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Decompositions of Two Player Games: Potential, Zero-Sum, and Stable Games

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Abstract

We introduce several methods of decomposition for two player normal form games. Viewing the set of all games as a vector space, we exhibit explicit orthonormal bases for the subspaces of potential games, zero-sum games, and their orthogonal complements which we call anti-potential games and anti-zero-sum games, respectively. Perhaps surprisingly, every anti-potential game comes either from the *Rock-Paper-Scissors* type games (in the case of symmetric games) or from the *Matching Pennies* type games (in the case of asymmetric games). Using these decompositions, we prove old (and some new) cycle criteria for potential and zero-sum games (as orthogonality relations between subspaces). We illustrate the usefulness of our decomposition by (a) analyzing the generalized Rock-Paper-Scissors game, (b) completely characterizing the set of all null-stable games, (c) providing a large class of strict stable games, (d) relating the game decomposition to the decomposition of vector fields for the replicator equations, (e) constructing Lyapunov functions for some replicator dynamics, and (f) constructing Zeeman games -games with an interior asymptotically stable Nash equilibrium and a pure strategy ESS.

Keywords: normal form games, evolutionary games, potential games, zero-sum games, orthogonal decomposition, null stable games, stable games, replicator dynamics, Zeeman games, Hodge decomposition.

JEL Classification Numbers: C72, C73

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1. Introduction

Two player normal form games (or bi-matrix games) are among the most simple and popular games. The symmetric games in which the two players do not distinguish between the different roles of the play have been widely used in evolutionary dynamics, and such dynamics have been extensively studied (Weibull, 1995; Hofbauer and Sigmund, 1998; Sandholm, 2010b). Special classes of games such as potential games, zero-sum games, and stable games have received a great deal of attention because of their respective analytical advantages. For instance, in potential games, all players' motivations to choose and deviate from a certain strategy are described by a *single* function, called a potential function (Monderer and Shapley, 1996).

The conditions under which a game belongs to these classes have been examined by several researchers (Hofbauer, 1985; Monderer and Shapley, 1996; Ui, 2000; Hofbauer and Sandholm, 2009; Sandholm, 2010a). For example Monderer and Shapley (1996) and Hofbauer and Sigmund (1998) provide four-cycle criteria for potential games and zero-sum games (See Theorem 11.2.2 and Exercise 11.2.9 in Hofbauer and Sigmund, 1998). Unlike existing approaches, our focus here is to examine the extent to which a given game fails to be a potential game or a zero-sum game.

Our basic insight is to view the set of all games as a vector space endowed with its scalar product. Natural classes of games form subspaces of this vector space and we systematically analyze these subspaces and their orthogonal complements. At the very basic level it provides an immediate intuition about the games and their dynamics. A game which consists of a potential game plus a small non-potential part is expected, generically, by stability to exhibit a dynamic close to the gradient-like dynamic of a potential game. On the contrary a game with a large non-potential part will be rather close to a volume-preserving dynamics with cycling behavior. In addition our decomposition will clarify the relationship between potential and zero-sum games by analyzing completely the class of games which are both potential and zero-sum.

We develop three decomposition methods of bi-matrix games. In the first decomposition, we consider the subspace of potential games and its orthogonal complement which we call "anti-potential" games (see Figure 1). Maybe surprisingly, anti-potential games are entirely described in terms of either the *Rock-Paper-Scissors* games in the case of symmetric games, and the *Matching Pennies* games in the case of bi-matrix games. In the space of symmetric games with three strategies, the only

