for leverage dynamics and bubbles

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Abstract

We examine the impact of regulation on repo, leverage, Ponzi schemes and bubbles. Repo Ponzi schemes can be done by creditors who collect haircut and then reuse the collateral that was pledged to them. In bilateral repo, only dealers have such haircut benefit and regulation consists in limiting dealers' positions. In exchanges, the segregation of the haircut avoids Ponzi schemes. However, as we illustrate, both cases allow for bubbles when agents are not uniformly impatient. We also show that all agents may be worse off if repo markets were absent and that bubbles are robust to the endogenous issuance of the securities.

Keywords: repo; re-hypothecation; broker-dealer; hedge fund; collateral reuse; regulation arrangements to limit leverage; bilateral repo; central repo; leverage dynamics; repo Ponzi schemes; rational bubbles.

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1 Introduction

1.1 Motivation

The crisis that followed Lehman Brothers' bankruptcy and, in particular, the run on funding markets, put in evidence how important repo markets are for the functioning of the financial system. In the aftermath of the crisis, policy makers became interested in knowing how to control excessive build up and also in developing the tools to break vicious circles in the failure of the funding multiplier process through which securities pledged as collateral get re-hypothecated. The Financial Stability Board (FSB), in its 29 August 2013 white paper, defined as a policy goal to limit the amount of re-hypothecation[1]. By re-hypothecation is meant the reuse of any security (including client assets) delivered in one transaction in order to collateralize another transaction. It encompasses both the narrower sense (see Aitken and Singh (2010)) and the broader sense where the collateral is short-sold and used by the buyer to secure another loan.

With a size in excess of 4, 6 and 1 trillion dollars just for USD, EURO and YEN repo markets, respectively, the worldwide repo market size easily exceeds 10 trillion dollars[2]. Still the repo market is smaller than more traditional credit markets. Also, the run on repo was only a small part of the whole funding run in the Lehman crisis - e.g., the run on ABCP was more important -, as was shown recently by Krishnamurthy, Nagel and Orlov (2014). However, understanding the implications of repo is fundamental in the policy arena because the collateral multiplier makes it a quasi-monetary phenomenon. In this paper we explore a few potential levers for regulators and policy makers to influence this multiplier in both directions[3].

It is not our goal in this paper to evaluate the role of repo in the recent financial crisis. Our purpose is instead to examine carefully some repo regulation measures which have been suggested to policy makers. Our analysis has a particular emphasis on the implications for leverage dynamics, rehypothecation, Ponzi schemes and bubbles.

1.2 The Financial Stability Board's proposed limits on rehypothecation

Repo has allowed for substantial leverage, but also for abrupt deleverage when repo markets froze. It all depends on how easily securities can be re-hypothecated but then turned down if they are perceived as being bad collateral or being provided by stressed counter-parties. Reducing the amplitude of such re-hypothecation cycles became a goal of policy makers

[1] On similar statements, see the European System Risk Board Occasional Paper 2, by Bouveret, Jardelot, Keller, Molitor, Theal, and Vital (2013).

[2] See http://www.federalreserve.gov/econresdata/notes/feds-notes/2014/repurchase-agreements-in-the-financial-accounts-of-the-united-states-20140630.html , http://www.icmagroup.org/Regulatory-Policy-and-Market-Practice/short-term-markets/Repo-Markets/repo/latest/, https://www.boj.or.jp/en/research/brp/ron_2013/data/ron130515a.pdf.

[3] An interesting example of such interventions was the non-recourse repo introduced at the peak of the crisis (through TALF). Just like the deposit insurance schemes to fight bank runs, we see here an attempt to maintain the multiplier (see http://www.newyorkfed.org/research/epr/forthcoming/1207ashc.pdf).

and regulators. The FSB's white paper highlighted some requirements to the extent in which re-hypothecation of securities should be limited.

First, if reused assets serve as collateral for financial transactions, they could potentially be subject in the future to proposals on minimum haircuts intended to limit the build-up of excessive leverage.

Second, financial intermediaries, including banks and securities broker-dealers, should be subject to regulatory capital and liquidity requirements.

Third, the FSB emphasized that "more safeguards are needed on re-hypothecation of client assets", and explicitly wrote that: "Clients assets may be re-hypothecated by an intermediary for the purpose of financing client long positions and covering short positions, but they should not be re-hypothecated for the purpose of financing the intermediary's own-account activities".

The present paper analyzes these requirements with a special focus on the long-run effects of re-hypothecation. We use an infinite horizon model since it is only in an open ended set-up that we can properly study leverage dynamics and check whether the re-hypothecation limitations preclude repo Ponzi schemes.

1.3 Ponzi Schemes and Rehypothecation

In the context of securities and repo markets, shorting and issuing are recognizably distinct transactions in real life. Short-selling without having borrowed the security - called "naked" short sale in market parlance - is not allowed but, formally, that is what previous models were contemplating, even though such negative security positions had the flavor of being a primary market issuance of debt. Bottazzi, Luque and Pascoa (2012) build a three-date general equilibrium model where short-sales and re-hypothecation are done as in real life security and repo markets, and examined the implications for security prices. We extend this new set-up to an infinite horizon economy with a focus on the pure trading aspects of securities. Rather than assuming solvency constraints or debt limits, we show that the way security and repo trades are articulated determines whether Ponzi schemes may or may not occur.

We find that a new form of Ponzi scheme, that we call a "haircut Ponzi scheme", is actually possible when repo is present in the economy. Its simplest version can be done by a dealer running a matched repo book. Dealers are exempt from paying haircut but collect haircuts from customers. Then, by borrowing and lending a same amount of the security, the dealer can get a net positive cashflow equal to the value of the haircut. As long as the full matched balance can be rolled over forever, the dealer constructs a positive cash flow that is never paid back. Without limits on dealers positions, the dealer could scale this up and unbounded gains would result.

Another version of the haircut Ponzi scheme could be done by short selling (rather than pledging) the security that was borrowed (accepted as collateral). Any haircut collecting agent could do this second scheme, irrespectively of paying or not haircut.

This paper studies two institutional arrangements that prevent the aforementioned Ponzi schemes by restricting the way a borrowed security can be reused. These two arrangements were already suggested in the FSB white paper. One of the two forms of limited re-hypothecation is based on the current market situation where the positions of dealers with haircut privilege are bounded. We call this the "*constrained dealers case*" (I), and consists on properly controlling the leverage in the economy by constraining the dealers adequately. An attempt to limit directly the leverage of the customers would run the risk of making ineffective any future policy of fighting a multiplier collapse in a run. In fact, the reaction function of non-dealers could be hampered by adding more constraints (trapping them at low leverage ceilings). Moving repo to exchanges, however, preserves options for the policy makers. This is actually the other form of regulation that we contemplate.

The second regulation scenario consists on the provision that the security borrowed through repo cannot be fully shorted or lent. More precisely, at least the haircut portion of the pledged security, paid for with client money, should not be reused. We call this the "*segregated haircut case*" (II). Anecdotal evidence suggests that moving to central clearing and new regulation are pushing repo exchanges towards the market practice of haircut segregation (this change has been specially relevant after the Lehman crisis).[4]

As our Example 1 below illustrates, the haircut alone is not enough to bound leverage (by the inverse of the haircut rate) - this is only true for the customers but not for the dealers themselves. It is also interesting to realize that bounding leverage node by node would not suffice for existence of an infinite horizon equilibrium. Non-arbitrage strategies that yield no gains in finite horizon may become improvement opportunities once extended over time. In the absence of a terminal date, which is a realistic feature of our model, nothing forces the unwinding and the return of the haircut by the dealer. Even more subtle is the observation that the segregated haircut policy could not be replaced by the requirement that the haircut can only be re-pledged but not short sold. In a finite horizon, the two policies have the same optima, but an infinite horizon equilibrium under segregated haircuts may fail to be an equilibrium under a "no shorting of haircut" policy. We illustrate this in Example 4.

1.4 Security bubbles

Keeping track of collateral involved in short sales suggests that Ponzi schemes are less of a problem in an open ended trading secondary market setting than in the primary market or naked short sales model. We see this as a confirmation that preventing naked short sales in secondary markets is a reasonable idea. Existence of equilibrium can be established under assumptions on preferences and endowments that are milder than those of models with "naked" and unsecured short sales. This is natural as the only money one can possibly raise in the secondary market is through haircut and this can only be done by dealers (or cannot be done at all in exchanges segregating the haircut). These milder assumptions of our model turn out to be less hostile to the occurrence of bubbles: we find bubbles when markets are incomplete and agents are not uniformly impatient.

While we focus on bubbles for securities whose net supply is positive and fixed, we also argue, in Example 3, that, depending on how trading and issuance are separated, the bubble can be robust to issuance. Our setting has issuance constrained by the present value of future endowments, while trading is just constrained by one of our limited re-hypothecation

[4]It is also interesting to observe that in the case of derivatives cleared on exchange the initial margins (the equivalent to haircuts) are segregated.

arrangements.

We refer to a bubble with shorting but fixed issuance as a "*trading bubble*" or a "*secondary markets bubble*". A "*primary market bubble*" refers to a speculative outcome when all or some agents have the freedom to issue the security. Note that trading Ponzi schemes are more difficult to construct when issuance is not possible, since raising money by means of trading alone (i.e. mostly through haircuts) is a more difficult strategy, actually impossible under the above regulation mechanisms.

We illustrate the incomplete markets trading bubble (in Examples 1 and 2) for each of the two forms of limited re-hypothecation regulation discussed above. In these examples, consumers are impatient, although not uniformly, and can do short-sales[5]. Consumers' deflators are given by their own marginal rates of substitution and yield a finite present value of wealth.

These are examples where there are no positive shadow prices for the constraint that keeps track of security positions, here referred to as the "box constraint"[6], i.e., the security is not on special (its repo rate is not below the General Collateral rate)[7].

To show the relevance of repo, we provide another example of a speculative equilibrium (Example 4), where box shadow prices are positive and show that such possession value for the security makes repo improve upon what consumers would achieve in the absence of repo and without any leverage.

It is important to understand how our work differs from general equilibrium models where "short sales" are collateralized by financial or physical assets - see, for example, Geanakoplos and Zame (2014) for a finite horizon economy, and Kubler and Schmedders (2003) and Fostel and Geanakoplos (2008) for the case of infinite horizon economies. Whereas these models capture money promises (of the type "i.o.u.") backed by securities, we model the shorting of the collateral, that is, the shorting of the securities themselves. And we model it as it is actually done in the markets: a short sale is a sale of a security that one does not own by borrowing it first through repo. The most important consequence of this new approach is that the security that serves as collateral can be now re-hypothecated infinitely many times.

Observe that, either in finite or in infinite horizon, existence of repo equilibrium is a more complicated issue and does not follow from any known results of models where money promises are collateralized by assets. Repo equilibrium requires instead an explicit regulation bounding re-hypothecation.

To make the comparison with previous models clearer, first notice that in finite horizon, the available amount of (financial or durable goods) collateral bounded short positions of money promises and that sufficed for existence of equilibrium. However, in a repo model, security short sales cannot be trivially bounded in that way, since the quantity that a trader borrows to short sell may have been lent by someone who got the security either by

[5]In Santos and Woodford (1997) and Páscoa, Petrassi and Torres-Martinez (2011), there were examples of incomplete market bubbles for non-uniformly impatient consumers but short sales were not allowed.

[6]The box constraint guarantees that the amount of titles collected of each security through repo or security trading is non-negative. The box constraint says that a short-sale must be backed by adequate borrowing of the security and that when the security is lent one must have an adequate long position.

[7]Duffie (1996) was the first to incorporate in an equilibrium model constraints that keep track of the security and did an extensive analysis of specialness.

borrowing it or by buying it from a short seller: the collateral is recycled. In an infinite horizon economy, on the other hand, Ponzi schemes could not be done with money promises backed by (physical or financial) collateral, since collateral costs had to exceed the promise price (see Araujo, Páscoa and Torres-Martinez (2002)), due to a non-arbitrage condition.[8] In a repo model, on the contrary, an agent that collects haircut can engage in a Ponzi scheme, in the absence of other constraints.

1.5 Why hold on to a bubble?

What is the basic intuition behind our results on bubbles and our examples? Why don't impatient agents sell the bubble as soon as they perceive it? The reason has to do with non-uniform impatience combined with market incompleteness (the diversity of agents' marginal rates of substitution across nodes of the event tree). All agents agree that the security has a bubble but they may want to sell it at different moments in time, and may even buy it again later on. But for this to occur in equilibrium, it must be the case that when some agents sell the security, others buy the security with the purpose of selling it later at a more adverse node. Thus, market incompleteness plays also a role: idiosyncratic endowment shocks and different beliefs on nodes imply the diversity of consumers' deflators.

Short selling can prevent the above bubbled outcome. In a conventional setting, without repo markets, there might be no reason to hold on to the bubble if a future adverse situation can always be dealt with by short selling at the time the adversity occurs. In an economy with repo markets, short selling might also hamper the bubble, if securities were shorted at a price higher than the haircutted price at which they were borrowed. Under the two regulatory arrangements that bound re-hypothecation we show that this can not happen: in case (I) non-dealers do not collect haircut while dealers have their positions bounded by regulation; in case (II) every agent can only short sell the non-haircut portion.

The main message of the paper is that the regulation arrangements or the tendency to move to central clearing in repo will succeed in ensuring a long-term equilibrium, by ruling out repo Ponzi schemes, but bubbles may occur and then burst, as we illustrate by means of examples. Moreover, even if the incomplete markets bubble found here is inefficient, it is not clear that everyone would be better off without it. In fact, our examples suggest the opposite. Nevertheless, policy makers or regulators may want to reduce leverage in the securities market and, therefore, reduce the size of the bubble, by increasing the haircut $1 - h$ or by decreasing the percentage H (possibly below h) of the collateral that can be re-hypothecated. Booms and busts can be soothed over the cycle by managing haircuts or dealers' constraints counter-cyclically. It stands to reason the relevance for monetary authorities of these tools as alternative instruments for macro prudential policies in an international context with interest rates close to the zero bound.

1.6 Structure of the Paper

The next sections are organized as follows. Section 2 starts with a brief introduction to the repo market and its aking market structure, the *securities lending market*. Section 3

[8] It is this non-arbitrage relation rather than the scarcity of collateral what ruled out Ponzi schemes - see Páscoa and Seghir (2009) for the case of utility penalties coupled with collateral.

addresses repo equilibrium and regulation when there are dealers, and examines the long-term implications for the occurrence of Ponzi schemes and bubbles. Section 4 deals with the repo exchanges and also the long-term implications.

We study several important issues as we discuss these two cases. First, we examine the busting of bubbles and the associated reduction in real leverage, in the context of bilateral repo between dealers and non-dealers, within Section 3. Then, in Section 5, we compare the exchanges case with the standard collateral model not allowing for re-hypothecation. In Section 6 we discuss the robustness of repo bubbles to endogenous issuance. Section 7 illustrates why repo constitutes a Pareto improvement: it never hurts agents and will actually, in general, make some (possibly all) agents better off.

2 Repo Roadmap

Repo markets have been an integral part of the financial economy for almost one century in the US alone. For example, the recent history of mortgage securitization is full of examples where traditional banks and conduit lenders have recurrently financed the purchase of large pools of mortgages by putting their (agency or non-agency) issued mortgage-based securities (MBS) as collateral in a repo. Gordon and Metrick (2012) coined such activity as "securitization banking". But repo goes beyond MBS, as it encompases the whole spectre of securities.

A repurchase agreement, or repo, consists in a security sale under the agreement of a future repurchase at a predetermined date and price. Thus, repo is a collateralized loan, where the security serves as collateral for a cash loan. The repo rate is the interest rate implicitly earned by the lender of cash, given the difference between the repurchase and the original prices. The collateral value usually exceeds the cash lent, the excess value is called the *haircut*. The original justification for the haircut is to compensate for the risk of collateral depreciation in case of default. But notice that the level of haircut influences leverage, and, therefore, for systemic policy it is reasonable to aim to adjust it counter-cyclically rather than to reflect that pro-cyclical risk. For previous work on haircut spirals that follow funding shocks, see Adrian and Shin (2009) and Brunnermeier and Pedersen (2009).

Repo can also be seen as a way to borrow or lend securities: the security that serves as collateral is lent to the creditor, who can keep the security in his balance-sheet, lend it further, or short-sell it. That is, repo allows traders to sell a security they do not own (i.e., "short selling"), by borrowing it first through repo. Then, by iterating the strategies of borrowing the security and short-selling it, the market builds leverage for the same settlement day. For the haircut paying customers of constrained dealers, this collateral multiplier, which shows of how many times the positions outstrip physical available collateral, gets tightly bounded from above as the haircut increases. The work of Bottazzi, Luque and Páscoa (2012) pointed out that, even when trading in a single security, there is a repo collateral multiplier, which is given by the inverse of the haircut. With this result in mind, it becomes reasonable that, in the face of a multiplier collapse, one may want to cap the increase in haircuts.

An akin market structure is the *securities lending market* (SLM) where securities can be borrowed against a collateral. In the U.S. the collateral is usually cash, whereas in the European SLM other securities tend to be pledged as collateral. In the case of cash collateral, the *lending fee* paid by the security borrower is the difference between the interest rate that the security lender earns by reinvesting the collateral and the *rebate rate* paid back to the security borrower. In SLM, the lender of the security always asks for an initial margin.

Repo and SLM are two forms of security financing transactions (SLT), where securities are used to borrow cash or vice-versa. In both forms, ownership of the security is temporarily exchanged and the pledger of the collateral may have to pay his counterparty an initial margin in the first leg of the transaction. The counterparty that is temporarily in possession of the security can re-use it. The market price of such reuse services measures how much the repo rate or the rebate rate are below the money markets interest rate.

Finally, notice that while bonds have been the preferred securities in repo, the SLM is dominant in the case of equity in two market segments: first, in the segment where beneficial owners (or their lending agents) trade bilaterally with prime broker-dealers, and second, in the segment where the these two intermediaries trade bilaterally with hedge funds. However, notice that, among dealers or between dealers and real money agents (e.g., commercial banks and money market funds), the dominant form of the SLT for equity is repo, traded either bilaterally or through central clearing (see the FSB (2012) report for details). In this paper we focus on repo but will also comment on how our results extend to the SLM.

3 Bilateral Repo Equilibrium and Regulation

3.1 Repo with dealers and non-dealers

We consider a discrete time infinite horizon economy where the set of dates is $T = \{0, 1, ...\}$. Date 0 is free of uncertainty. However, at following dates different states of nature may affect preferences, endowments and security returns. A node occurring at date t is specified by a history of state realizations up to that date, $\bar{s}_t = (s_0, ..., s_t)$. The set of nodes, also referred as **the event-tree**, is D.

The subtree with root ξ is $D(\xi) = \{\mu \in D : \mu \geq \xi\}$ and the set of nodes with date T in $D(\xi)$ is denoted by $D_T(\xi)$. Usual notation for successors and predecessors applies.[9] The unique predecessor of ξ is ξ^- .

There is a finite set $\mathbf{I}$ of *infinite lived agents* and there are L commodities that can be consumed and traded at every node. Each agent $i \in \mathbf{I}$ has commodity endowments $\omega^i \in \mathbb{R}^{L\times D}_{++}$ and an utility function, $U^i : \mathbb{R}^{L\times D}_{+} \to \mathbb{R}_+ \cup \{+\infty\}$.

Agents trade J infinite lived *securities*. In general, we conceive the initial node ξ_0 as a situation where issuance has already happened and issued securities have been placed. Thus, each agent i has initial endowments of securities $e^i \in \mathbb{R}\mathbb{R}^J_{++}$, describing his holdings when trading starts. Securities are traded at every node in the event-tree. We denote a *trade* in security j at node $\xi \in D$ by $y^i_{j\xi}$. Agent i's security j *position* at node ξ is $\varphi^i_{j\xi}$.

[9] $\mu = \bar{s}_{t'}$ is a successor of $\xi = \bar{s}_t$, i.e., $\mu \geq \xi$, if $t' \geq t$ and $\bar{s}_{t'} = (\bar{s}_t, s, ...)$. We denote by ξ^+ the set of immediate successors of node ξ.

At the initial node ξ_0 the position is $\varphi^i_{j\xi_0} = e^i_j + y^i_{j\xi_0}$. For node $\xi > \xi_0$, the corresponding position is $\varphi^i_{j\xi} = \varphi^i_{j\xi^-} + y^i_{j\xi}$. A *short sale* at node ξ occurs when $\varphi^i_{j\xi} < 0$.

We model securities as real assets. The real proceeds of security j at node $\xi > \xi_0$ are given by a non-zero vector $B_{j\xi} \in \mathbb{R}^L_+$. Moreover, we assume that each good has at least some security paying in that good[10]. Formally, we assume that for any $\xi \in D$, the real returns matrix B_ξ of type $L \times J$ does not have null rows. Given spot prices at node $\xi \in D$, $p_\xi \in \mathbb{R}^L_+$, the nominal return of security j is then given by $p_\xi B_{j\xi}$. By taking into account security proceeds, we can write the total physical resources at node ξ as $\Omega_\xi = \sum_i \omega^i_\xi + \sum_j B_{j\xi} \sum_i e^i_j$. Security prices are denoted by $q_{j\xi}$.

Nominal securities can also be accommodated. Say that security j pays $A_{j\xi}$ units of account in each node ξ, we let $B_{1j\xi} = \frac{A_{j\xi}}{p_{1\xi}}$ and $B_{kj\xi} = 0$ for $k \neq 1$, so that $p_\xi B_{j\xi} = A_{j\xi}$. The assumptions we will make on preferences ensure $p_{1\xi} > 0$ in equilibrium. For this trick to work, when security j is in positive net supply, we need to close the model by explaining how $A_{j\xi} \sum_i e^i_j$ is being generated, possibly from the issuer's physical resources. We illustrate this in Examples 1 and 3. It is important to accommodate this case, given that fixed income securities tend to be commonly used in repo markets.

Finally, let us describe the *repo trades.* An agent is said to be *long* in repo in security j at ξ if he borrows an amount $\theta_{j\xi} > 0$ of this security in exchange for giving a cash loan. In other words, $\theta_{j\xi} > 0$ is the amount of the security that was accepted as collateral for a repo cash loan. On the other hand, when the agent is the borrower of cash (the lender of the security), with a repo trade $\psi_{j\xi} > 0$, we say he is *short* in repo. We consider over-night repos, where the repurchase of a repo contract signed at node ξ occurs at the following date $t(\xi) + 1$ at some, usually higher, repurchase price and the repo rate (or interest rate) on a repo loan at node ξ captures that variation.

We allow for an *haircut* in a repo on security j, here denoted by $(1 - h_{j\xi}) \in [0, 1]$. The haircut is exogenously given and may be imposed to compensate the lender of funds for the risk associated with a simultaneous default and adverse market move of the security lent[11]. When the lender of funds collects an haircut, the loan associated with the collateral $\theta_{j\xi}$ becomes $h_{j\xi} q_{j\xi} \theta_{j\xi}$, where $h_{j\xi} q_{j\xi}$ is the "haircutted" price of the loan signed at node ξ.

Evidence from repo markets suggests that not all agents get the privilege of collecting a haircut when borrowing a security, i.e., receiving a security as collateral that is worth more than the loan value. Dealers and prime brokers, whose business is intermediation, have this cash benefit, but in practice have their positions bounded in value by regulation through mechanisms that often are, among other things, BIS ratios limits. Their customers (e.g., hedge funds, mutual funds, retail securities brokers, private banks and insurance companies) do not face such regulation on their positions, but must pay haircut when lending securities. We refer to the former set of agents with cash benefits as *dealers* (**D**) and to the latter as *non-dealers* (**ND**). As it usually occurs, we assume that non-dealers only engage in repo with dealers.

The asymmetry in haircut treatment implies that the interest rates in repo may depend

[10] For example, if there is a forward term contract for each good.

[11] For the sake of simplicity and following typical market practice, we assume the haircut that may affect repos depends on the security and the node $\xi \in D$ but does not depend on the credit of the trading entities. This can and should be relaxed if we were to focus more on default risk- something we do not go into here.

on whether the lender of funds is a dealer or a non-dealer[12]. When a dealer (non-dealer) is long (short) in repo the interest rate that applies is $R_{1\xi}$, while if the dealer (non-dealer) is short (long) in repo the interest rate that applies is $R_{2\xi}$. Let $r_{1\xi} = 1+R_{1\xi}$ and $r_{2\xi} = 1+R_{2\xi}$ be the ratios of repurchase prices to purchase prices. Notice that, just by looking at the budget constraint of each node, dealers would like to take both long and short positions, whereas non-dealers would rather have either a long or a short position. However, regulation prevents dealers from taking extreme long and short positions.

Non-dealer $i \in \mathbf{ND}$ has budget constraints at nodes ξ_0 and $\xi > \xi_0$ given, respectively, by

$$p_{\xi_0}(x_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\varphi_{\xi_0} - e^i + \theta_{\xi_0} - h_{\xi_0}\psi_{\xi_0}) \leq 0 \tag{1}$$

$$p_\xi(x_\xi - \omega^i_\xi) + q_\xi(\varphi_\xi - \varphi_{\xi^-} + \theta_\xi - h_\xi\psi_\xi) - q_{\xi^-}(r_{2\xi^-}\theta_{\xi^-} - r_{1\xi^-}h_{\xi^-}\psi_{\xi^-}) - p_\xi B_\xi \varphi_{\xi^-} \leq 0 \tag{2}$$

The non-dealer cannot borrow against the whole value of the collateral: when borrowing money a non-dealer has to set aside a haircut $q_{j\xi}(1 - h_{j\xi})\psi_{j\xi}$.

Dealer $i \in \mathbf{D}$ has budget constraints at nodes ξ_0 and $\xi > \xi_0$ given[13], respectively, by

$$p_{\xi_0}(x_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\varphi_{\xi_0} - e^i + h_{\xi_0}\theta_{\xi_0} - \psi_{\xi_0}) \leq 0 \tag{3}$$

$$p_\xi(x_\xi - \omega^i_\xi) + q_\xi(\varphi_\xi - \varphi_{\xi^-} + h_\xi\theta_\xi - \psi_\xi) - q_{\xi^-}(r_{1\xi^-}h_{\xi^-}\theta_{\xi^-} - r_{2\xi^-}\psi_{\xi^-}) - p_\xi B_\xi \varphi_{\xi^-} \leq 0 \tag{4}$$

The haircut is a benefit for the dealer who gets to reverse in collateral while only posting the value of the collateral minus haircut. Just by looking at the budget constraint of each node, dealers would like to take both long and short positions, whereas non-dealers would rather have either a long or a short position (but may end up mixing as well once the repo settlement at next nodes is taken into account).

In addition, there is a constraint that keeps track of the security quantities. At each node ξ, the amount of each security j that an agent holds must be non-negative, which means, on one hand, that what is pledged as collateral cannot be greater than the security long position and, on the other hand, that what is short sold cannot be greater than the repo net long position. This is *the box constraint* :

$$\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi} \geq 0 \tag{5}$$

This constraint allows for the full reuse of the collateral held by a creditor in a repo loan. It can be entirely re-hypothecated in other repo loans (so that the net repo position $\theta_{j\xi} - \psi_{j\xi}$ becomes zero) or short-sold (so that $0 > \varphi_{j\xi} = \psi_{j\xi} - \theta_{j\xi}$).

[12] It is also this asymmetry that forces us to use different variables for security borrowing and lending, denoted by θ_ξ and ψ_ξ, respectively, subject to non-negativity constraints.

[13] For notation brevity, when two vectors $a = (a_1, ..., a_N)$ and $b = (b_1, ..., b_N)$, with the same dimension, appear multiplied, ab, we mean the vector $a\Box b = (a_1b_1, ..., a_Nb_N)$.

3.2 Repo Ponzi schemes

In the absence of institutional arrangements that would limit the benefit from the reuse of the collateral, dealers could do Ponzi schemes, resulting in a new type of infinite horizon arbitrage combining security and repo positions. In fact, as a dealer collects haircut, the value of the collateral that he accepted in a repo loan exceeds the value of the loan he gave. A dealer that is long in repo can reuse the collateral at the security price $q_{\xi,j}$, whereas this collateral was obtained by just delivering the haircutted $h_{\xi,j}q_{\xi,j}$ amount of cash.

Let us construct a Ponzi scheme. To simplify things, we consider a deterministic economy, although the argument extends to stochastic economies in a straightforward way. Given any plan $(x, \varphi, \theta, \psi)$ satisfying budget and box constraints for a dealer i, we reduce at some date t the security position φ_t and match this by increasing the repo long position θ_t in the same amount, with a free cash flow due to the haircut, and then repeat this procedure at all following dates, but possibly with different amounts in order to accommodate the changes in debt and dividends. Such variation can then be scaled up arbitrarily.

To be more precise, we consider the following variation: at date t, the repo long position θ_t is increased by $\varepsilon_t > 0$ and the security position φ_t is decreased by ε_t. The box constraint remains satisfied at date t, but this joint operation results in a gain $q_t(1-h_t)\varepsilon_t$ at date t that can be spent on extra consumption. At the following date $t+1$ the dealer can accommodate the variation in dividends and debt (net of the settlement of the repo variation) by finding $\varepsilon_{t+1} > 0$, so that θ_{t+1} increases by ε_{t+1} and φ_{t+1} decreases by ε_{t+1}, while preserving at the same time the budget constraint at date $t+1$. That is, ε_{t+1} must satisfy

$$q_{t+1}(1 - h_{t+1})\varepsilon_{t+1} - (q_{t+1} + p_{t+1}B_{t+1})\varepsilon_t + q_t h_t r_{1t}\varepsilon_t \geq 0$$

Hence, we must have

$$\varepsilon_{t+1} \geq \frac{(q_{t+1} + p_{t+1}B_{t+1} - q_t h_t r_{1t})\varepsilon_t}{q_{t+1}(1 - h_{t+1})}$$

This condition is trivially satisfied by any positive ε_{t+1} when the numerator on the right hand side of the inequality is negative. If the numerator is positive, then $\varepsilon_{t+1} > 0$ must be large enough. We then just need to repeat the procedure at the following dates and obtain a vector of increments $(\varepsilon_t, \varepsilon_{t+1}, ...)$ that determines an increase in the utility of dealer i. By multiplying this vector by an arbitrarily large scalar $\alpha > 0$ we get unbounded gains. As non-dealer does not collect haircut, he could not do this Ponzi scheme.

However if everybody was the same, paying haircut when borrowing cash and receiving it when lending cash against collateral, the above Ponzi scheme could still be done, in the absence of other constraints. This is why, as we will see later, exchanges should not pass on the full haircut the agents if they want to be universally used.

A *second type of dealer's Ponzi scheme* could be done by pledging the increase ε_t in the repo long position. That is, both ψ_t and θ_t are increased in the amount ε_t. The dealer gets also a cash flow since he collects but does not pay haircut. The variation should be such that $\varepsilon_{t+1} \geq \frac{q_t\varepsilon_t(r_{2t}-h_t r_{1t})}{q_{t+1}(1-h_{t+1})}$, being trivially satisfied for $r_{2t} < h_t r_{1t}$ and otherwise demanding ε_{t+1} large enough. This second arbitrage is only accessible to dealers and would not be compatible with a symmetric treatment of agents. Dealers simply run a matched repo book forever and collect the haircut. In the next subsection we consider a regulatory arrangement that precludes dealers from doing any of the above Ponzi schemes.

3.3 Equilibrium with Constrained Dealers

Before we characterize optimal plans for dealers and non-dealers, we need to introduce some standard assumptions. Denote by $\mathcal{D}$ the differential operator.

(A1) *The utility function of each agent i is separable in time and states of nature, i.e., for any $x \in \mathbb{R}^{L\times D}_{++}$, $U^i(x) \equiv \sum_{\xi\in D} u^i_\xi(x_\xi)$. At any node ξ, u^i_ξ is twice continuously differentiable and strictly increasing, $u^i_\xi(0) = 0$ and $\mathcal{D}u^i_\xi(x_\xi) \in \mathbb{R}^L_{++}$, $\forall x_\xi \in \mathbb{R}^L_+$.*

(A2) *Commodity endowments are uniformly bounded away from zero: for any $i \in I$, $\omega^i \ggg 0$.*

We say that a plan $(x^i, \varphi^i, \theta^i, \psi^i)$ is optimal for non-dealer i at prices (p, q, r_1, r_2) if it maximizes U^i subject to (1), (2) and (5). Next we characterize individual optimality. Euler conditions are derived in a standard way in Kuhn-Tucker form and are reported after Lemma A.1 in the Appendix. More interesting is the *transversality condition*. Recall that in the absence of repo the transversality condition said that (in terms of deflated average values) the agent should not be a creditor at infinity, formally $\liminf_{T\to\infty} \sum_{\xi:t(\xi)=T} \lambda^i_\xi q_\xi \varphi^i_\xi \le 0$, for the Kuhn-Tucker budget multipliers λ^i_ξ. With repo, the *transversality condition* of a non-dealer requires $\limsup_{T\to\infty} \sum_{\xi:t(\xi)=T} \lambda^i_\xi q_\xi(\varphi^i_\xi + \theta^i_\xi - h_\xi \psi^i_\xi) \le 0$. It says that, in the limit, the non-financed long security position constitutes a waste and the non-reused part of the long repo position is also a waste.

Lemma 1.I(n-d) (**Optimality**): *Suppose assumptions (A1) and (A2) hold. A plan $a^i \equiv (x^i, \varphi^i, \theta^i, \psi^i)$ verifying constraints (1), (2) and (5) is optimal for non-dealer i at prices (p, q, r_1, r_2) if and, when $U^i(x^i) < \infty$, only if also, there are budget multipliers $\lambda^i_\xi > 0$ and box multipliers $\mu^i_\xi \ge 0$ at all nodes, for which $\mu^i_{j\xi}[\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi}] = 0$, satisfying the Euler and transversality conditions.*

For a proof see the Appendix. Necessity is not surprising and extends to repo markets well known results in the literature. Sufficiency is more innovative. In fact, in the absence of repo markets, Euler and transversality conditions were necessary but not sufficient: a plan satisfying Euler and transversality was optimal only among plans satisfying an inequality that was the converse of transversality. That constraint stated that no one should be a debtor at infinity, formally $\liminf_{T\to\infty} \sum_{\xi:t(\xi)=T} \lambda^i_\xi q_\xi \varphi^i_\xi \ge 0$. Now, with repo markets and in the case of a non-dealer, the converse to the transversality condition is that, for each security j, $\liminf_{T\to\infty} \sum_{\xi:t(\xi)=T} [\lambda^i_\xi q_{j\xi}(\varphi_{j\xi} + \theta_{j\xi} - h_\xi \psi_{j\xi}) - \mu^i_{j\xi}(\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi})] \ge 0$, which always holds for any plan satisfying the budget and box constraints of a non-dealer.

We assume that dealers' borrowing and lending of securities are bounded by regulation in the following way (refered to as case (I) of regulation):

(A3) *The real values of dealer i's long repo positions have a uniform bound and the nominal values of his short repo positions are uniformly bounded, i.e., for each j,*

$$\frac{q_{j\xi}}{\sum_l p_{l\xi}} \theta_{j\xi} - M_j \le 0 \tag{6}$$

$$q_{j\xi} \psi_{j\xi} - N_j \le 0 \tag{7}$$

This policy does not suffer from monetary illusion as the regulator prevents real values of long repo positions from exploding . Also notice that (A3) implies that the real value of dealers' short sales is also uniformly bounded. In fact, the box constraint together with (A3) imply $(q_{j\xi}/\sum_l p_{l\xi})\varphi_{j\xi} \geq -M_j$. The weaker version also holds: nominal values of security borrowing and short-sales are clearly (uniformly) bounded as (p_ξ, q_ξ) can be normalized to be in the simplex. This implies that *feasible* security and repo positions have bounded nominal values, which suffices for existence of equilibrium in finite horizon.