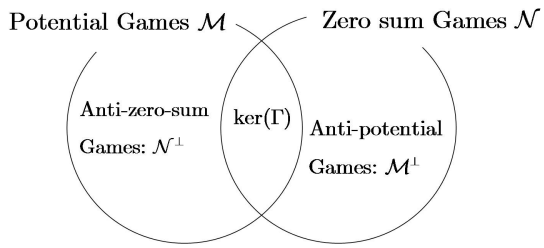


Figure 1: **Decomposition Diagram.**

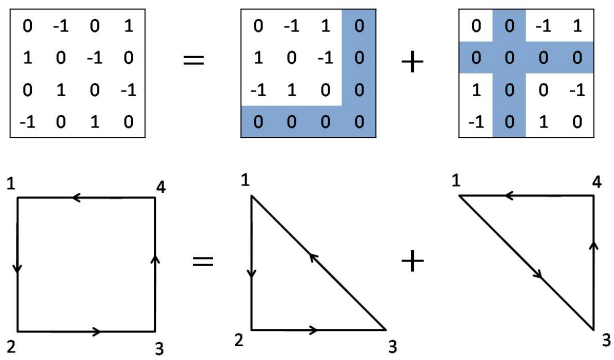


Figure 2: **Decomposition of Games and Representations.** The upper panel shows the decomposition of a symmetric game into the two extended Rock-paper-scissors games. In the first extended Rock-paper-scissors game, strategy 4 is "null", while in the second one strategy 2 is null. The lower panel shows the representation of these games.

anti-potential game is the Rock-Paper-Scissors games, up to a constant multiple. For symmetric games with more than three strategies, the extended Rock-Paper-Scissors game, which involves three strategies as Rock, Paper, and Scissors, forms a basis for the anti-potential games (see the upper panels of Figure 2). Similarly, (extended) Matching Pennies games provide a basis for bi-matrix anti-potential games.

In our second decomposition we start with the subspace of zero-sum games (see Figure 1 again) and find that the orthogonal complement is a special subspace of the potential game subspace (potential games for which the sums of rows are all zero). This class of potential games plays an important role in understanding the structures of stable games.

Finally to understand the relationship between these two decompositions and hence potential games and zero-sum games we use the projection mapping P onto the tangent space of the simplex (See Sandholm, 2010b; Hofbauer and Sandholm, 2009). We derive a third decomposition of the space of all games by considering the mapping $\Gamma(A) = PAP$ for any matrix A. The kernel of this mapping turn out to

consist of games which are both zero-sum and potential games (See Figure 1). We also show that the kernel coincides with the set of all games with dominant strategies. Thus, the simplest dynamics, namely the one induced by the Prisoner's Dilemma, is the only possible type of the dynamics that games belonging to both potential and zero-sum spaces can exhibit. In addition, we show that the subspace of all potential games has an orthogonal decomposition into the subspace of all anti-zero-sum games and the kernel of the mapping Γ. Similarly, the subspace of all zero-sum games is the direct sum of the subspace of anti-potential games and the kernel of Γ (See Figure 1). This implies that the space of two player games can be uniquely decomposed into three orthogonal subspaces: the subspaces of anti-potential games, of anti-zero-sum games, and of the kernel of Γ. The map Γ has been used in the recent work Sandholm (2010a) on the decomposition of normal form games: our decomposition implies that the range of Γ can be further decomposed into two nice classes of games; the anti-potential games and anti-zero-sum games.

We illustrate the effectiveness of these decompositions by considering several applications:

Algorithmic methods of identifying potential and zero-sum games: The decompositions provide algorithmic methods to test whether a game is a potential game, a zero-sum game, or both. Providing explicit bases for subspaces of games allows an easy numerical implementation even for a large number of strategies.

Representation of games: The decompositions allow to understand easily the structure of well known games such as the generalized Rock-Paper-Scissors game, universally cycling games (Hofbauer and Sigmund, 1998, p.98). In particular, the simple anti-potential games have a graphical representation using the orthonormal basis of games (See the lower panels of Figure 2).

Stable games: We provide a complete characterization of the null-stable games; every null stable game is a zero-sum game. The converse of the statement is obvious from the definition; using the decomposition we show that there is no null stable game which is not a zero-sum game, a non-trivial claim. Since there are games that are both potential and zero-sum, this shows that some potential games are null-stable. In addition, because every bi-matrix stable game is null stable (Hofbauer and Sandholm, 2009), this provides a complete characterization of two-player asymmetric stable games. We also present an explicit class of strict stable games using the decompositions.