For a dealer, we say that $(x^i, \varphi^i, \theta^i, \psi^i)$ is optimal for dealer i at prices (p, q, r_1, r_2) if it maximizes U^i subject to (3), (4), (5) and the dealer's bounds on repo. The characterization is analogous: necessity and sufficiency also hold once we add for each node and each security multipliers $(c^i_{j\xi}, k^i_{j\xi})$ which are null when the respective dealer's bounds M_j or $N_{j\xi}$ are not attained. Euler conditions are also addressed in the Appendix and the *transversality condition* of a dealer requires $\limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}\lambda^i_\xi q_\xi(\varphi^i_\xi + h_\xi\theta^i_\xi - \psi^i_\xi) \leq 0$. This transversality condition says that, in the limit, a long security position that is not entirely funded or a collateral value that is not entirely re-invested (lent or short-sold) are both a waste.

Lemma 1.I(d) (**Optimality**): *Suppose assumptions (A1), (A2) and (A3) hold. A plan* $a^i \equiv (x^i, \varphi^i, \theta^i, \psi^i)$ *verifying constraints (3), (4), (5), (6) and (7) is optimal for dealer* i *at prices* (p, q, r_1, r_2) *if and, when* $U^i(x^i) < \infty$*, only if also, there exist multipliers* $(\lambda^i_\xi, \mu^i_\xi, c_{j\xi}, k_{j\xi}) \geq 0$ *for which* $\mu^i_{j\xi}[\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi}] = 0$, $c_{j\xi}(\frac{q_{j\xi}}{\sum_l p_{l\xi}}\theta_{j\xi} - M_j) = 0$, $k_{j\xi}(q_{j\xi}\psi_{j\xi} - N_\xi) = 0$ *and the Euler and transversality conditions are satisfied.*

For proof see the Appendix. In the case of a dealer, Euler and transversality conditions are sufficient for optimality since $\liminf_{T\to\infty}\sum_{\xi:t(\xi)=T}[\lambda^i_\xi q_{j\xi}(\varphi_{j\xi} + h_{j\xi}\theta_{j\xi} - \psi_{j\xi}) - \mu^i_{j\xi}(\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi}) + c^i_{j\xi}\frac{q_{j\xi}}{\sum_l p_{l\xi}}\theta_{j\xi} + k^i_{j\xi}q_{j\xi}\psi_{j\xi}] \geq 0$ always holds for any plan satisfying the constraints. However, contrary to what happened in the case of a non-dealer, combining box constraints with the way haircut is paid and collected is not enough and we need to use the fact that dealers' repo positions are bounded according to constraints (6) and (7) (see the Appendix, proof of Lemma A.3).

Definition 1: *An equilibrium is a process of prices* $(p, q, r_1, r_2) \in \mathbb{R}^{L\times D}_+ \times \mathbb{R}^{J\times D}_+ \times \mathbb{R}^{J\times D}_+$ *together with individual plans* $(\bar{a}^i)_{i\in\,\mathbf{I}}$*, such that, (i) for each agent* $i \in \mathbf{I}$*, the plan* $\bar{a}^i$ *is optimal at prices* (p, q, r_1, r_2)*, and (ii) at any node* $\xi \in D$*, commodity, security and repo markets clear, i.e.* $\sum_{i\in\mathbf{I}}\bar{x}^i_\xi - \Omega_\xi = 0$, $\sum_{i\in\mathbf{I}}(\bar{\varphi}^i_\xi - e^i) = 0$ *and* $\sum_{i\in\mathbf{ND}}\bar{\theta}^i_\xi = \sum_{i\in\mathbf{D}}\bar{\psi}^i_\xi$ *and* $\sum_{i\in\mathbf{D}}\bar{\theta}^i_\xi = \sum_{i\in\mathbf{ND}}\bar{\psi}^i_\xi$.

For our existence result we will also assume the utility taking a finite value at aggregate resources and that indifference sets do not touch the axis and that utility is differentiably strictly concave. However, in all examples, we work with economies with linear utilities, for which it is easier to compute the equilibrium, and in our last example we don't require the utility to be finite at the unbounded aggregate resources.

(A1') $U^i(\Omega_\xi) < +\infty$.

(A1") $\forall c \in \mathbb{R}$, *the set* $[u^i_\xi]^{-1}(c)$ *is closed in* $\mathbb{R}^L_{++}$*, and* $\forall b \in \mathbb{R}^L_{++}$, $h' \cdot \mathcal{D}^2 u^i_\xi(b) \cdot h < 0$, $\forall h \neq 0$.

Using the above characterization of optimality we establish the following existence result (for the proof see the Appendix):

Theorem 1: *Equilibrium exists, in the constrained dealers case under (A1), (A1'), (A1"), (A2) and (A3).*

3.4 Bubbles in equilibrium

Using Euler conditions on security positions, recursively, for any non-dealer, we obtain the following pricing formula for security j at node η

$$q_{j\eta} = \overbrace{\underbrace{\sum_{\xi>\eta} \frac{\lambda^i_\xi}{\lambda^i_\eta} p_\xi B_{j\xi}}_{\text{discounted dividends}} + \underbrace{\sum_{\xi\geq\eta} \frac{\mu^i_{j\xi}}{\lambda^i_\eta}}_{\text{specialness}}}^{\text{fundamental security value}} + \underbrace{\frac{1}{\lambda^i_\eta} \lim_T \sum_{\xi\geq\eta : t(\xi)=T} \lambda^i_\xi q_\xi}_{\text{bubble}} \tag{8}$$

where the sum of the two series is the fundamental value and the last term is the bubble.

A positive shadow price of the box constraint at a node ξ reveals a possession value for the security which is related to the security being on special. Possession value implies that the repo rate is below an hypothetical risk free rate, if it would exist. Specialness means that the repo rate is below the General Collateral (GC) rate, the prevailing repo market interest rate if the borrower of funds can choose the security to pledge. GC is below and frequently close to the rate for one-period uncollateralized borrowing free of default risk, which can be taken, for simplicity, to be the interest rate on a risk free one-period bond. In Example 4 we show that the bubble can occur on top the specialness overpricing [14].

Next, we recover in the repo set-up well known results by Santos and Woodford (1997) on the absence of bubbles when markets are complete or agents are uniformly impatient. Recall that their result relied on the use of the standard borrowing constraints that were known to rule out Ponzi schemes under uniform impatience. Ours rests on the asymmetric haircut treatment for dealers and non-dealers, together with the bounds on dealers' repo positions. As in Santos and Woodford (1997), by complete markets we mean the full efficiency scenario, where all agents marginal rates of inter-node substitution coincide[15].

Let us define what uniform impatience is.

Definition 3: *Consumer i is uniformly impatient if there exist $\pi \in (0,1)$ and Δ_ξ for each $\xi \in D$ such that, for $0 \leq x \leq \Omega$, we have $U^i(\tilde{x}(\xi,\pi')) > U^i(x)$, $\xi \in D$, where $\tilde{x}(\xi,\pi')$ differs from x on $D(\xi)$ in the following way: $\tilde{x}_\xi(\xi,\pi') = x_\xi + \Delta_\xi$ and $\tilde{x}_\eta(\xi,\pi') = \pi' x_\eta$* for *$\eta > \xi$ with $\pi' \in [\pi, 1)$.*
Moreover, $\exists k > 0$ such that $\omega^i_\xi \geq k\Delta_\xi > 0$, $\xi \in D$.

It is a joint assumption on preferences and endowments, stating that, uniformly on consumption plans and on nodes, the consumer is willing to sacrifice a constant fraction π

[14]For a dealer, the following additional term appears in the fundamental value of a security: $\sum_{\xi\geq\eta} \frac{(c^i_{j\xi}+k^i_{j\xi})}{\lambda^i_\eta}$. This term captures the series of shadow prices for the bounds on dealers' positions.

[15]When preferences are not time-separable, complete market bubbles may occur under constraints that are appropriate for non-impatient agents. See Araujo, Novinski and Páscoa (2011).

of the consumption beyond node ξ in exchange for an additional bundle at ξ and that this additional bundle does not exceed some constant fraction k of the agent's node endowment. A consumer is not uniformly impatient if his endowments are unbounded or if his discount factor is not stationary (for example, when the discounting is hyperbolic).

Proposition 1: *In the constrained dealers case, securities in positive net supply are free of bubbles under (i) complete markets or (ii) uniform impatience for deflators with finite present value of individual endowments.*

The proof of Proposition 1 is left for the Appendix. The result applies in particular to deflators given by the personal marginal rates of intertemporal substitution, as established by the following lemma, proven in the Appendix. For complete markets, such personal deflators are common to all consumers.

Lemma 2 : *The present value of ω^i is finite for the Lagrange multipliers deflator process λ^i, provided that $U^i(x^i)$ is finite.*

Corollary: *If there is $k' > 0$ such that $\omega^i > k' \sum_h \omega^h$, for each i, then the result (ii) in Proposition 1 holds for any deflator with a finite present value of aggregate endowments.*

The result in the Corollary also holds if in Definition 3 the condition $\omega^i_\xi \geq k\Delta_\xi > 0$ is replaced by $\sum_h \omega^h_\xi \geq k\Delta_\xi > 0$, for all ξ. The next section shows that under incomplete markets we may have $\lim_T \sum_{\xi \geq \eta : t(\xi)=T} \lambda_\xi q_\xi > 0$ if uniform impatience is not assumed.

3.5 An example of a bubble with quite asymmetric leverage

A central result in the literature of asset pricing bubbles asserts that, if short sales were allowed, the standard no-Ponzi schemes conditions (debt constraints coupled with uniform impatience) end up ruling out also bubbles, for deflators yielding finite present value of wealth and when assets were in positive net supply (see Santos and Woodford (1997) and Magill and Quinzii (1996)). However, as we will illustrate now, when repo markets are contemplated, a bubble can occur in a trading environment (with fixed issuance), when markets are incomplete and agents are not uniformly impatient, which is not required for existence of equilibrium with constrained dealers.

Example 1: This is an example of a security and repo equilibrium for an economy with two infinite lived agents, $\mathcal{A}$ and $\mathcal{B}$, trading one commodity and one security in sequential incomplete markets. Agent $\mathcal{A}$ is a non-dealer and agent $\mathcal{B}$ is a dealer. The regulation framework here is as in the constrained dealers' case.

Preferences and endowments are adapted from an example of a monetary equilibrium in Páscoa, Petrassi and Torres-Martinez (2011), but fiat money (with a no-short-sales constraint) is now replaced by a security paying real dividends. Portfolios must satisfy the box constraint (5): security purchases can be funded and what has been borrowed of the security can be reused.

The infinite tree D is such that each node ξ has two followers, up (ξ_u) or down (ξ_d). We denote by $\xi_{\bar{s}_t u}$ the node attained by going up after the history of node realizations $\bar{s}_t$ (and similarly for $\xi_{\bar{s}_t d}$). Preferences are given by $U^i(x) = \sum_{\xi \in D} \beta^{t(\xi)} \rho^i_\xi x_\xi$, for $i = \mathcal{A}, \mathcal{B}$ where $\beta \in (0,1)$ is the discount factor and $\rho^i_\xi \in (0,1)$ is the probability belief at node ξ satisfying

$\rho^i_{\xi_0} = 1$, $\rho^i_\xi = \rho^i_{\xi_u} + \rho^i_{\xi_d}$, $\rho^{\mathcal{A}}_{\xi_u} = (1/2^{t(\xi)+1})\rho^{\mathcal{A}}_\xi$ and $\rho^{\mathcal{B}}_{\xi_u} = (1-(1/2^{t(\xi)+1}))\rho^{\mathcal{B}}_\xi$. We denote by $\xi_{\bar{s}_t ud}$ the node attained after the history of node realizations $\bar{s}_t$ by going up and then down, and similarly for other pairs of branches.

Commodity endowments have a trend component g_t which is constant and equal to 1 for both agents. Endowment shocks benefit agent $\mathcal{A}$ when down is followed by up, while agent $\mathcal{B}$ gets also a positive shock but when up is followed by down and also at ξ_{0d}. More precisely, agent $\mathcal{A}$'s endowment is $\omega^{\mathcal{A}}_\xi = 1 + P_{t(\xi)}$ if $\xi = \sigma_{du}$ for some $\sigma \in D$ and equals 1 otherwise. Agent $\mathcal{B}$'s endowment is $\omega^{\mathcal{B}}_\xi = 1 + P_{t(\xi)}$ if $\xi = \xi_{0d}$ or $\xi = \sigma_{ud}$ for some $\sigma \in D$ and equals 1 otherwise. The security pays B_t units of the commodity at every node occurring at date t. Agent $\mathcal{A}$ is endowed with 1 unit of the security at ξ_0, while agent $\mathcal{B}$ is not endowed.

The dealer is constrained as follows: $q_\xi \theta^B_\xi / p_\xi \leq 1$ and $q_\xi \psi_\xi \leq 1$. We construct an equilibrium where, at every node, $q_\xi = 1$ (the security is the numeraire) and $P_\xi = 1/p_\xi$, so that endowment shocks are worth one unit of the security. Then, $\theta^B_\xi \leq p_\xi$, whereas $\psi_\xi \leq 1$.

We set the shadow prices for the box constraint and the dealer's upper bounds all equal to zero. Then, the Euler condition on security positions becomes $\lambda^i_\xi = \sum_{\eta \in \xi^+} \lambda^i_\eta (1 + p_\eta B_\eta)$. The Euler conditions on repo positions hold if, for each ξ, the two repo rates $r_{1\xi} - 1$ and $r_{2\xi} - 1$, coincide with $p_\eta B_\eta$, for any $\eta \in \xi^+$. Then, agents $\mathcal{A}$ and $\mathcal{B}$ have budget constraints given by, respectively,

$$x^{\mathcal{A}}_\xi = \omega^{\mathcal{A}}_\xi - P_\xi[\varphi^{\mathcal{A}}_\xi + \theta^{\mathcal{A}}_\xi - h\psi^{\mathcal{A}}_\xi] + (P_\xi + B_{t(\xi)})[\varphi^{\mathcal{A}}_{\xi^-} + \theta^{\mathcal{A}}_{\xi^-} - h\psi^{\mathcal{A}}_{\xi^-}] \tag{9}$$

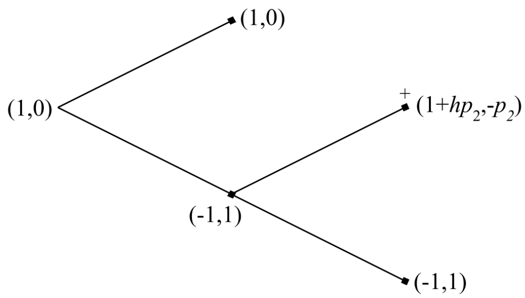
$$x^{\mathcal{B}}_\xi = \omega^{\mathcal{B}}_\xi - P_\xi[\varphi^{\mathcal{B}}_\xi + h\theta^{\mathcal{B}}_\xi - \psi^{\mathcal{B}}_\xi] + (P_\xi + B_{t(\xi)})[\varphi^{\mathcal{B}}_{\xi^-} + h\theta^{\mathcal{B}}_{\xi^-} - \psi^{\mathcal{B}}_{\xi^-}] \tag{10}$$

We look for *deterministic prices*, $p_{t(\xi)}$, and positive consumption at every node. So, $\lambda^i_\xi = \rho^i_\xi \beta^{t(\xi)} / p_\xi$ and the Euler condition on security positions holds if $\lambda^i_\xi = \beta^{t(\xi)+1} \rho^i_\xi (B_{t(\xi)+1} + 1/p_{t(\xi)+1})$, that is, if $(1/p_{t(\xi)+1}) - (1/\beta)(1/p_{t(\xi)}) = -B_{t(\xi)+1}$. Hence, $P_{t(\xi)}$ solves $P_{t(\xi)+1} - (1/\beta)P_{t(\xi)} = -B_{t(\xi)+1}$. For $B_t = k^t$, with $k \neq 1/\beta$, and positing $P_0 = 1$ we have

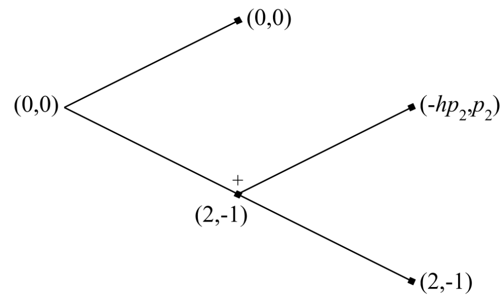
$$P_t = \frac{(1/\beta - 2k)\beta^{-t} + k^{t+1}}{1/\beta - k} \tag{11}$$

As argued below, for the sake of having a bubble, we are interested in the case $k < \frac{1}{2\beta}$, for which endowment shocks P_t are positive and unbounded, which implies that consumers are not uniformly impatient. We allow, however, for k to be either less or greater than one. The endowment shocks and equilibrium positions for agents $\mathcal{A}$ and $\mathcal{B}$ are summarized in graphs 1 and 2, respectively. Nodes where the agent receives a positive endowment shock $P_{t(\xi)}$ are marked with $+$.

Agents use positive endowment shocks in low probability nodes to buy the security (with credit) and, at the next date, short-sell it in a node with higher probability and keep the same long security position in a node with lower probability. Let us see in detail how the *equilibrium positions* are constructed. At the initial node ξ_0 there are no securities nor repo trades, agent $\mathcal{A}$ still holds the unit of the security he was endowed with and agents consume their endowments. At the first node where an endowment shock occurs, node ξ_{0d}, agent $\mathcal{B}$ uses the endowment shock to purchase the unit of the security that agent $\mathcal{A}$ had and pledges it as collateral. Then the counterparty short-sells this unit to $\mathcal{B}$, so that $\mathcal{B}$ ends up with two units as his long position in the security (but just holding one unit in



Graph 1: Agent A's positions $(\varphi^A, \theta^A - \psi^A)$.



Graph 2: Agent B's positions $(\varphi^B, \theta^B - \psi^B)$.

his box, since the other unit was encumbered). Consumptions are $x^A_{\xi_{0d}} = 1 + P_1 + B_1$ and $x^B_{\xi_{0d}} = 1$. At node ξ_{0u}, there are neither securities nor repo trades, agent $\mathcal{B}$ consumes his endowment while agent $\mathcal{A}$ consumes his endowment plus the dividends.

At node ξ_{0du}, agent $\mathcal{A}$ uses his endowment shock to purchase the security and post as most as he can as collateral. His counterparty, the dealer, cannot be long in repo more than p_2. The value of the endowment shock should be equal to what the loan doesn't cover: the security long position $\varphi^{\mathcal{A}}_{\xi_{0du}}$ should be such that $1 = \varphi^{\mathcal{A}}_{\xi_{0du}} - hp_2$, so $(\varphi^{\mathcal{A}}_{\xi_{0du}}, \theta^{\mathcal{A}}_{\xi_{0du}}, \psi^{\mathcal{A}}_{\xi_{0du}}) = (1 + hp_2, 0, p_2)$. For the dealer we have $(\varphi^{\mathcal{B}}_{\xi_{0du}}, \theta^{\mathcal{B}}_{\xi_{0du}}, \psi^{\mathcal{B}}_{\xi_{0du}}) = (-hp_2, p_2, 0)$. The box constraint is satisfied and non-binding for both agents (as $1 > p_2(1-h) > 0$)[16]. Consumption is given by $x^{\mathcal{A}}_{\xi_{0du}} = 1$ and $x^{\mathcal{B}}_{\xi_{0du}} = 1 + P_2 + B_2$.

At node ξ_{0dd}, there are neither security nor repo trades, agent $\mathcal{A}$ consumes his endowment while agent $\mathcal{B}$ consumes his endowment plus the dividends net of repo repayments. At node ξ_{0duu}, portfolio positions are the same as in the preceding node (agent $\mathcal{B}$ consumes his endowment while agent $\mathcal{A}$ consumes his endowments plus the dividends net of repo repayments). At ξ_{0dud} positions are the same as in node ξ_{0d}. We have determined positions at all types of nodes. Notice that at the initial node the non-dealer's box is non-binding while the dealer's is. At subsequent nodes the dealer's box is always non-binding whereas the non-dealer's is binding when he is short in repo.

The asymmetric haircut treatment implies *asymmetric leverage*. Since the node's endowment shock is worth one unit of the security, the leverage coefficient is just equal to the long position in the security. For a dealer this leverage coefficient stays constant, equal to 2. For a non-dealer, the leverage coefficient tends to 1, that is, leverage fades away over time, in nominal terms. However, the purchasing power of the security is growing exponentially (at the same rate as the economy's resources), so the real values of long positions ($2P_t$ for a dealer and $P_t + h$ for a non-dealer) are exploding.

It remains to show that the *transversality condition* holds for both agents. For $\mathcal{A}$ and

[16] For the dealer to pledge what is left in the box, the non-dealer would have to borrow and then, not to lose money, short sell that amount $(1-h)p_2$ to the dealer (this is a peculiarity of a two-agent example). Then, the dealer would end up with a lower short sale ($(1-2h)p_2 < 0$) and the non-dealer with a lower long position $(1 + p_2(2h-1))$, assuming $h > 1/2$. This would be another equilibrium, with the same real allocation, illustrating that both agents can go long and short in repo in a same node.

$\mathcal{B}$, these conditions are, respectively,

$$\limsup_T \sum_{\xi:t(\xi)=T} \lambda^i_\xi \left(\varphi^{\mathcal{A}}_\xi + \theta^{\mathcal{A}}_\xi - h\psi^{\mathcal{A}}_\xi\right) \leq 0 \tag{12}$$

and

$$\limsup_T \sum_{\xi:t(\xi)=T} \lambda^i_\xi \left(\varphi^{\mathcal{B}}_\xi + h\theta^{\mathcal{B}}_\xi - \psi^{\mathcal{B}}_\xi\right) \leq 0 \tag{13}$$

We now notice that $\varphi^{\mathcal{A}}_\xi + \theta^{\mathcal{A}}_\xi - h\psi^{\mathcal{A}}_\xi$ and $\varphi^{\mathcal{B}}_\xi + h\theta^{\mathcal{B}}_\xi - \psi^{\mathcal{B}}_\xi$ coincide with the money position in the example of Páscoa, Petrassi and Torres-Martinez (2011), and that we have

$$\lambda^i_\xi = \rho^i_\xi \frac{1/\beta - 2k + k^{T+1}\beta^T}{1/\beta - k} \tag{14}$$

where ρ^i_ξ was the budget multiplier in that example[17]. If $k\beta < 1$ then $\lambda^i_\xi < \rho^i_\xi$ and, therefore, the transversality conditions will hold. Recall that Euler and transversality conditions (together with all constraints) are sufficient for individual optimality.

Bubble: In this equilibrium the security has a price bubble. In fact, at each node η the security price 1 is equal to the fundamental value plus the bubble $(1/\lambda^i_\eta)\lim_T \sum_{\xi \geq \eta: t(\xi)=T} \lambda^i_\xi$. To evaluate this limit we use (14). Let $m_T \equiv \frac{1/\beta - 2k + k^{T+1}\beta^T}{1/\beta - k}$. Now, $\lim_T m_T > 0$ when $k < \frac{1}{2\beta}$ and notice that $\lim_T \sum_{\xi \geq \eta: t(\xi)=T} \rho^i_\xi = \rho^i_\eta > 0$[18].

Finally, observe that, as (m_T) is bounded, the process $(\lambda^i_\xi)_\xi$ for which the bubble occurs yields a *finite present value of aggregate endowments*. In fact, the present value of aggregate endowments was finite in the example in Páscoa, Petrassi and Torres-Martinez (2011) and $\lambda^i_\xi p_\xi$ is as in that example. The endowment trend is also the same, while in that example the endowment shocks were $\beta^{-t(\xi)}$ but in our example the shocks are $\nu_{t(\xi)}\beta^{-t(\xi)}$.

If the security paid *nominal returns* A_ξ instead, we make $B_\xi = A_\xi/p_\xi$ and commodity market clearing can be ensured by specifying how the nominal returns of the positive net supply (one unit in this example) are being generated. This can be done if we add an issuer, agent $\mathcal{C}$, facing at each node the budget constraint $A_\xi = p_\xi(\omega^{\mathcal{C}}_\xi - x_\xi)$, in case $\mathcal{C}$ would abstain from trading in financial markets. The bubble is as before and, for an appropriate choice of $\omega^{\mathcal{C}}_\xi$, the Lagrange deflators λ^i of the three agents still yield finite present values of aggregate endowments and the three agents are still uniformly impatient (say $\mathcal{C}$ has linear utility with uniform beliefs $\rho^i_\xi = \rho^i_{\xi^-}/2$ and $\omega^{\mathcal{C}} = \omega^{\mathcal{A}} + \omega^{\mathcal{B}}$.). In Example 3 with provide more details on the issuer's optimization problem and allow for issuance to change along the event-tree.

3.6 Bubble bursting and deleverage

A variant of Example 1 allows for the bubble to burst. This happens due to a change in the pattern of the relation between security dividends and endowments, as we enter a subtree.

[17] that is, the multiplier that would make the Euler equation in consumption hold if p_t were β^t and for which we had the following $\lim_T \sum_{\xi:t(\xi)=T} \rho^i_\xi \left(\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi)\right) = 0$

[18] The latter follows from the fact that fiat money had price 1 and zero fundamental value in the example of Páscoa, Petrassi and Torres-Martinez (2011).

We modify on a subtree the rate k governing the evolution of dividends while keeping the fundamentals of the economy are not kept unchanged: as we change the dividends variation rate k, the purchasing power P_t of the security will change but, by the way we construct our examples (both Example 1 and this new one), consumers' endowments shocks are specified to be worth one unit of the security (that is, coincide with P_t).

Take some node η and suppose that for $\xi > \eta$ all security dividends, in real terms, are given by $\tilde{k}^{t(\xi)}$, where $\tilde{k}$ is assumed to be still less than $1/\beta$. Then, depending on the relation between the former k and the new $\tilde{k}$, the bubble may burst on the subtree with root at the node η. Let $T = t(\eta)$. The general solution to P_t on this subtree is given by $\alpha\beta^{-T} + \frac{\tilde{k}}{1/\beta-\tilde{k}}\tilde{k}^t$, where the constant α is determined from the initial condition $P_T = \beta^{-T}(1 - \frac{k}{1/\beta-k}) + \frac{k}{1/\beta-k}k^T$, implying that $\alpha = \beta^T(\frac{k^{T+1}}{1/\beta-k} - \frac{\tilde{k}^{T+1}}{1/\beta-\tilde{k}}) + 1 - \frac{k}{1/\beta-k}$.

Now, for $\xi > \eta$ we have $\lambda^i_\xi = (\alpha + \frac{\tilde{k}}{1/\beta-\tilde{k}}\tilde{k}^t\beta^t)\rho^i_\xi$ and the bubble bursts if $\alpha = 0$.

Let us consider two numerical examples. First, suppose $\beta = \frac{9}{10}$, $k = \frac{4}{9} < \frac{1}{2\beta}$ and that the root for the subtree is $\eta = \xi_{0dd}$. We have $T = 2$ and α becomes zero when $\tilde{k} = 0.7128$, which is larger than $k \approx 0.4444$. In this first numerical example, the endowment shocks in the subtree beyond ξ_{0dd} are now bounded, given by $P_t(\xi) = \frac{\tilde{k}}{1/\beta-\tilde{k}}\tilde{k}^t$. Hence, both agents are uniformly impatient for $\xi > \xi_{0dd}$ and the bursting of the bubble in the subtree is consistent with Proposition 1.

Consider another example, with $\beta = \frac{2}{5}$, $k = \frac{9}{8} < \frac{1}{2\beta}$ and the same root for the subtree[19]. Now, $\alpha = 0$ for $\tilde{k} = 1.44$, which is larger than $k \approx 1.125$. Endowment shocks $P_t(\xi) = \frac{\tilde{k}}{1/\beta-\tilde{k}}\tilde{k}^t$ are still unbounded in the subtree and, therefore, both agents are still non-uniformly impatient in the subtree. Impatience becoming uniform is sufficient but not necessary for the bursting of incomplete market bubbles.

In both numerical examples, the security's dividends evolution rate reaches some upper threshold that makes the bubble burst. At the threshold there is still an equilibrium for which consumers' endowment shocks are worth one unit of the security and, therefore, consumers can use these shocks to alternate between themselves the purchase the security aggregate net supply of one unit (and do some leverage on top of that), but the equilibrium has no longer a bubble. Beyond the threshold endowments shocks P_ξ are too low for the high rate k of variation in dividends. Intuitively, the security becomes too productive and, therefore, too pricey, compared with the economy's resources that have to be used to sustain the purchase of the security's aggregate net supply[20].

Actually, already when the threshold is attained, the bursting of the bubble causes a fall in P_ξ and a *severe deleverage*. As the bubble bursts, the real leverage becomes quite different. Security long positions, in real terms, are the positions reported in graphs 1 and 2 multiplied by $P_t(\xi)$. Under the new rate $\tilde{k}$ (at nodes after the bubble bursts) we have $P_t(\xi) = \frac{\tilde{k}}{1/\beta-\tilde{k}}\tilde{k}^t$, whereas at contemporaneous nodes where the bubble did not burst (under

[19]Since nodes are repo settlement moments, one unit of time is actually the repo settement period and, therefore, the interpretation of the discount factor β will depend on how long repo contracts are.

[20]We are not suggesting non-existence of equilibrium beyond the threshold, but just that endowment shocks P_ξ can no longer have value 1 in the numeraire (the security).

the old rate k) we have $P_t(\xi) = \frac{(\beta-2k)\beta^{-t}+k^{t+1}}{1/\beta-\tilde{k}}$. In the first numerical example, already for $t(\xi) \geq 4$ and $\xi > \xi_{0dd}$, we see that $P_t(\xi)$ is smaller than at contemporaneous nodes where the bubble did not burst (the burst makes P_4 drop from 0.534 to 0.462). Far away along the subtree the reduction in real leverage is much more significant (P_{12} drops from 1.180 to 0.031). The amplitude of the leverage build up during a boom and the deleverage during the crash can be moderated by managing the haircut $1-h$ countercyclically.

4 Bringing Repo on Exchanges

Recent years saw a development of repo trades done in exchanges, dispensing with the presence of dealers. In this different environment all agents have a symmetric treatment in terms of haircut, as everybody pays haircut to the exchanges and the exchanges segregate such haircuts without any trading on its own. The haircut is no longer justified on the basis of protection of the creditor. The haircut is more like an initial margin, it is paid by both the borrower and the lender of securities. It is meant to protect the exchange against adverse market move during the repo transaction.

In practice exchanges collect and segregate haircuts - there is an evolution of practice in this direction even with dealers (see the Financial Stability Board (2013) white paper) and current interpretations of existing law (as haircut is paid for with customer money)[21]. The haircut posted by counter-parties, and paid for with their own funds, should be set aside at the exchange. We model the exchange as a passive agent (with no objective function) that can only collect collateral.

Any trader's i's *box constraint* for security j at node ξ is as before, given by inequality (5). However, trader i's budget constraints at nodes ξ_0 and $\xi > \xi_0$ are now

$$p_{\xi_0}(x_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}[\varphi_{\xi_0} - e^i + (\theta_{\xi_0} - \psi_{\xi_0}) + (1-h_{\xi_0})(\theta_{\xi_0} + \psi_{\xi_0})] \leq 0 \tag{15}$$

$$p_\xi(x_\xi - \omega^i_\xi) + q_\xi[\varphi_\xi - \varphi_{\xi^-} + (\theta_\xi - \psi_\xi) + (1-h_\xi)(\theta_\xi + \psi_\xi)] - p_\xi B_\xi \varphi_{\xi^-} - q_{\xi^-} r_{\xi^-}(\theta_{\xi^-} - \psi_{\xi^-} + (1-h_{\xi^-})(\theta_{\xi^-} + \psi_{\xi^-})) \leq 0 \tag{16}$$

The exchange devolves the haircut and pays the repo rate r on the haircut. The haircut collected from a repo long $(1-h_\xi)q_\xi\theta_\xi$ enters with a positive sign in (16), whereas it entered with a minus sign in the dealer's budget constraint (4), as it was then a benefit to the dealer. The haircut that the exchange collects from a repo short $(1-h_\xi)q_\xi\psi_\xi)$ already entered with a positive sign in a non-dealer's budget constraint (2).

We say that a plan $(x^i, \varphi^i, \theta^i, \psi^i)$ is optimal for agent i trading in exchanges, at prices $(p, q, \tilde{r})$, if it maximizes U^i subject to (15), (16) and (5). The traders set is still **I**. Say each security has its own repo exchange. To close the model we allow for exchanges to consume one good, say the first good (although we can assume their endowments to be zero). What the exchange for security j spends on good 1 at node ξ is passively determined by

[21] This assumption is not dissimilar to the segregation of initial margin by exchanges for vanilla derivatives like interest rate swaps.

$$p_{1\xi}(x_{1\xi}^{E_j} - \omega_{1\xi}^{E_j}) = 2q_{j\xi}(1-h_{j\xi})\sum_{i\in\mathbf{I}}\psi_\xi^i - 2q_{\xi^-}r_{\xi^-}(1-h_{j\xi^-})\sum_{i\in\mathbf{I}}\psi_{\xi^-}^i \tag{17}$$

Definition E: *An exchanges equilibrium is a process of prices* $(p,q,r)\in\mathbb{R}_+^{L\times D}\times\mathbb{R}_+^{J\times D}\times\mathbb{R}_+^{J\times D}$ *together with individual plans* $(\bar{a}^i)_{i\in\mathbf{I}}$*, such that, (i) for each agent* $i\in\mathbf{I}$*, the plan* $\bar{a}^i$ *is optimal at prices* (p,q,r)*, and (ii) at any node* $\xi\in D$*, commodity, security and repo markets clear, i.e.* $\sum_{i\in\mathbf{I}}\bar{x}_\xi^i=\Omega_\xi$*,* $\sum_{i\in\mathbf{I}}(\bar{\varphi}_\xi^i-e^i)=0$ *and* $\sum_{i\in\mathbf{I}}\bar{\theta}_\xi^i=\sum_{i\in\mathbf{I}}\bar{\psi}_\xi^i$*.*

Exchanges equilibria can be reinterpreted as equilibria of a model without apriori explicit intermediaries and where the haircut that repo longs collect from repo shorts has to be segregated. We will use this fact to establish existence of equilibrium.