Dynamics: The decompositions are useful to analyze the evolutionary dynamics of the underlying game. (a) Lyapunov functions arise naturally from the perspective of decompositions. (b) Our decomposition yields the Hodge decomposition for the vector field of the replicator dynamics, i.e. the decomposition into a gradient-like part (anti-zero-sum), a monotonic part (kernel of Γ), and a circulation part (anti-potential) (See Abraham et al. (1988) and equation (1) in Tong et al. (2003)). (c) We construct games with special properties: for example we will explain how to construct Zeeman games, namely games with an interior attracting fixed point and a strict pure NE (hence an ESS) and we provide such an example for four strategy games.

These decompositions turn out to be extremely useful also for the stochastic updating mechanism of games with finite populations. These applications will be studied elsewhere.

This paper is organized as follows: in Section 2 we present the three decompositions of the bi-matrix games and of the symmetric games, give several examples and discuss also the decomposition of $n-$player normal form games. In Section 3 we characterize stable games using the decompositions, explain the implications of the decompositions on the dynamics, and provide a four-strategy Zeeman game. We provide in the main text some proofs that are important in the expositions of the paper; tedious and book-keeping proofs are relegated to the Appendix.

2. Decompositions of the Space of Games into Orthogonal Subspaces

2.1. Potential games and zero-sum games decompositions

To illustrate the idea of our first decomposition, we decompose the well-known generalized Rock-Paper-Scissors game by performing a simple calculation.

$$\begin{pmatrix} \gamma_1 & -a+\gamma_2 & b+\gamma_3 \\ b+\gamma_1 & 0+\gamma_2 & -a+\gamma_3 \\ -a+\gamma_1 & b+\gamma_2 & \gamma_3 \end{pmatrix} \tag{1}$$

$$= \underbrace{\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}}_{\text{Passive Game}} + \frac{1}{2}(b-a)\underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{\text{Potential Part}} + \frac{1}{2}(b+a)\underbrace{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}}_{\text{Anti-potential Part}} \tag{2}$$

It is easy to see that the game (1) is a potential game if and only if $a = -b$ and is equivalent to the Rock-paper-scissors game if and only if $a = b$. In this section we show that such a decomposition as in (2) holds for any game.

We start with symmetric games: let us denote the space of all $l \times l$ matrices by $\mathcal{L}$, and let us endow $\mathcal{L}$ with the inner product $\langle A, B \rangle_{\mathcal{L}} = \mathrm{tr}(A^T B)$. A *passive game* (in the terminology of Sandholm (2010b)) is a game in which players' payoffs do not depend on the choice of strategies. Let $E_\gamma^{(j)} \in \mathcal{L}$ be the matrix given by

$$E_\gamma^{(j)}(k,l) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} ;$$

i.e., $E_\gamma^{(j)}$ is a matrix which has 1's in its jth column and 0's at all other entries. Then the set of all symmetric passive games is given by $\mathcal{I} := \mathrm{span}\{E_\gamma^{(i)}\}_{j.}$ It is well-known that the set of Nash equilibria for a symmetric game is left invariant under the addition of a passive game to the payoff matrix.

To characterize the spaces of all potential games and all zero-sum games, we define the following special matrices:

$$K^{(ij)} = \begin{array}{c} \\ i\text{-th} \rightarrow \\ \\ j\text{-th} \rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} i\text{-th} & & j\text{-th} \end{array} \\ \boxed{\begin{array}{ccc} -1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & -1 \end{array}} \end{array}, \quad N^{(ij)} = \begin{array}{c} \\ 1\text{st} \\ \\ i\text{-th} \rightarrow \\ \\ j\text{-th} \rightarrow \end{array} \begin{array}{c} \begin{array}{ccccc} 1\text{st} & & i\text{-th} & & j\text{-th} \end{array} \\ \boxed{\begin{array}{ccccc} 0 & \cdots & -1 & \cdots & 1 \\ \vdots & & \vdots & & \vdots \\ 1 & \cdots & 0 & \cdots & -1 \\ \vdots & & \vdots & & \vdots \\ -1 & \cdots & 1 & \cdots & 0 \end{array}} \end{array},$$

where all other elements in the matrices are zeros. Note that $N^{(ij)}$ is a game whose restriction on the strategy set $\{1, i, j\} \times \{1, i, j\}$ is the Rock-Paper-Scissors game. Monderer and Shapley (1996)

Recall that a symmetric game A is a *potential game* (Monderer and Shapley (1996)) if there exist a symmetric matrix S and a passive game $\sum_j \gamma_j E_\gamma^{(j)} \in \mathcal{I}$ such that

$$A = S + \sum_j \gamma_j E_\gamma^{(j)} . \tag{3}$$

We will use the word "exact " to indicate that a game is a potential game with no passive part, i.e., all $\gamma_j = 0$ (exact potential games are called full potential games

in Sandholm (2010b)). We denote by $\mathcal{M}$ the linear subspace of all potential games and we have the orthogonal decomposition $\mathcal{L} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ with respect to the inner product $<,>_{\mathcal{L}}$. We call a game in $\mathcal{M}^{\perp}$ an *anti-potential game*.

Note that the dimension of the subspace of $\mathcal{L}$ consisting of all symmetric matrices is $\frac{1}{2}l(l+1)$ and the dimension of the subspace of passive games is l. Since the sum of all E_{γ} is an exact potential game, namely the game whose payoffs are all 1's, the dimension of the intersection between the subspace of all symmetric matrices and $\mathcal{I}$ is at least 1. Conversely if a matrix belongs to this intersection, then the entries of this matrices should be all the same (see also the discussion in Sandholm (2010a, p.15)) and so the dimension of the intersection is exactly 1. Hence the dimension of $\mathcal{M}$ is given by

$$\dim(\mathcal{M}) = \frac{l(l+1)}{2} + l - 1 = l^2 - \frac{(l-1)(l-2)}{2}. \tag{4}$$

Note that the extended Rock-Paper-Scissors game, $N^{(ij)}$, is an anti-symmetric matrix whose column sums and row sums are all 0's. Thus, we have

$$\left\langle A, N^{(ij)} \right\rangle_{\mathcal{L}} = 0 \ ,$$

for all $A \in \mathcal{M}$, because $\left\langle S, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ and $\left\langle P, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ for all symmetric matrix S and all passive game P (See the appendix for the properties of $\langle\rangle_{\mathcal{L}}$). In other words, $N^{(ij)} \in \mathcal{M}^{\perp}$ for all i, j. The set $\{N^{(ij)} : j > i, \ i = 2, \cdots, l-1\}$ has $\frac{(l-1)(l-2)}{2}$ elements and they are linearly independent since each $N^{(ij)}$ is uniquely determined by the property of having 1 in its (i, j) th position. This set forms a basis for $\mathcal{M}^{\perp}$. If a matrix B is antisymmetric and the sums of elements in each column in B are all zeros, $\langle S, B\rangle_{\mathcal{L}} = 0$ for a symmetric matrix and $\langle P, B\rangle_{\mathcal{L}} = 0$ for a passive game P. Therefore $B \in \mathcal{M}^{\perp}$. On the other hand, if $B \in \mathcal{M}^{\perp}$, B can be written as a linear combination of $N^{(ij)}$, and hence B is antisymmetric and the sums of elements in each column in B are all zeros. Thus we obtain

Proposition 2.1 (Anti-potential games). *We have*

$$B \in \mathcal{M}^{\perp} \textit{ if and only if } \ B^T = -B \text{ and } \sum_j B(i,j) = \sum_i B(i,j) = 0\,.$$

Moreover the set $\{N^{(ij)} : j > i, \, i = 2, \cdots, l\}$ *forms a basis for* $\mathcal{M}^{\perp}$.

Proposition 2.1 shows that a basis for $\mathcal{M}^{\perp}$ can be obtained from the extended Rock-Paper-Scissors. As a corollary of Proposition 2.1 we obtain immediately the criterion for potential games given by Hofbauer and Sigmund (1998).