4.1 Segregated haircut

Let us consider a no-intermediaries environment which serves as an auxiliary model but may also have an interest on its own. Repo shorts pay haircuts to their counterparties but the repo longs are prevented from reusing the haircut. Then, no agents can do any of the Ponzi schemes considered in subsection 3.2. The fraction $H_{j\xi}$ of a security j that can be sold or lent out in repo after being borrowed at node ξ must satisfy $H_{j\xi}\le h_{j\xi}$. Agent i's *box constraint* for security j at node ξ is now:

$$\varphi_{j\xi}+H_{j\xi}\theta_{j\xi}-\psi_{j\xi}\ge 0 \tag{18}$$

Agent i's budget constraints are

$$p_{\xi_0}(x_{\xi_0}-\omega_{\xi_0}^i)+q_{\xi_0}(\varphi_{\xi_0}-e^i+h_{\xi_0}(\theta_{\xi_0}-\psi_{\xi_0}))\le 0 \tag{19}$$

$$p_\xi(x_\xi-\omega_\xi^i)+q_\xi[\varphi_\xi-\varphi_{\xi^-}+h_\xi(\theta_\xi-\psi_\xi)]-p_\xi B_\xi\varphi_{\xi^-}-q_{\xi^-}r_{\xi^-}h_{\xi^-}(\theta_{\xi^-}-\psi_{\xi^-})\le 0 \tag{20}$$

An equilibrium for the segregated haircuts economy induces an exchanges equilibrium. Denote the repo long positions in the exchanges equilibrium by $\tilde{\theta}_\xi^i$. A repo long in security j at node ξ effectively spends $(2-h_{j\xi})q_{j\xi}\tilde{\theta}_{j\xi}$. Denoting $H_{j\xi}=\frac{h_{j\xi}}{2-h_{j\xi}}$, we get $\frac{h_{j\xi}}{H_{j\xi}}=2-h_{j\xi}>1$. Denote by $\theta_{j\xi}^i$ the repo long positions in the segregated haircuts equilibrium. We make $\tilde{\theta}_{j\xi}^i=H_{j\xi}\theta_{j\xi}^i$. The market clearing condition for exchanges requires the sum over traders of $H_{j\xi}\theta_{j\xi}^i-\psi_{j\xi}^i$ to be zero. The exchanges can be thought of as being repo shorts with positions $\psi_{j\xi}=(\frac{1}{H_{j\xi}}-1)\sum_i\tilde{\theta}_{j\xi}^i$ and, therefore, the sum over all agents i (traders and exchanges) of $\theta^i-\psi^i$ is zero.

Let us see that equilibrium exists for the segregated haircuts economy, referred to as case (II). We say that a plan $(x^i,\varphi^i,\theta^i,\psi^i)$ is optimal for agent i in case (II), at prices (p,q,r), if it maximizes U^i subject to (19), (20) and (18). Euler conditions are reported after Lemma A.1 in the Appendix and imply $\theta_\xi^i\psi_{j\xi}^i=0$. The *transversality condition* requires $\limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}[\lambda_\xi^i q_{j\xi}(\varphi_{j\xi}^i+h_{j\xi}(\theta_{j\xi}^i-\psi_{j\xi}^i)]\le 0$.

Lemma 1.II (**Optimality**): *Suppose assumptions (A1) and (A2) hold. A plan* $a^i\equiv(x^i,\varphi^i,\theta^i,\psi^i)$ *verifying constraints (18), (19) and (20) is optimal for agent* i *at prices*

(p, q, r) *if and, when* $U^i(x^i) < \infty$, *only if also, there exist multipliers* $(\lambda^i_\xi, \mu^i_\xi) \geq 0$, *for which* $\mu^i_{j\xi}[\varphi_{j\xi} + H_{j\xi}\theta_{j\xi} - \psi_{j\xi}] = 0$ *and the Euler and transversality conditions are satisfied.*

The proof of Lemma 1.II is provided in the Appendix. Sufficiency follows from the fact that any plan (φ, θ, ψ) satisfying (18), (19) and (20) satisfies the inequality which is the converse of the transversality condition, $\liminf_{T\to\infty} \sum_{\xi:t(\xi)=T} \lambda^i_\xi q_\xi(\varphi_\xi + h_{j\xi}(\theta_{j\xi} - \psi_{j\xi})) - \mu^i_{j\xi}(\varphi_{j\xi} + H_{j\xi}\theta_{j\xi} - \psi_{j\xi})] \geq 0$, for every j.

Definition 2: *A segregated haircuts equilibrium is a process of prices* $(p, q, r) \in \mathbb{R}^{L\times D}_+ \times \mathbb{R}^{J\times D}_+ \times \mathbb{R}^{J\times D}_+$ *together with individual plans* $(\bar{a}^i)_{i\in\,\mathbf{I}}$*, such that, (i) for each agent* $i \in \mathbf{I}$*, the plan* $\bar{a}^i$ *is optimal under (18), (19) and (20) at prices* (p, q, r)*, and (ii) at any node* $\xi \in D$*, commodity, security and repo markets clear, i.e.* $\sum_{i\in\mathbf{I}} \bar{x}^i_\xi = \Omega_\xi$, $\sum_{i\in\mathbf{I}}(\bar{\varphi}^i_\xi - e^i) = 0$ *and* $\sum_{i\in\mathbf{I}} \bar{\theta}^i_\xi = \sum_{i\in\mathbf{I}} \bar{\psi}^i_\xi$.

Theorem 2: *Equilibrium exists, in the case of segregated haircuts, under (A1), (A1'), (A1") and (A2).*

Corollary E: *An exchanges equilibrium exists, under (A1), (A1'), (A1") and (A2).*

Bottazzi, Luque and Páscoa (2012) considered (in a case designated then as direct limited re-hypothecation) a constraint allowing for full reuse of the collateral only through lending. Let $z_{j\xi} \equiv \theta_{j\xi} - \psi_{j\xi}$. Such an arrangement, best described in terms of *no short-selling of the haircut*, requires

$$\varphi_{j\xi} + h_{j\xi}z^+_{j\xi} - z^-_{j\xi} \geq 0 \tag{21}$$

In a finite horizon the optima under (18) and (21) coincide. However, in infinite horizon, under the less stringent box constraint, Euler and transversality conditions are only know to be sufficient if $\mu^i_{j\xi} = 0$ when $z^i_{j\xi} \leq 0$. Otherwise, it may not be possible to dominate plans with $z_{j\xi} > 0$ and $\varphi_{j\xi} < 0$ (see Example 4)[22].

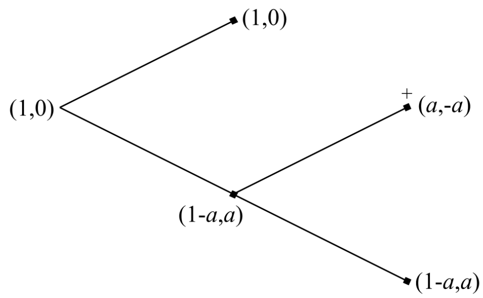
4.2 Bubbles in exchanges equilibrium

Proposition 1 on absence of bubbles extends to exchanges.

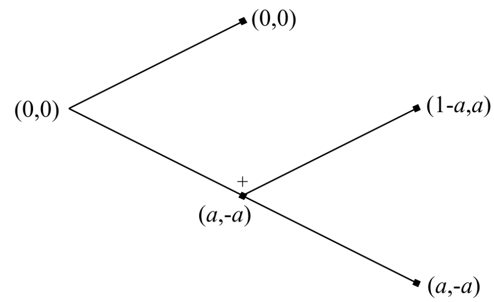
Proposition 2: *In the case of exchanges equilibria (or segregated haircuts equilibria), securities in positive net supply are free of bubbles, under either uniform impatience, for deflators with finite present value of wealth, or complete markets.*

See the Appendix for a proof. In incomplete markets, when uniform impatience does not hold, there is room for bubbles, as our next example shows.

[22]Notice that the Ponzi schemes reported in subsection 3.2 are a simple strategies with unbounded gains. There may also exist improvement strategies with bounded gains. That is why the impossibility of doing the former might not imply existence of equilibrium.



Graph 3: Agent A's positions
$(\varphi^{\mathcal{A}}, \theta^{\mathcal{A}} - \psi^{\mathcal{A}})$



Graph 4: Agent B's positions
$(\varphi^{\mathcal{B}}, \theta^{\mathcal{B}} - \psi^{\mathcal{B}})$

4.3 Examples of bubble under segregated haircuts or in exchanges, with symmetric leverage

Example 2: We examine first a segregated haircuts case with $h = H$ and then we relax this assumption to accommodate exchanges. Preferences, endowments, security returns and initial holdings are as in Example 1. Recall that $\theta_\xi \psi_\xi = 0$ at any node ξ.

As in Example 1, we take the security as the numeraire, choose endowment shocks to be worth one unit of the security (that is $P_\xi = 1/p_\xi$), look for deterministic prices and pick the repo rate paid at nodes of date t to be $p_t B_t$. Equation (11) describes the shocks P_t. Equilibrium positions are shown in the next two graphs, where $a \equiv \frac{1}{1-h}$.

Agents use positive endowment shocks to purchase the security with funding, thus consuming just the trend endowment at such a node. At the next nodes, when endowments are back at the trend level 1, a short sale is done at the node that has a higher probability, whereas at the other node the security position remains the same long one.

If an agent does not trade in repo at a node ξ (that is, $\theta^i_\xi - \psi^i_\xi = 0$), the agent's security position remains what it was before ($\varphi^i_\xi = \varphi^i_{\xi^-}$) When the endowment shock with value 1 occurs, it is spent on $\varphi^i_\xi - h\psi^i_\xi = (1-h)\varphi^i_\xi$, so we get $\varphi^i_\xi = a$ at a node where i gets an endowment shock. The counterparty has $\theta^i_\xi = a$ and therefore $\varphi^i_\xi = -ha = 1 - a$. Then, $\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi)$ is equal to 1 or 0 depending on whether $\theta^i_\xi - \psi_\xi$ is negative or positive.

Let us now see what the consumption plans are. The choice for the repo rate implies that the budget constraint at node ξ can be written as follows

$$x^i_\xi = \omega^i_\xi - P_\xi[\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi) - (\varphi^i_{\xi^-} + h(\theta^i_{\xi^-} - \psi^i_{\xi^-}))] + B_{t(\xi)}(\varphi^i_{\xi^-} + h(\theta^i_{\xi^-} - \psi^i_{\xi^-})) \quad (22)$$

Hence, $(x^A_{\xi_0}, x^B_{\xi_0}) = (1,1)$, $(x^A_{\xi_{0d}}, x^B_{\xi_{0d}}) = (\omega^A_{\xi_{0d}} + P_\xi + B_1, \omega^B_{\xi_{0d}} - P_\xi)$ and $(x^A_{\xi_{0u}}, x^B_{\xi_{0u}}) = (\omega^A_{\xi_{0d}} + B_1, \omega^B_{\xi_{0d}})$. The transversality condition $\limsup_T \sum_{\xi: t(\xi)=T} \lambda^i_\xi \left(\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi)\right) \leq 0$ holds. In fact, $\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi)$ coincides with what was in Example 1 the relevant combined position (for the dealer $\varphi^{\mathcal{B}}_\xi + h\theta^{\mathcal{B}}_\xi - \psi^{\mathcal{B}}_\xi$ and for the non-dealer $\varphi^{\mathcal{A}}_\xi + \theta^{\mathcal{A}}_\xi - h\psi^{\mathcal{A}}_\xi$). The bubble in the security price is as in Example 1, but leverage is now symmetric.

Example 2A: *In the case of exchanges,* $H_\xi < h_\xi$ and Example 2 needs to be modified. Prices are as before, but equilibrium positions will be different. At node ξ_{0d}, the first node where an endowment shock occurs, agent $\mathcal{B}$ uses this shock (worth one unit of the security) to go long in the security but does not pledge all of $\varphi^{\mathcal{B}}$. Let $u_1 = \varphi^{\mathcal{B}}_{\xi_{0d}} - \psi^{\mathcal{B}}_{\xi_{0d}}$ be the unencumbered portion, then we must have $1 = (1-h)\psi^{\mathcal{B}}_{\xi_{0d}} + u_1$. Hence, $\psi^{\mathcal{B}}_{\xi_{0d}} = \epsilon_1 a$, where $\epsilon_1 \equiv 1 - u_1 \in (0,1)$ and $a \equiv \frac{1}{1-h}$. We have $\varphi^{\mathcal{B}}_{\xi_{0d}} = 1 + \epsilon_1 ha$.

The counterparty's long repo position, in the exchanges equilibrium, is $\tilde{\theta}^{\mathcal{A}}_{\xi_{0d}} = \epsilon_1 a$. In terms of the underlying segregate haircut equilibrium the long repo position is $\theta^{\mathcal{A}}_{\xi_{0d}} = \epsilon_1 a/H$ (in this auxiliary equilibrium the exchange is seen as an agent going short in repo in the amount $(1/H - 1)\epsilon_1 a$). Agent $\mathcal{A}$ short sells just $\epsilon_1 ha$. Both traders have the box constraint non-binding. At other nodes where endowment shocks occur positions are constructed similarly and when there are no shocks the positions are as in the preceding node.

The combined position $\varphi^i_\xi - h_\xi(\theta^i_\xi - \psi^i_\xi)$ of a repo short is 1 as in Example 2, but for a repo long this combined position is now positive (equal to $\epsilon_t ah(1/H - 1)$) rather than zero. Transversality conditions hold if and only if ϵ_t goes to zero. The nominal leverage coefficient is $1 + \epsilon_t ha$, which tends to 1, for any agent (just like for the non-dealer in Example 1), but in real terms the leveraged long security positions are growing exponentially.

5 A comparison with the standard collateral model

Let us look at the standard general equilibrium model of collateralized credit and default, as conceived by Geanakoplos and Zame (see Geanakoplos (1997) and Geanakoplos and Zame (2014)) within our framework. We focus on the case where the collateral is a security or several securities.

Take the box constraint for the segregated haircut case, $\varphi_{j\xi} + H_{j\xi}\theta_{j\xi} - \psi_{j\xi} \geq 0$. If $H_{j\xi} = 0$ the collateral cannot be re-hypothecated and the box becomes a standard collateral constraint. It just requires the amount of the security pledged as collateral ($\psi_{j\xi}$) to be less than or equal to the security position ($\varphi_{j\xi}$). Moreover, it implies that the security position must be non-negative, that is, the security can not be short sold. That is, we can capture the collateral constraint of standard short-term loans backed by securities in the same way we modeled repo loans, provided that we rule out the reuse of the collateral.

In our model we did not allow for default and we have exogenous haircuts. So a very important feature of the model by Geanakoplos and Zame would be lost. In Geanakoplos and Zame (2014) loans are non-recourse, that is, the effective payment is $min\{(1 + R_\xi)q_\xi h_\xi, q_\eta + p_\eta B_\eta\}$ and the haircut $1 - h_\xi$ is endogenous, but by non-arbitrage, less than one. However, in repo loans, default usually entails bankruptcy. When the debtor (the repo short) goes bankrupt, the creditor (the repo long) keeps the collateral, does not pay manufactured dividends or receive the repo repayment. If the agent going bankrupt has a significant share of the market, a fire sale of the securities may follow (either by his creditors or by others holding the securities), as was observed after Lehman Brothers collapse. So, in repo markets, default is a much more serious and rare event. It is nevertheless an interesting subject, with relevance for financial crisis, and deserves to be studied in a general equilibrium model allowing for bankruptcy and its chain effects.

6 Endogenous issuing

Are the bubbles that we found robust to having an endogenous issuing of the securities? The following example illustrates that they are, provided some separation exists between issuing and trading decisions.

Example 3: Let us consider a variant of Example 2 where there are two traders, agents $\mathcal{A}$ and $\mathcal{B}$, and one issuer, agent $\mathcal{C}$. As before, there is just one consumption good and one security. The issuer won't be able to do issuing Ponzi schemes since what he has issued up to each node ξ is constrained by the present value of his future endowments, computed with respect to some non-arbitrage deflator process. Such constraint is akin to a constraint that Hernandez and Santos (1996) imposed on the value of the portfolio, but we now imposed it only on the accumulated issuance. Apart from that, it is also a weaker constraint since we do not require it to hold for the whole set of non-arbitrage deflator process but just for some process to be endogenously determined and, in this respect, it is in the spirit of the constraints with endogenous deflators considered by Magill and Quinzii (1994).

We will see that the issuing constraint implies a "'transversality-type"' constraint preventing the issuer from having a positive limit for the deflated value of his issuance (net of shadow values of the issuance restriction). More precisely, let i_ξ be the issuance at node ξ. The issuer is allowed to repurchase outstanding securities, so i_ξ can be negative, but we impose a non-negativity constraint on accumulated issuance:

$$\gamma_\xi \equiv \sum_{\eta \leq \xi} i_{\eta \leq \xi} \geq 0 \tag{23}$$

Now, we require the accumulated issuance to be bounded by the present value of the issuer's future endowments, computed using some deflator process $\alpha >> 0$ which will be chosen endogenously within the set of non-arbitrage deflators for q.

$$q_\xi \gamma_\xi \leq PVF^{\mathcal{C}}_\xi(\alpha) \equiv \frac{1}{\alpha_\xi} \sum_{\eta > \xi} \alpha_\eta p_\eta \omega^i_\eta \tag{24}$$

We denote by $\nu^{\mathcal{C}}_\xi$ the shadow price of constraint (24). If the issuer does not trade, his budget constraint reduces to:

$$p_\xi(x_\xi - \omega^{\mathcal{C}}_\xi) + p_\xi B_\xi \gamma_{\xi^-} = q_\xi(\gamma_\xi - \gamma_{\xi^-}) \tag{25}$$

The issuer's constraint set is defined by budget, box and upper-bound on issuance constraints. Market clearing requires $\gamma_\xi = \sum_i \varphi^i_\xi$. Thinking of ξ_0 literally as the initial node, we have to drop the initial holdings (that is, $e^{\mathcal{A}} = 0$)[23].

Agents $\mathcal{A}$ and $\mathcal{B}$ are not allowed to issue and they trade the security under the segregated haircut constraint. This constraint prevents repo Ponzi schemes and allows us to find individually optimal plans without having to assume uniform impatience and, for this reason, a bubble occurs.

[23] Alternatively, if we think of ξ_0 as just the first node we look at (and beyond which issuance starts changing), we can keep $e^{\mathcal{A}} = 1$, interpret it as the position that $\mathcal{A}$ held before and make $\gamma_{\xi_{0^-}} = \gamma_{\xi_0}$.

Preferences are described by $U^i(x) = \sum_{\xi \in D} \beta^{t(\xi)} \rho^i_\xi x_\xi$. The event tree and probability beliefs for agents $\mathcal{A}$ and $\mathcal{B}$ are as in Example 2. Agent $\mathcal{C}$ has uniform beliefs $\rho^i_\xi = \rho^i_{\xi^-}/2$. Commodity endowments of agents $\mathcal{A}$ and $\mathcal{B}$ are as in Example 2 and $\omega^{\mathcal{C}} = \omega^{\mathcal{A}} + \omega^{\mathcal{B}}$. When the issuer does not trade, his *transversality condition* requires

$$\limsup_T \sum_{\xi : t(\xi) = T} (-\lambda^{\mathcal{C}}_\xi q_\xi + \nu^{\mathcal{C}}_\xi q_\xi) \gamma^{\mathcal{C}}_\xi \leq 0 \tag{26}$$

The following lemma is proven in the Appendix (section 9.3.1).

Lemma 1.III
(i) the issuer's transversality condition is satisfied when the security is always in positive net supply.
(ii) in the segregated haircuts context, when the issuer's box shadow prices are null and for $\alpha = \lambda^{\mathcal{C}}$, a consumption and issuance plan $(x^{\mathcal{C}}, \gamma^{\mathcal{C}})$ (together with a no-trade plan in security and repo markets) satisfying (24), (25), Euler and transversality conditions, will be optimal for the issuer.

The example of an equilibrium with a bubble and endogenous issuance can be completed along the lines of Example 2. The security is the numeraire, its deterministic returns are $B_t = k^t$ and the repo rate paid at node ξ will coincide with $p_{t(\xi)} B_{t(\xi)}$. We look for an equilibrium where the box constraints, the sign constraints on issuance and the upper bound constraint on issuance have null shadow values. Then, Euler equations imply the same solution for P_t as in Examples 1 and 2. Marginal utilities of income λ^i_ξ are the same as in those examples. The following claim is proven in the Appendix (section 9.3.1).

Claim (i) *Let $I \equiv \frac{1-2k\beta}{(1-k\beta)^2} > 0$, for $k\beta < 1/2$. Then, $PVF^{\mathcal{C}}_\xi(\lambda^{\mathcal{C}}) \geq I/2$ and we can make $\gamma^{\mathcal{C}}_\xi = (I/2) \sum_{s=0}^{t(\xi)} 1/2^{s+1}$, so that the security net supply $\gamma^{\mathcal{C}}_\xi$ is deterministic, uniformly bounded away from zero and increases towards $I/2 > 0$ as $t(\xi) \to \infty$.*

Let us see what are the traders positions (for the proof see the Appendix, section 9.3.1).

Claim (ii) *(a) $(\varphi^{\mathcal{A}}_{\xi_0}, \theta^{\mathcal{A}}_{\xi_0}, \psi^{\mathcal{A}}_{\xi_0}) = (I/4, 0, 0)$ and $(\varphi^{\mathcal{B}}_{\xi_0}, \theta^{\mathcal{B}}_{\xi_0}, \psi^{\mathcal{B}}_{\xi_0}) = (0, 0, 0)$.*
(b) *A trader that gets an endowment shock uses it to go long in the security and short in repo: $\varphi^i_\xi = \psi^i_\xi = a_\xi \equiv \frac{\gamma^{\mathcal{C}}_\xi}{1-h}$. For his repo counterpart, $\varphi^i_\xi = -ha_\xi$.*
(c) *At non-initial nodes where there are no endowment shocks, the trader that was long in the security before will go long again in the security and will go short in repo only if a shock has already occurred in the past.*

The security bubble is as in Example 2. It is shown in the Appendix (section 9.3.1) that

Claim 3: *the three agents are not uniformly impatient.*

Notice that as $\gamma^{\mathcal{C}}_\xi < I/2$ for every node ξ we could replace constraint (24) by an exogenous constraint $q_\xi \gamma_\xi \leq M$ where $0 < M \leq I/2$ and such constraint would never bind in equilibrium. The separation of issuing and trading decisions is crucial to get existence of equilibrium dispensing with uniform impatience. If we had merely required the portfolio value $q_\xi \varphi_\xi$ to be bounded by the present value of future endowments of the trader, $PVF^i_\xi(\lambda^i)$, computed with respect to an endogenous deflator λ^i (to coincide with the Euler deflator of this agent), then individual optimality might not be achieved. In fact, we should show that $\liminf_T \sum_{\xi:t(\xi)=T}[\lambda^i_\xi q_\xi(\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi)) - \tilde{\nu}^i_\xi q_\xi \varphi_\xi] \geq 0$ for any budget feasible plan, where $\tilde{\nu}^i_\xi$ stands for the shadow price of the constraint on the portfolio value. If these shadow prices were zero, individual optimality would follow (by an argument similar to the one we used above). However, for the two traders, $PVF^i_\xi(\lambda^i)$ is not uniformly bounded away from zero and, therefore, there is no reason to expect $\tilde{\nu}^i_\xi$ to be zero [24].

In this one-good and one-security setting, if the security had nominal returns A_ξ instead, we do the trick mentioned earlier, making $B_\xi = A_\xi/p_\xi$ and we close the model (ensuring commodity market clearing) using (25), which requires in this case that $A_\xi \gamma_{\xi^-} = q_\xi(\gamma_\xi - \gamma_{\xi^-}) + p_\xi(\omega^{\mathcal{C}}_\xi - x_\xi)$.

7 On the relevance of repo markets and short sales: a Pareto improvement

Being able to trade on repo constitutes a Pareto improvement: it never hurts agents and will actually, in general, make some (possibly all) agents better off. To illustrate this consider an example where dividends are stochastic and box shadow prices are not always zero.

7.1 An example with possession value for the security

Example 4: The economy in this example differs from the one in Example 2 in only two features: the trend endowment is now an unbounded sequence g_t and dividends B_ξ are stochastic. Notice that assumption (A1') doesn't hold as utility is not finite when evaluated at aggregate commodity endowments. However, this does not prevent us from constructing, for the segregated haircuts regulation (with $H = h$), an equilibrium with positive shadow prices μ^i_ξ for the box constraints. Security and repo positions are described as in graphs 3 and 4. Budget multipliers λ^i_ξ are as in Example 2 and the security is the numeraire.

Euler conditions with respect to security and repo positions (see the Appendix) imply that when $\theta^i_\xi > 0$ we must have $\sum_{\eta\in\xi^+} \lambda^i_\eta p_\eta B_\eta / \sum_{\eta\in\xi^+} \lambda^i_\eta = R_\xi$, whereas when $\psi^i_\xi > 0$ and $\mu^i_\xi > 0$ we must have $\sum_{\eta\in\xi^+} \lambda^i_\eta p_\eta B_\eta / \sum_{\eta\in\xi^+} \lambda^i_\eta > R_\xi$. When $\theta^i_\xi = \psi^i_\xi = 0$ we can have either the first or the second condition. This fact motivates the following choice for the stochastic dividends.

There is a dividend trend D_t to be specified below. Dividends are given by $B_\xi = D_{t(\xi)}(1+\epsilon_\xi)$, where $\epsilon_\eta = 0$ for $t(\eta) = 1$. For dates $t \geq 2$, the dividend shocks may be positive or null. For instance, we pick $\epsilon_{0du} > 0$ and $\epsilon_{0dd} = 0$ so that $\sum_{\eta\in 0d^+} \lambda^B_\eta p_\eta B_\eta / \sum_{\eta \in 0d^+} \lambda^B_\eta = p_2 D_2(1 + \frac{3}{4}\epsilon_{0du})$, whereas $R_{0d} = p_2 D_2(1 + \frac{1}{4}\epsilon_{0du})$.

[24] This observation is consistent with the existence argument in Magill and Quinzii (1994): uniform impatience is crucial in order to pass from an equilibrium with endogenous "transversality-type" constraints to an equilibrium with an implicit portfolio constraint and then to an equilibrium with exogenous portfolio constraints that never bind.

For a non-initial node ξ and $\xi^+ = \{\eta, \nu\}$, the shock ϵ_η is positive if the repo short i at ξ has $\rho^i_\eta > \rho^i_\nu$, and zero otherwise. The magnitude of the dividend shock, which is the same at contemporaneous nodes where dividend shocks occur, is $\epsilon_{t+1} = \frac{g_{t+1}\delta_{t+1}}{(1-1/2^{t+1})haD_{t+1}}$, with δ_{t+1} chosen in $(0,1)$ so that consumption is non-negative. The following claims, proven in the Appendix, complete the characterization of equilibrium. Let $\rho_t \equiv (2^{t(1+t)/2})^{-1}$ be the minimum of ρ^i_ξ over all contemporaneous nodes and both agents.

Claim (iv) *Suppose the trend endowment is $g_t = (\beta^t \rho_{t-1})^{-1} \to \infty$. Then box shadow prices are $\mu^i_\xi = (\rho^i_\xi/\rho_{t(\xi)})\mu_t$, where $\mu_t = \delta_{t+1}(1-1/2^t)(1-1/2^{t+1})^{-1}$. If $\delta_t = g_t^{-1}k^t$, for $t > 1$, and $D_{t+1} = k^{t+1}[1 - \frac{\beta(1-1/2^t)+(1-h)/2^{t+1}}{(1-1/2^{t+1})h}]$, then P_t is as in Example 2, assuming $h > 1/2$, $h > \beta/2 + 1/4$* and *$h > \beta + 1/8$ (say $h = 0.9$ and $\beta = 0.7$).*

It is possible to show that the above equilibrium under segregated haircuts fails to be an equilibrium for the weaker regulation consisting in no short selling of haircuts.

7.2 Repo improves upon

We use Example 4 to argue that without repo markets consumers would be worse off. An agent i that is short in repo at some node ξ has a positive shadow price for the box at this node and the present value of the coming dividends, $\sum_{\eta\in\xi^+} \frac{\lambda^i_\eta p_\eta B_\eta}{\lambda^i_\xi}$, exceeds the repo rate R_ξ at node ξ. His counterpart, the agent that is long in repo has a null shadow price and for him the repo rate coincides with the present value of the coming dividends.

If repo markets were not available, the agent that gets an endowment shock would purchase less of the security, as he would not be able to post this as collateral. For instance, at node ξ_{0d}, agent $\mathcal{B}$ would purchase just one unit of the security (the aggregate net supply), instead of $a = \frac{1}{1-h} > 1$ units that he is purchasing at this node. That is, leverage would not be done. His counterpart would have now a null position in the security, instead of short selling ($\varphi^{\mathcal{A}}_{\xi_{0d}}$ was $1 - a = \frac{-h}{1-h}$ in Example 4). In terms of consumption, there would be no impact at node $0d$ (since the security position net of the haircut repo position reported in Example 2 is the same as the new security position), but at the next nodes consumption would be affected, since the change in dividends is not cancelled out by the repo interest.

In the absence of repo markets, we have $x^{\mathcal{A}}_{\xi_{0du}} = g_2$, $x^{\mathcal{B}}_{\xi_{0du}} = g_2 + P_2 + B_{\xi_{0du}}$, $x^{\mathcal{A}}_{\xi_{0dd}} = g_2$ and $x^{\mathcal{B}}_{\xi_{0dd}} = g_2 + B_{\xi_{0dd}}$, where $P_2 = 1/p_2$. Comparing with the consumption levels reported in Example 4 we see that the introduction of repo makes agent $\mathcal{A}$ change his consumption in both nodes by $\frac{R_1 ha}{p_2} - B_\eta ha \equiv \Delta_\eta$, whereas the consumption of agent $\mathcal{B}$ changes by $-\Delta_\eta$ at $\eta \in \xi_{0d}$. Given the relation between the repo rate and the present value of dividends that we identified above, it follows that when repo markets are available agent $\mathcal{B}$ (the one that will be short in repo) will gain whereas his counterparty will be indifferent.

Then, at the next node where an endowments shock occurs, node ξ_{0du}, it will be agent $\mathcal{A}$ instead that will be short in repo and benefit from the change in consumption taking place at nodes ξ_{0duu} and ξ_{0dud}, while agent $\mathcal{B}$ stays indifferent. At nodes that immediately follow $\xi = (..., d, u)$ agent $\mathcal{A}$ gains and $\mathcal{B}$ stays indifferent, while at nodes that immediately follow $\xi = (..., u, d)$ agent $\mathcal{B}$ gains and $\mathcal{A}$ stays indifferent. On the whole both agents gain and the introduction of repo markets constitutes a Pareto improvement.

The trend endowments g were chosen in Example 4 to be unbounded so that box shadow prices would not tend to zero, with the purpose of showing that an equilibrium with segregated haircuts may fail to be an equilibrium under no short-selling of the haircut. However, we can choose the trend endowment to be as in Examples 1, 2 and 3, $g_t = 1$, for any date t, and we get an example where there is a bubble for deflators yielding finite present values of aggregate endowments, under segregated haircuts (or in the dealers/non-dealers context), where repo creates leverage and constitutes a Pareto improvement with respect to pure trading (with non-negative positions) of the securities.

8 Conclusions

Once we explicitly take into account the way securities are actually shorted, by borrowing them first in repo markets, we find that there are mechanisms that bound leverage and prevent infinite lived agents from doing Ponzi schemes. Existence of equilibrium dispenses with any uniform impatience assumptions. In this context, we see reappearing the main insight in Santos and Woodford (1996): bubbles in positive net supply securities cannot occur when markets are complete, but may occur in incomplete markets (for deflators with finite present value of wealth) when consumers are not uniformly impatient. However, that room for bubbles seemed to be quite narrow in models without repo markets, as, in the absence of uniform impatience, short sales apparently had to be ruled out (as in the examples by Santos and Woodford (1996) or Páscoa, Petrassi and Torres-Martinez (2011)).

We consider two procedures that limit the re-hypothecation of the security (and the resulting leverage) and prevent Ponzi schemes. One, which has been increasingly advocated after Lehman's bankruptcy, consists in not reusing (shorting or lending) the haircut collected when borrowing a security. The other is the current arrangement that limits by regulation the positions of dealers. We illustrate the bubble in both cases and discuss how the bubble might burst.

We show that consumers might be worse off if repo markets were absent and also show that bubbles can be robust to endogenous issuance. In the literature, the assets' positive net supply results from initial holdings at the first date and issuance at other nodes of the event-tree is not being considered. We provide an example where the separation of trading and issuance decisions allows for the bubble, under non-uniform impatience. We are far from having examined all the implications of issuance, in particular that one should take into account that a large issuance may decrease the security price - raise the interest paid on debt. Price taking might be questionable in that context. Repo fails or the counterparties' default were also not addressed, but there may be interesting substitution effects between not honoring repo agreements and running a Ponzi scheme.

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9 Appendix

9.1 On individual optimality

To prove Lemmas 1.I and 1.II we define agent i's Lagrangian at node ξ, $L^i_\xi(a_\xi, a_{\xi^-}; \lambda^i_\xi, \mu^i_\xi, c^i_\xi)$. We denote by $g^i_\xi(a_\xi, a_{\xi^-}; p, q, r)$ the function on the left hand side of the budget constraint and by $f^i_\xi(a_\xi)$ the function on the left hand side of the box constraint, where $a_\xi \equiv (x_\xi, \varphi_\xi, \theta_\xi, \psi_\xi))$ denotes an agent's plan at node ξ. The functions on the left hand side of dealers' upper bound constraints are denoted by $C_{j\xi}(a_\xi, M_j)$ and $G_{j\xi}(a_\xi, N_j)$. Multipliers for the budget, box and dealers' bounds constraints are λ^i_ξ, μ^i_ξ, c^i_ξ and k^i_ξ respectively.

$$L^i_\xi(a_\xi, a_{\xi^-}; \lambda^i_\xi, \mu^i_\xi, c^i_\xi) \equiv u^i_\xi(x_\xi) - \lambda^i_\xi g^i_\xi(a_\xi, a_{\xi^-}) + \mu^i_\xi f^i_\xi(a_\xi) - c^i_\xi C_\xi(a_\xi, M) - k^i_\xi G_\xi(a_\xi, N) \quad (27)$$

For non-dealers, in the constrained dealers case, or for all agents in the case II, we take $c^i_\xi \equiv 0$ and $k^i_\xi \equiv 0$ as for these agents there are no bounds on positions. Finally, let $L^i_{1\xi}$ and

$L^i_{2\xi}$ denote super-gradient vectors of L^i_ξ with respect to present and previous plans a_ξ and a_{ξ^-}, respectively. Let us state first three results and then we prove them.

Lemma A.1 (**Necessary conditions**): *Suppose assumptions (A1) and (A2) hold. If $\bar{a}^i \equiv (x^i, \varphi^i, \theta^i, \psi^i)$ is an optimal plan for agent i such that $U^i(x^i) < \infty$, then there exist multipliers $(\lambda^i_\xi, \mu^i_\xi, c^i_\xi) \geq 0$ such that $\mu^i_{j\xi} f^{i,H}_{j\xi}(a_\xi) = 0$, $c^i_{j\xi} C_\xi(a_{j\xi}, M_j) = 0$ and $k^i_{j\xi} G_{j\xi}(a_{j\xi}, N_{j\xi}) = 0$, such that the following Euler conditions are satisfied*

$L^i_{1\xi}(\bar{a}^i) + \sum_{\eta\in\xi^+} L^i_{2\eta}(\bar{a}^i) \leq 0$, $(L^i_{1\xi}(\bar{a}^i) + \sum_{\eta\in\xi^+} L^i_{2\eta}(\bar{a}^i))\bar{a}^i_\xi = 0$

and the following transversality condition holds

$$\lim\sup_{T\to\infty} \sum_{\xi:t(\xi)=T} \left(\sum_{\eta\in\xi^+} L^i_{2\eta}(\bar{a}^i)\bar{a}^i_\xi \right) \leq 0 \tag{28}$$

Let us see what the Euler conditions look like. In case (I), let $A^i_\xi = \lambda^i_\xi - r_{2j\xi}\sum_{\eta\in\xi^+}\lambda^i_\eta + \mu^i_\xi q^{-1}_{j\xi}$ and $B^i_\xi = \lambda^i_\xi - r_{1j\xi}\sum_{\eta\in\xi^+}\lambda^i_\eta + \mu^i_\xi (q_{j\xi}h_{j\xi})^{-1}$. Then, *in case (I), Euler conditions* hold for a non-dealer if (i) $(u^i_\xi)' = \lambda_\xi p_\xi$ and (ii) $\lambda^i_\xi q_{j\xi} = \sum_{\eta\in\xi^+}\lambda^i_\eta(p_\eta B_{j\eta} + q_{j\eta}) + \mu^i_{j\xi}$ hold, together with $A^i_\xi \leq 0$, $A^i_\xi\theta^i_\xi = 0$, $B^i_\xi \leq 0$ *and* $B^i_\xi\psi^i_\xi = 0$.