Corollary 2.2 (Potential games). *A is a potential game if and only if*

$$a(l,m) - a(k,m) + a(k,l) - a(m,l) + a(m,k) - a(l,k) = 0 \quad \textit{for all } l,m,k \in S \tag{5}$$

Proof. First note from Proposition 2.1 that A is a potential game if and only if $\left\langle A, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ for all i,j. Then note that

$$a(l,m) - a(k,m) + a(k,l) - a(m,l) + a(m,k) - a(l,k) = \langle A, E \rangle_{\mathcal{L}}$$

where

$$E = \begin{array}{c|ccc|} & k & l & m \\ \hline k & 0 & 1 & -1 \\ l & -1 & 0 & 1 \\ m & 1 & -1 & 0 \\ \hline \end{array} \text{ and all other entries in } E \text{ are 0's.}$$

Then clearly (5) implies $\left\langle A, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ for all i,j. Conversely, the matrix E is anti-symmetric and its row sums and column sums are zero, so $E \in \mathcal{M}^{\perp}$. Therefore E can be uniquely written as $N^{(ij)}$ and thus $\left\langle A, N^{(ij)} \right\rangle_{\mathcal{L}} = 0$ for all i,j implies (5). ■

We provide next a similar decomposition starting with zero-sum games. We call an anti-symmetric matrix A an *exact zero-sum game* and call a game *zero-sum* if it can be written as the sum of a antisymmetric matrix and a passive game. Let us denote by $\mathcal{N}$ the subspace of all zero-sum games. The dimension of the subspace all anti-symmetric matrices is $\frac{(l-1)l}{2}$ and the dimension of the intersection between the subspace of anti-symmetric matrices and $\mathcal{I}$ is 0 (the diagonal elements of anti-symmetric matrices are all zeros and hence all off-diagonal elements are again all zeros if this game is also a passive game). Thus

$$\dim(\mathcal{N}) = \frac{(l-1)l}{2} + l = l^2 - \frac{(l-1)l}{2}. \tag{6}$$

We decompose the space of game as $\mathcal{L} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ and we call a game in $\mathcal{N}^{\perp}$ an *anti-zero-sum* game. Note that $K^{(ij)}$ is a symmetric matrix whose row sums and column sums are zeros, so $K^{(ij)} \in \mathcal{N}^{\perp}$. The set $\{K^{(ij)} : j > i,\ i = 1, \cdots, l\}$ has $\frac{(l-1)l}{2}$

elements which are linearly independent since each $K^{(ij)}$ is uniquely determined by having 1 in its (i,j)th entry. Thus we obtain

Proposition 2.3 (Anti-zero-sum games). *We have*

$$B \in \mathcal{N}^{\perp} \quad \textit{if and only if} \quad B^T = B \text{ and } \sum_j B(i,j) = \sum_i B(i,j) = 0\,.$$

Moreover the set $\{K^{(ij)} : j > i,\, i = 1, \cdots, l-1\}$ *forms a basis for* $\mathcal{N}^{\perp}$.

Using this orthogonal decomposition we obtain a new criterion to identify a zero-sum game similar to the criterion in Corollary 2.2.

Corollary 2.4 (Zero-sum games). *A is a zero-sum game if and only if*

$$a(j,i) - a(i,i) + a(i,j) - a(j,j) = 0 \text{ for all } i,j \in S\,. \tag{7}$$

Proof. If $A \in \mathcal{N}$ then $\left\langle A, K^{(ij)} \right\rangle_{\mathcal{L}} = 0$ which yields (7). ■

2.2. Decomposition using the projection mapping Γ

The subspaces of potential games and zero-sum games have a non-trivial intersection $\mathcal{M} \cap \mathcal{N}$. In order to understand this set let $P = I - \frac{1}{l}\mathbf{1}\mathbf{1}^T$ where I is the identity matrix and $\mathbf{1}$ the constant vector with entries equal to 1. It is easy to see that P is the orthogonal projection onto the subspace $T\Delta = \{x \in \mathbb{R}^l\,; \sum_i x_i = 0\}$, i.e., onto the tangent space to the unit simplex $\Delta = \{x \in \mathbb{R}^l\,; x_i \geq 0\,, \sum_i x_i = 1\}$. Let us define a linear transformation Γ on $\mathcal{L}$ by