For a dealer, replace A^i_ξ by $\lambda^i_\xi - r_{1j\xi}\sum_{\eta\in\xi^+}\lambda^i_\eta + \mu^i_\xi(q_{j\xi}h_{j\xi})^{-1} - c^i_\xi h^{-1}_{j\xi}/\sum_k p_{k\xi}$ and replace B^i_ξ by $\lambda^i_\xi - r_{2j\xi}\sum_{\eta\in\xi^+}\lambda^i_\eta + \mu^i_\xi q^{-1}_{j\xi} - k^i_\xi$.

In case (II), let $V^i_{j\xi} = \lambda^i_\xi q_{j\xi}h_{j\xi} - r_{j\xi}q_{j\xi}h_{j\xi}\sum_{\eta\in\xi^+}\lambda^i_\eta$. *In case II, Euler conditions* hold if (i) and (ii) hold, together with $V^i_{j\xi} - \mu^i_{j\xi}H_{j\xi} \geq 0$, $(V^i_{j\xi} - \mu^i_{j\xi}H_\xi)\theta^i_{j\xi} = 0$, $V^i_{j\xi} - \mu^i_{j\xi} \leq 0$ and $(V^i_{j\xi} - \mu^i_{j\xi})\psi^i_{j\xi} = 0$.

In case (I), (28) is equivalent to the transversality condition claimed for a non-dealer, since $\mu^i_{j\xi}(\varphi^i_{j\xi} + z^i_\xi) = 0$. For a dealer, (28) is also equivalent to the claimed transversality condition, since $\lim_{T\to\infty}\sum_{\xi:t(\xi)=T}[c^i_{j\xi}(q_{j\xi}/\sum_k p_{k\xi})\theta^i_{j\xi} + k^i_{j\xi}q_{j\xi}\psi^i_{j\xi}] = 0$ due to (6) and (7) and the fact that $c^i_j \in l^1$ and $k^i_j \in l^1$ (by fact (a) below). In case (II), we see that (28) becomes the transversality condition claimed for case (II)

Lemma A.2 (**Sufficient conditions**): *Suppose assumption (A1) holds. Given prices (p, q, r), an admissible plan $\bar{a}^i$satisfying Euler and transversality conditions is optimal among all plans a^i, that satisfy*

$$\lim\sup_{T\to\infty} \sum_{\xi:t(\xi)=T} L^i_{1\xi}(\bar{a}^i)a^i_\xi \leq 0 \tag{29}$$

Lemma A.3: *In case (II) and, when $U^i(x^i) < \infty$, also in case (I), any plan a^i which is admissible for agent i at prices (p, q, r), is such that* (29) *will be satisfied.*

Lemma 1.I (for both dealers and non-dealers) and Lemma 1.II follow from Lemma A.3.

To prove Lemmas A.1-A.3 we define an optimization problem with finite horizon T. Let $D^T(\xi) = \cup^T_{k=t(\xi)} D_k(\xi)$ and $U^{iT}(x) = \sum_{\xi\in D^T(\xi_0)} u^i_\xi(x_\xi)$. At $t = T$ commodities can be traded, securities pay dividends but are no longer traded. At $T-1$ repo trades cannot be done and securities are traded under no-short-sales. Denote by a^{iT} a solution to this truncated problem, by x^{iT} its consumption component and by $(\lambda_\xi^{iT}, \mu_\xi^{iT}, c_\xi^{iT}, k_\xi^{iT})$ the associated multipliers. Then,

(a) $\lambda_\xi^{iT}\sum_l p_{l\xi} \leq U^i(x^{iT})/\underline{\omega}_\xi^i$, where $\underline{\omega}_\xi^i = \min_l \omega_{l\xi}^i$ and, furthermore, in case (I), for a dealer, both $\sum_{\xi:t(\xi)\leq T}\sum_j c_{j\xi}^{iT} M_j$ and $\sum_{\xi:t(\xi)\leq T}\sum_j k_{j\xi}^{iT} N_j$ are bounded by $U^i(x^{iT})$ (this follows by the saddle point property[25]).

(b) For each node ξ we have in case I that $\mu_{j\xi}^{iT} \leq \lambda_\xi^{iT} q_{j\xi} + c_{j\xi}^{iT}\frac{q_{j\xi}}{\sum_l p_{l\xi}}$, and in case II that $\mu_{j\xi}^{iT} \leq \lambda_\xi^{iT} q_{j\xi} h_{j\xi}/H_{j\xi}$ (this follows from the first order condition on $\theta_{j\xi}$).

Proof of Lemma A.1: Euler conditions follow from the Kuhn-Tucker conditions of the truncated problem and by noticing **(a)** and **(b)** imply that $(\lambda_\xi^{iT}, (\mu_{j\xi}^{iT})_j, (c_{j\xi}^{iT})_j, (k_{j\xi}^{iT})_j)_T$ has a cluster point $(\lambda_\xi^i, (\mu_{j\xi}^i)_j, (c_{j\xi}^i)_j, (k_{j\xi}^i)_j)$ for the countable product topology.

To prove (28) we use the saddle point property to obtain $-\sum_{\xi:t(\xi)=t}\lambda_\xi^{iT}\cdot\mathcal{D}_2 g_\xi^i(p,q,r)\cdot \bar{a}_{\xi^-}^i + \sum_{\xi:T\geq t(\xi)>t}\lambda_\xi^{iT} p_\xi \omega_\xi^i \leq \sum_{\xi:t(\xi)\geq t} u_\xi^i(\bar{x}_\xi^i)$. By (A1)(ii) the series of utilities converges for feasible plans. Then, $L_{2\xi}^i(a_\xi, a_{\xi^-}) = -\lambda_\xi^i \cdot \mathcal{D}_2 g_\xi^i(p,q,r)$ implies (28). ■

Proof of Lemma A.2: Consider any plan a^i satisfying the budget, box and sign constraints. Let x^i its respective consumption plan. Notice that

$$U^{iT}(x) - U^{iT}(\bar{x}^i) \leq \sum_{\xi:t(\xi)\leq T}(L_\xi^i(a^i) - L_\xi^i(\bar{a}^i)) \leq \sum_{\xi:t(\xi)\leq T}(L_{1\xi}^i(\bar{a}^i), L_{2\xi}^i(\bar{a}^i))((a_\xi^i, a_{\xi^-}^i) - (\bar{a}_\xi^i, \bar{a}_{\xi^-}^i)) =$$

$$= \sum_{\xi:t(\xi)<T}(L_{1\xi}^i(\bar{a}^i) + \sum_{\eta\in\xi^+} L_{2\eta}^i(\bar{a}^i))(a_\xi^i - \bar{a}_\xi^i) + \sum_{\xi:t(\xi)=T} L_{1\xi}^i(\bar{a}^i)(a_\xi^i - \bar{a}_\xi^i)$$

By Lemma A.1, we get $\limsup_{T\to\infty}(U^{iT}(x) - U^{iT}(\bar{x}^i)) \leq \limsup_{T\to\infty}\sum_{\xi:t(\xi)=T} L_{1\xi}^i(\bar{a}^i) a_\xi$. ■

Proof of Lemma A.3:

In case I:

For a non-dealer, (29) requires that for any plan (φ, θ, ψ) satisfying budget and box constraints we must have $\limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}\mathcal{E}_{j\xi}^i \leq 0$, where $\mathcal{E}_{j\xi}^i \equiv -\lambda_\xi^i q_{j\xi}(\varphi_{j\xi} + \theta_{j\xi} - h_{j\xi}\psi_{j\xi}) + \mu_{j\xi}^i(\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi})$. Now, $\mathcal{E}_{j\xi}^i \leq (-\lambda_\xi^i q_{j\xi} + \mu_{j\xi}^i)(\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi})$ since $(\lambda_\xi^i q_{j\xi} h_{j\xi} - \mu_{j\xi}^i)\psi_{j\xi} \leq (\lambda_\xi^i q_{j\xi} - \mu_{j\xi}^i)\psi_{j\xi}$. Now, $-\lambda_\xi^i q_{j\xi} + \mu_{j\xi}^i < 0$ (from the Euler condition in φ_ξ), so $\mathcal{E}_\xi^i \leq 0$ and (29) holds.

Let us now look at the dealers' case. For any (φ, θ, ψ) satisfying budget and box constraints, with $(\theta, \psi) \geq 0$, $|q_{j\xi}\varphi_{j\xi}^i| \leq M_j$ and $q_{j\xi}\psi_{j\xi} \leq N_j$, the constraint (29) holds if, for any j, $\limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}\mathcal{B}_{j\xi}^i \leq 0$, where $\mathcal{B}_{j\xi}^i \equiv -\lambda_\xi^i q_{j\xi}(\varphi_{j\xi} + h_{j\xi}\theta_{j\xi} - \psi_{j\xi}) + \mu_{j\xi}^i(\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi}) - c_{j\xi}^i q_{j\xi}\theta_{j\xi}/\sum_l p_{l\xi} - k_{j\xi}^i q_{j\xi}\psi_{j\xi} = (-\lambda_\xi^i q_{j\xi} + \mu_{j\xi}^i)(\varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi}) + \lambda_\xi^i(1 - h_{j\xi})q_{j\xi}\theta_{j\xi} - c_{j\xi}^i q_{j\xi}\theta_{j\xi}/\sum_l p_{l\xi} - k_{j\xi}^i q_{j\xi}\psi_{j\xi} \leq \lambda_\xi^i\sum_l p_{l\xi}(1 - h_{j\xi})q_{j\xi}\theta_{j\xi}/\sum_l p_{l\xi}$. By Lemma B, (6) and (7) we get $\limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}\sum_j \mathcal{B}_{j\xi}^i \leq 0$.

In case II, as $H_{j\xi} \leq h_{j\xi}$, constraint (29) holds. In fact, $-\lambda_\xi^i q_\xi(\varphi_\xi + h_{j\xi}(\theta_{j\xi} - \psi_{j\xi}) + \mu_{j\xi}^i(\varphi_{j\xi} + H_{j\xi}\theta_{j\xi} - \psi_{j\xi}) \leq (-\lambda_\xi^i q_\xi + \mu_{j\xi}^i)(\varphi_\xi + h_{j\xi}\theta_{j\xi} - \psi_{j\xi}) \leq 0$, since $-\lambda_\xi^i q_\xi + \mu_{j\xi}^i < 0$ by the Euler conditions on $\varphi_{j\xi}^i$.■

[25]Notice that Uzawa (1958) constraint qualification holds by making $x_{\xi_0} = \omega_{\xi_0}^i$, $\varphi_\xi = e_\xi^i$ and for $\xi \neq \xi_0$, $x_\xi = \omega_\xi^i + B_\xi e^i$ (so that budget constraints hold with equality and box constraints with strict inequality).

9.2 Proof of Theorems 1 and 2

Existence of equilibrium for a finite horizon T economy (under assumptions (A1) and (A1")) was established in Bottazzi, Luque and Páscoa (2012) for the case of constrained dealers and the case of no short-selling of the haircut (which in a finite horizon is not distinguishable from the segregated haircuts case). We denote such finite horizon equilibrium by $(p^T, q^T, r^T, \bar{a}^T)$ and normalize (p^T_ξ, q^T_ξ) in the simplex. Facts **(a)** and **(b)** hold for the associated sequence of multipliers and we also have the following three facts:

(c) $\sum_l p^T_{l\xi_0}$ is bounded away from zero (by monotonicity).

(d) For each node ξ the sequence $(\lambda^{iT}_\xi)_T$ is bounded.

For the initial node, this follows from (a) and (c). For the next nodes, say, we use the Euler equation on security positions: if $\lambda^{iT}_\eta \to \infty$ for some $\eta \in \xi_0^+$, then, $q^T_{j\eta}$ should go to 0 for every j, implying that $\sum_l p^T_{l\eta} \to 1$ and (by **(a)**) that $\lambda^{iT}_\eta \nrightarrow \infty$, a contradiction.

(e) Under (A1"), the sequence $(r^T_\xi)_T$ is bounded for each ξ.

This follows from Euler conditions on repo trades and the fact that marginal rates of substitution are bounded from above and below.

Hence, we can find a cluster point $(p, q, r, \bar{a}, (\lambda^i_\xi, \mu^i_\xi, c^i_\xi, k^i_\xi)_{i,\xi})$, (for the countable product topology), which satisfies Euler and transversality condition, by the argument in the proof of Lemma A.1. Then, by Lemmas A.2 and A.3, individual optimality holds. ■

9.3 Bubbles

Proof of Proposition 1, the complete markets case:

The transversality condition requires $\limsup_T \sum_{\xi:t(\xi)=T} \left(-L^i_{1\xi}(\bar{a}^i) \cdot \bar{a}^i_\xi\right) \leq 0$, which implies under complete markets (for null shadow prices of box constraints and dealers' bounds)

$$\limsup_T \sum_{\xi \geq \eta: t(\xi)=T} \lambda_\xi q_\xi (\sum_i \bar{\varphi}^i_\xi + h_\xi(\sum_d \theta^d_\xi - \sum_{nd} \psi^{nd}_\xi) + (\sum_{nd} \theta^{nd}_\xi - \sum_d \psi^d_\xi)) \leq 0$$

By market clearing $\limsup_T \sum_{\xi\geq\eta:t(\xi)=T} \lambda_\xi q_\xi e \leq 0$, then $\lim_T \sum_{\xi\geq\eta:t(\xi)=T} \lambda_\xi q_\xi = 0$. ■

Proof of Proposition 1, the uniform impatience case:

We take the equilibrium plans of *non-dealers* and modify them as follows, $(\varphi^i_\eta, \theta^i_\eta, \psi^i_\eta) \mapsto \pi(\varphi^i_\eta, \theta^i_\eta, \psi^i_\eta)$,for $\eta \geq \xi$. Under uniform impatience an appropriate choice of (Δ_ξ, π) requires $(1-\pi) q_\xi (\varphi^i_\xi + \theta^i_\xi - h_\xi \psi^i_\xi) < p_\xi \Delta_\xi$, where $\varphi^i_\xi + \theta^i_\xi - h_\xi \psi^i_\xi \geq 0$ by the box constraint. So, it follows that $(q_\xi / p_\xi \omega^m_\xi) \sum_{i\in\mathbf{ND}} (\varphi^i_\xi + \theta^i_\xi - h_\xi \psi^i_\xi) < k^{-1}(\sharp\mathbf{ND})/(1-\pi)$, for any $m \in \mathbf{ND}$.

Now, by (A3) we know that $(q_\xi / \sum_l p_{l\xi}) \sum_{i\in\mathbf{D}} \theta^i_\xi$ is uniformly bounded and, therefore, $(q_\xi / \sum_l p_{l\xi}) \sum_{i\in\mathbf{D}} \varphi^{i-}_\xi$ is also uniformly bounded (by the box constraint). It follows that $(q_\xi / \sum_l p_{l\xi}) \sum_{i\in\mathbf{D}} (\varphi^i_\xi - \psi^i_\xi + h_\xi \theta^i_\xi)$ is uniformly bounded.[26] Putting the two results together we have that $(q_\xi / p_\xi \omega^m_\xi)(\sum_{i\in\mathbf{I}} \varphi^i_\xi + \sum_{i\in\mathbf{ND}} \theta^i_\xi - \sum_{i\in\mathbf{D}} \psi^i_\xi - h_\xi(\sum_{i\in\mathbf{ND}} \psi^i_\xi - \sum_{i\in\mathbf{D}} \theta^i_\xi))$ is

[26] The uniform bounds from below and from above follow, respectively, from $\varphi^i_\xi - \psi^i_\xi \geq -\theta^i_\xi$ and $\sum_{i\in\mathbf{D}} (\varphi^i_\xi - \psi^i_\xi) = \sum_{i\in\mathbf{ND}} (\varphi^i_\xi - \theta^i_\xi) < \sum_{i\in\mathbf{ND}} (\varphi^{i+}_\xi - \theta^i_\xi) = \sum_{i\in\mathbf{D}} (\varphi^{i-}_\xi - \psi^i_\xi)$.

uniformly bounded (as $\sum_l p_{l\xi}/p_\xi\omega^m_\xi \leq 1/\inf_\xi \omega^m_\xi$). Hence, $(q_\xi/p_\xi\omega^m_\xi)_\xi \in l^\infty$ and we get $\lim_{t\to\infty}\sum_{\xi:t(\xi)=t}\gamma_\xi q_\xi = 0$ as claimed for γ with $\sum_{t=0}^\infty\sum_{\xi:t(\xi)=t}\gamma_\xi p_\xi\omega^m_\xi < \infty$. ■

Proof of Proposition 2, the complete markets case:

We write the transversality condition as

$$\limsup_T \sum_{\xi:t(\xi)=T}[(-\mathcal{D}u^i_\xi + \lambda^i_\xi p_\xi)\bar{x}^i_\xi + (\lambda^i_\xi q_\xi - \mu^i_\xi)\bar{\varphi}^i_\xi + (\lambda^i_\xi q_\xi h_\xi - \mu^i_\xi H_\xi)\bar{\theta}^i_\xi - (\lambda^i_\xi q_\xi h_\xi - \mu^i_\xi)\bar{\psi}^i_\xi] \leq 0$$

Now, $\mu^i_\xi(\bar{\varphi}^i_\xi + H_\xi\bar{\theta}^i_\xi - \bar{\psi}^i_\xi) = 0$ implies $\limsup_T\sum_{\xi:t(\xi)=T}\lambda^i_\xi(q_\xi\bar{\varphi}^i_\xi + q_\xi h_\xi(\bar{\theta}^i_\xi) - \bar{\psi}^i_\xi))$ is bounded by $\limsup_T\sum_{\xi:t(\xi)=T}\lambda^i_\xi(\mathcal{D}u^i_\xi - \lambda^i_\xi p_\xi)\bar{x}^i_\xi \leq 0$. Given any η, $\lambda^i_\xi/\lambda^i_\eta = \lambda_\xi$ for $\xi \geq \eta$ and all i,

$$\limsup_T \sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi(\sum_i \bar{\varphi}^i_\xi + h_\xi\sum_i \bar{z}^i_\xi) \leq \sum_i \limsup_T \sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi(\bar{\varphi}^i_\xi + h_\xi\bar{z}^i_\xi) \leq 0$$

Hence, $\limsup_T\sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi e \leq 0$, implying $\limsup_T\sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi \leq 0$. So $0 \leq \lim_T\sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi \leq \lambda_\eta q_\eta < \infty$ and $\lim_T\sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi \leq \limsup_T\sum_{\xi:t(\xi)=T}\lambda_\xi q_\xi$. Then, $\lim_T\sum_{\xi\geq\eta:t(\xi)=T}\lambda_\xi q_\xi = 0$.