$$\Gamma : \mathcal{L} \to \mathcal{L},\ A \mapsto PAP.$$

To characterize the kernel and the range of the map Γ, let us say that a game is a *constant game* if the player's payoff does not depend on his opponent's strategy, that is the payoff matrix is constant on each row. The matrices $E_\eta^{(i)} := (E_\gamma^{(i)})^T$ form an orthonormal basis of the subspace of constant games. Note that $E_\eta^{(i)}$ has a strictly

dominant strategy. Furthermore let us define

$$E_\kappa^{(ij)} = \begin{array}{c} \\ \\ i-\text{th} \\ i+1-\text{th} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & j-\text{th} & j+1-\text{th} & \end{array} \\ \boxed{\begin{array}{cccc} & \vdots & \vdots & \\ \cdots & \text{-1} & 1 & \cdots \\ \cdots & 1 & \text{-1} & \cdots \\ & \vdots & \vdots & \end{array}} \end{array} \text{ where all other entries are 0's.}$$

It is easy to see that

$$\text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\} \subset \ker \Gamma. \tag{8}$$

Conversely, one can show that the left and right actions of the projection matrices makes only this class belongs to $\ker \Gamma$. Then note that

$$\sum_i E_\gamma^{(l)} = \sum_i E_\eta^{(l)},$$

so by throwing away one element from the spanning set (8), we may obtain the independent spanning set, hence a basis for the kernel of Γ. Concerning the range of Γ, by counting the basis elements, we have $\dim(\ker \Gamma) = 2l-1$ and, thus, $\dim(\text{range}\Gamma) = l^2 - (2l-1) = (l-1)^2$. Since $\mathbf{1}E_\kappa^{(ij)} = \mathbf{0}$ and $E_\kappa^{(ij)}\mathbf{1} = \mathbf{0}$,

$$\{E_\kappa^{(ij)} : i = 1, \cdots, l-1, j = 1, \cdots l-1\}$$

provides a natural candidate for the basis of the range. These observations lead to Proposition 2.5 whose formal proof is elementary but tedious, and hence relegated to the Appendix.

Proposition 2.5 (Characterizations of $\ker(\Gamma)$ **and range(**Γ**)).** *We have*
(1) $\{E_\eta^{(i)}\}_{i\neq 1} \cup \{E_\gamma^{(j)}\}_j$ *form a basis for* $\ker \Gamma$.
(2) $\{E_\kappa^{(ij)} : i = 1, \cdots, l-1, j = 1, \cdots l-1\}$ *form a basis for range*(Γ).

Next, we study the relationship among these subspaces. First every game in the subspace $\mathcal{N}^\perp$ is a symmetric matrix and thus a potential game. Similarly every anti-potential game is a zero-sum game, so we have $\mathcal{N}^\perp \subset \mathcal{M}$ and $\mathcal{M}^\perp \subset \mathcal{N}$. To

understand the relationship among these spaces further, note the following facts:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

This example illustrates the fact that any game in $\ker(\Gamma)$ which is not a passive game is both a potential game and zero-sum game; i.e., every constant game is both a potential games and zero-sum games. As Proposition 2.6 shows the converse holds: a game which is both a potential and a zero-sum game is equivalent to a constant game.

Proposition 2.6. $\ker(\Gamma) = \mathcal{M} \cap \mathcal{N}$ *and range*$(\Gamma) = \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp}$.

Proposition 2.6 provides the essential characterization of the relationship among spaces. Since $\mathcal{L} = \ker(\Gamma) \oplus \text{range}(\Gamma)$, from Proposition 2.6, we obtain the decomposition of a given game into three parts; $\mathcal{L} = \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp} \oplus \ker(\Gamma)$. Also since $\mathcal{N} \cap (\mathcal{M}^{\perp} \cup \mathcal{N}^{\perp}) = \mathcal{M}^{\perp}$, we will have $\mathcal{N} \cap \text{range}(\Gamma) = \mathcal{M}^{\perp}$ and this provides another characterization of $\mathcal{M}^{\perp}$ as follows. From Proposition 2.1, we know that a game is anti-potential if and only if it is an antisymmetric matrix whose row sums and column sums are zeros. We know that all row sums and column sums of games belonging to $\text{range}(\Gamma)$ are zeros and the zero sum game is the sum of an antisymmetric matrix and a passive game; thus we can show that $\mathcal{M}^{\perp} = \mathcal{N} \cap \text{range}(\mathbf{\Gamma})$. In this way we obtain the following key result in the paper.