Proof of Proposition 2, the uniform impatience case:

As in the proof of Proposition 1, $(1-\pi)q_\xi(\varphi^i_\xi + h_\xi(\theta^i_\xi - \psi^i_\eta)) < p_\xi\Delta_\xi$, so that optimality is not contradicted. Notice that in case II (see proof of Lemma A.3) we have $\varphi^i_\xi + h_\xi(\theta^i_\xi - \psi^i_\eta) \geq 0$. It follows that $0 \leq (q_\xi/p_\xi\Delta_\xi)(\varphi^i_\xi + h_\xi(\theta^i_\xi - \psi^i_\eta)) < 1/(1-\pi)$. Adding across consumers we see that $q_\xi/p_\xi\Delta_\xi$ is uniformly bounded. If γ is such that $\sum_{t=0}^\infty\sum_{\xi:t(\xi)=t}\gamma_\xi p_\xi\omega^i_\xi < \infty$, using again Definition 3, we get $\lim_{t\to\infty}\sum_{\xi:t(\xi)=t}\gamma_\xi q_\xi = 0$ as claimed.

Proof of Lemma 2: By the saddle point property, take $a_\eta = (0,0,0)$ for every node η, then, $\sum_{\eta:t(\eta)\leq T}\lambda^i_\eta p_\eta\omega^i_\eta + \lambda^i_{\xi_0}q_{\xi_0}e^i + \sum_j\sum_{\eta:t(\eta)\leq T}(c^i_{j\eta}M_j + k^i_{j\eta}N_{j\xi} \leq U^i(\bar{x}^i)$, which implies that $\sum_{\eta:t(\eta)\leq T}\lambda^i_\eta p_\eta\omega^i_\eta$ converges. ■

9.3.1 Details of Example 3

Proof of Lemma 1.III

Denote the shadow prices of the upper-bounds on issuance at node ξ by $\nu^{\mathcal{C}}_\xi$. The issuer's Euler condition on issuance is $E^{\mathcal{C}}_\xi \equiv \lambda^{\mathcal{C}}_\xi q_\xi - \sum_{\eta\in\xi^+}\lambda^{\mathcal{C}}_\eta(q_\eta + p_\eta B_\eta) - \nu^{\mathcal{C}}_\xi q_\xi \leq 0$, together with $E^{\mathcal{C}}_\xi\gamma^C_\xi = 0$. Notice that Euler conditions imply that when the accumulated issuance is always positive we have $\mu^{\mathcal{C}}_\xi = \nu^{\mathcal{C}}_\xi q_\xi$, where $\mu^{\mathcal{C}}_\xi \leq \lambda^{\mathcal{C}}_\xi q_\xi$. Hence, the transversality condition is always satisfied when the security is always in positive net supply (actually, even if the issuer would trade in security or repo markets, in both cases I or II).

In the segregated haircuts context, $(x^{\mathcal{C}}, \gamma^{\mathcal{C}})$, together with a no-trade plan in security and repo, satisfying (25), (24), the Euler conditions on issuance and (26), together with the usual Euler conditions on consumption, security and repo trades, will be optimal for the issuer among all budget and box feasible plans $(x,\gamma,\varphi,\theta,\psi)$ that satisfy (24) and the following (to simplify we assume H=h):

$$\liminf_T \sum_{\xi:t(\xi)=T}[\lambda^{\mathcal{C}}_\xi q_\xi(\varphi_\xi + h\theta_\xi - h\psi_\xi) - \mu^{\mathcal{C}}_\xi(\varphi_\xi + h\theta_\xi - \psi_\xi) + (-\lambda^{\mathcal{C}}_\xi q_\xi + \nu^{\mathcal{C}}_\xi q_\xi)\gamma^{\mathcal{C}}_\xi)] \geq 0 \quad (30)$$

When box shadow prices are null, (24) ensure that (30) is satisfied when $\alpha = \lambda^{\mathcal{C}}$. In fact, $\lambda^{\mathcal{C}} \in NA(q)$ and $\limsup_T \sum_{\xi:t(\xi)=T} \lambda^{\mathcal{C}}_\xi q_\xi \gamma^{\mathcal{C}}_\xi = 0$ since $\lambda^{\mathcal{C}}_\xi q_\xi \gamma_\xi \leq \sum_{\eta>\xi} \lambda^{\mathcal{C}}_\eta p_\eta \omega^{\mathcal{C}}_\eta$, which tends to zero as $t(\xi)$ increases if $U^{\mathcal{C}}(x^{\mathcal{C}})$ is finite (see Lemma 1).

Proof of Claim (i):

Notice that we can find a plan $\gamma^{\mathcal{C}}$ which is such that $\gamma^{\mathcal{C}}_\xi$ is uniformly bounded away from zero and actually increasing. Notice that as $\omega^{\mathcal{C}} = W/2$ and $\lambda^{\mathcal{C}} = (\lambda^{\mathcal{A}} + \lambda^{\mathcal{B}})/2$ we have that $PVF^{\mathcal{C}}_\xi(\lambda^{\mathcal{C}})$ is the average of $n^{\mathcal{A}}_\xi \equiv (PV^{\mathcal{A}}_\xi - p_\xi W_\xi)/2$ and $n^{\mathcal{B}}_\xi \equiv (PV^{\mathcal{B}}_\xi - p_\xi W_\xi)/2$. Now, $n^{\mathcal{A}}_\xi$ and $n^{\mathcal{B}}_\xi$ coincide with what were the present values for agents $\mathcal{A}$ and $\mathcal{B}$ of future aggregate endowments in Examples 1 and 2.

In fact, $n^i_\xi = F_{t(\xi)} + J^i_\xi Y_{t(\xi)+1}]Y^{-1}_{t(\xi)}$, where $Y_t = 1 - \frac{k\beta}{1-k\beta}(1-(k\beta)^t$, $J^A_\xi = 1 - 1/2^{t(\xi)}$ at $\xi \in D^{du} \cup D^{uu}$ and $J^A_\xi = 1/2^{t(\xi)}$ at $\xi \in D^{ud} \cup D^{dd}$ and $F_t = 2\frac{\beta^{t+1}}{1-\beta} + \frac{3}{2^{t+1}} - \frac{1/3}{4^t}$. For the other trader, $J^B_\xi = 1 - J^A_\xi$.

Now, for $I = \frac{1-2k\beta}{(1-k\beta)^2} > 0$, where $k\beta < 1/2$, we have the following. When $\xi \in D^{uu}$ or $\xi \in D^{du}$ we get $n^{\mathcal{A}}_\xi \geq I(1-2^{-t(\xi)-1})$ and $n^{\mathcal{B}}_\xi \geq I(2^{-t(\xi)-1})$, but when $\xi \in D^{dd}$ or $\xi \in D^{ud}$ we get $n^{\mathcal{B}}_\xi \geq I(1-2^{-t(\xi)-1})$ and $n^{\mathcal{A}}_\xi \geq I(2^{-t(\xi)-1})$. So, $PVF^{\mathcal{C}}_\xi(\lambda^{\mathcal{C}})$ is bounded from below by $I/2$. We make $\gamma^{\mathcal{C}}_\xi = (I/2)\sum_{s=0}^{t(\xi)} 1/2^{s+1}$, so that the security net supply $\gamma^{\mathcal{C}}_\xi$ is deterministic and increases towards $I/2 > 0$ as $t(\xi) \to \infty$ (assuming $k\beta < 1/2$).

Proof of Claim (ii)

For a trader i that at node ξ is short in repo we have $\psi^i_\xi = a_\xi$ and $\varphi^i_\xi = a_\xi$, implying $\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi) = (1-h)a_\xi$. For his repo counterpart we get $\varphi^i_\xi = -ha_\xi$ and $\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi) = 0$. Now, $\varphi^{\mathcal{A}}_\xi + \varphi^{\mathcal{B}}_\xi = \gamma^{\mathcal{C}}_\xi$ and, therefore, $a_\xi = \frac{\gamma^{\mathcal{C}}_\xi}{1-h}$.

For a trader i the budget constraint implies that $x^i_\xi = \omega^i_\xi - P_{t(\xi)}[\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi) - (\varphi^i_{\xi^-} + h(\theta^i_{\xi^-} - \psi^i_{\xi^-}))] + B_{t(\xi)}(\varphi^i_{\xi^-} + h(\theta^i_{\xi^-} - \psi^i_{\xi^-}))$. At the initial node, the claimed financial positions imply $x^{\mathcal{A}}_{\xi_0} = 1 - I/4$ (notice that $I < 4$) and $x^{\mathcal{B}}_{\xi_0} = 1$, while $x^{\mathcal{C}}_{\xi_0} = 2 + I/4$.

At a node where there are endowment shocks, the trader that gets that shock uses it to go long in the security and short in repo, consuming $x^i_\xi = \omega^i_\xi - P_{t(\xi)}\gamma^{\mathcal{C}}_\xi$. His repo counterpart consumes $x^i_\xi = \omega^i_\xi + (P_{t(\xi)} + B_{t(\xi)})\gamma^{\mathcal{C}}_{\xi^-}$ and $x^{\mathcal{C}}_\xi = \omega^{\mathcal{C}}_\xi + P_{t(\xi)}\gamma^{\mathcal{C}}_\xi - (P_{t(\xi)} + B_{t(\xi)})\gamma^{\mathcal{C}}_{\xi^-}$.

At non-initial nodes where there are no shocks, the trader that was long in the security at the preceding node will go long again in the security, consume $x^i_\xi = \omega^i_\xi - P_{t(\xi)}\gamma^{\mathcal{C}}_\xi + (P_{t(\xi)} + B_{t(\xi)})\gamma^{\mathcal{C}}_{\xi^-}$ and will go short in repo only if a shock has already occurred in the past. His repo counterpart consumes the endowment and the issuer consumes as in the other nodes.

Let us check the non-negativity of consumption. We need to show that if trader i gets an endowment shock at node ξ, we have $x^i_\xi = 1 - P_{t(\xi)}(\sum_{s=0}^{t(\xi)} 1/2^{s+1} I/2 - 1) \geq 0$. For $\beta = k = 1/2$, we get $I/2 = 4/9$ and $\sum_{s=0}^{t(\xi)} 1/2^{s+1} I/2 < 1$, so that $x^i_\xi > 0$. At a node where there are no shocks, the trader that does not consume the endowment has his consumption given by $1 + (1/2)^{t(\xi)}(1-(1/2)^t)I/2 - P_t(1/2)^{t+1})I/2 = 1 + (4/9)(1/2^t - (1/4)^{t+1}) - 2/9(2/3 + 1/3(1/4)^t) > 0$. For the issuer, $x^{\mathcal{C}} \geq 0$ follows from $k^t\gamma^{\mathcal{C}}_{t-1} \leq 4/9 < \omega^{\mathcal{C}}_\xi$.

Finally, the transversality condition of a trader holds since $\limsup_T \sum_{\xi:t(\xi)=T} \lambda^i_\xi(\varphi^i_\xi + hz^i_\xi) = \frac{(1-2k\beta)^2}{2(1-k\beta)^3} \limsup_T \sum_{\xi:t(\xi)=T} \rho^i_\xi \delta^i_\xi$ where $\delta^i_\xi = 1 - (1/2)^{t(\xi)+1}$ if $\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi) > 0$ and is zero otherwise. Hence, the result follows since we know already from Example 1 that

$\limsup_T \sum_{\xi:t(\xi)=T} \rho^i_\xi \zeta^i_\xi = 0$ where $\zeta^i = 1$ when $\varphi^i_\xi + h(\theta^i_\xi - \psi^i_\xi) > 0$ and is zero otherwise. It is immediate to see that for the issuer the transversality condition holds, since $\nu^C = 0$.

Proof of Claim (iii)

In fact, the uniform impatience criterion, applied at the aggregate endowment plan, requires the existence of $\pi \in (0,1)$ and $k > 0$ so that at any node ξ the following inequality holds: $\frac{\omega^i_\xi}{k} > \frac{1-\pi}{\beta^{t(\xi)}\rho^i_\xi} \sum_{\eta>\xi} \beta^{t(\eta)} \rho^i_\eta W_\eta$. Then we must have $P^{-1}_{t(\xi)} \frac{\omega^i_\xi}{k(1-\pi)} + W_\xi > \frac{1}{\beta^{t(\xi)}\rho^i_\xi P_{t(\xi)}} \sum_{\eta \geq \xi} \beta^{t(\eta)} \rho^i_\eta W_\eta \equiv PV^i_\xi$. Now, for $\mu \in D^{uu}$ we have $PV^{\mathcal{A}}_\xi$ and $PV^{\mathcal{C}}_\xi$ both greater than $I/2$, whereas for $\mu \in D^{dd}$ we have $PV^{\mathcal{B}}_\xi$ and $PV^{\mathcal{C}}_\xi$ both greater than $I/2$, where $I = \frac{1-2k\beta}{(1-k\beta)^2} > 0$. At both $\xi \in D^{uu}$ and $\xi \in D^{dd}$ we have $\omega^{\mathcal{A}}_\xi = \omega^{\mathcal{B}}_\xi = 1$ and $\omega^{\mathcal{C}}_\xi = 2$. Now, given any T arbitrarily large there are always nodes $\xi \in D^{uu}$ and $\xi \in D^{dd}$ such that $t(\xi) > T$. Then we get a contradiction since P^{-1}_T tends to zero.

9.4 Details of Example 4

At the nodes at immediately follow ϵ_{0d}, we have $x^A_{0du} = g_2 - \frac{3}{4} D_2 \epsilon_{0du} ha$, $x^B_{0du} = g_2 + P_2 + D_2(1+\epsilon_{0du}) + \frac{3}{4} D_2 \epsilon_{0du} ha$, $x^A_{0dd} = g_2 + \frac{1}{4} D_2 \epsilon_{0du} ha$ and $x^B_{0dd} = g_2 + D_2 - \frac{1}{4} D_2 \epsilon_{0du} ha$. Non-negativity of consumption holds for $\epsilon_{0du} = \frac{g_2 \delta_{0du}}{(1-1/4) D_2 ha}$, for some $\delta_{0du} \in (0,1)$.

Proof of Claim (iv) in Example 4:

Let us compute the box *shadow prices*. Euler conditions imply that for agent i with $\psi^i_\xi > 0$ we have $\mu^i_\xi = ha \sum_{\eta \in \xi^+} \lambda^i_\eta (p_{t(\eta)} B_\eta - R_\xi)$. For instance, $\mu^{\mathcal{B}}_{0d} = ha\beta^2 D_2 \epsilon_{0du} \rho^B_{0d} (3/4 - 1/4)$[27]. Euler condition on repo implies that both must have positive shadow prices for the box constraint at every node, even when they don't trade in repo.

We can make δ_ξ to be common to all contemporaneous nodes and we get $\epsilon_{t+1} = \frac{g_{t+1}\delta_{t+1}}{(1-1/2^{t+1}) ha D_{t+1}}$. Notice that ρ_t is equal to $1/2$ to the power of the sum of the first t terms of the arithmetic progression $1, 2, 3, ...$ and, therefore, $\rho_t = (2^{t(1+t)/2})^{-1}$. Then $\mu^i_\xi \geq \beta^{t(\xi)+1} \rho_t g_{t+1} \delta_{t+1} (1 - 1/2^t)(1 - 1/2^{t+1})^{-1} \equiv \mu_t$. We choose $g_t = (\beta^t \rho_{t-1})^{-1} \to \infty$. Then, $\mu^i_\xi = (\rho^i_\xi / \rho_{t(\xi)}) \mu_t$, where $\rho^i_\xi / \rho_{t(\xi)} \geq 1$ and $\mu_t = \delta_{t+1}(1 - 1/2^t)(1 - 1/2^{t+1})^{-1}$.

It remains to see what *prices* are. From the Euler condition on repo positions we get the following difference equation $P_t = \beta(P_{t+1} + D_{t+1}(1 + \epsilon_{t+1}/2^{t+1})) + \frac{\mu^i_t}{h_t \beta^t \rho_t}$. We make $\delta_t = g^{-1}_t k^t$, for $t > 1$. Then the difference equation becomes $P_{t+1} = \frac{1}{\beta} P_t - f_{t+1}$, where $f_{t+1} \equiv D_{t+1} + k^{t+1} \frac{\beta(1-1/2^t) + (1-h)/2^{t+1}}{(1-1/2^{t+1})h}$. We set $D_{t+1} = k^{t+1}[1 - \frac{\beta(1-1/2^t)+(1-h)/2^{t+1}}{(1-1/2^{t+1})h}]$, so that P_t is as in Example 1, assuming $h > 1/2$, $h > \beta/2 + 1/4$ and $h > \beta + 1/8$ (in order for $D_1 > 0$, $D_2 > 0$ and $D_t > 0$ for $t > 2$, respectively).

[27] In general, $\mu^i_\xi = ha\beta^{t(\xi)+1} D_{t(\xi)+1} \epsilon_\eta \rho^i_\xi (1 - 1/2^t)$, where $\eta \in \xi^+$.