Theorem 2.7. *We have*

(1) $\mathcal{M} = \mathcal{N}^{\perp} \oplus \ker(\Gamma)$ *and* $\mathcal{M}^{\perp} = \mathcal{N} \cap$ *range*$(\mathbf{\Gamma})$

(2) $\mathcal{N} = \mathcal{M}^{\perp} \oplus \ker(\Gamma)$ *and* $\mathcal{N}^{\perp} = \mathcal{M} \cap$ *range*$(\mathbf{\Gamma})$

(3) $\mathcal{L} = \mathcal{M}^{\perp} \oplus \mathcal{N}^{\perp} \oplus \ker(\Gamma)$

Proof. (1) From Proposition 2.6, we have $\mathcal{N}^{\perp} + \ker(\Gamma) = \text{span}(\mathcal{N}^{\perp} \cup \ker(\Gamma)) = \text{span}((\mathcal{N}^{\perp} \cup \mathcal{M}) \cap (\mathcal{N}^{\perp} \cup \mathcal{N})) = \mathcal{M}$. Since $\mathcal{N}^{\perp} \perp \ker(\Gamma)$, we have $\mathcal{M} = \mathcal{N}^{\perp} \oplus \ker(\Gamma)$.

From proposition 2.6, we have $\mathcal{M}^\perp \subset \mathcal{M}^\perp \oplus \mathcal{N}^\perp = \text{range}(\Gamma)$ and see that $\mathcal{M}^\perp \subset \mathcal{N} \cap \text{range}(\Gamma)$. Conversely again from proposition 2.6, we have

$$\mathcal{N} \cap \text{range}(\Gamma) = \mathcal{N} \cap (\text{span}(\mathcal{M}^\perp \cup \mathcal{N}^\perp)) \supset \text{span}(\mathcal{N} \cap (\mathcal{M}^\perp \cup \mathcal{N}^\perp)) = \mathcal{M}^\perp.$$

By changing the roles of $\mathcal{M}$ and $\mathcal{N}$, we obtain (2). (3) follows from $\mathcal{L} = \mathcal{M}^\perp \oplus \mathcal{M} = \mathcal{M}^\perp \oplus \mathcal{N}^\perp \oplus \ker(\Gamma)$. ■

Sandholm (2010a) provides a method of decomposing normal form games by using the orthogonal projection P : for a given A write

$$A = \underbrace{PAP}_{\in \text{ range}(\Gamma)} + \underbrace{(I-P)AP + PA(I-P) + (I-P)A(I-P)}_{\in \ker(\Gamma)}. \tag{9}$$

The first term in (9) belongs to the range of Γ and the remaining three terms belong to the kernel of Γ. Our decompositions (Proposition 2.6) show that PAP can be further decomposed into games having nice properties − potential games and zero-sum games − and every game in $\ker(\Gamma)$ is a game which is both a potential and a zero-sum game and possesses (generically) a dominant strategy.

Theorem 2.7 also provides a convenient way to compute the anti-zero-sum part (anti-potential part, resp.) of a game when the anti-potential part (anti-zero-sum part, resp.) is known. Suppose that A is a symmetric game and its anti-potential part is Z. Then the part of A that belongs to $\ker(\Gamma)$ is $A - PAP$. Hence from (3) of Theorem 2.7, its anti-zero-sum part is given by $A - Z - (A - PAP) = PAP - Z$; in fact Theorem 2.7 shows that $PAP - Z$ is a symmetric matrix in $\mathcal{L}$ and its all row sums and column sums are zeros.

2.3. Decompositions of bi-matrix games

In this section we prove a decomposition theorem for bi-matrix games and elucidate the relations between the decomposition of symmetric and bi-matrix games. Most results in this section generalize the corresponding results in sections 2.1-2.2. We denote (with a slight abuse of notation) by $\mathcal{L}$ the space of all $l_r \times l_c$ matrices with the inner product $\langle A, B \rangle_{\mathcal{L}} := tr(A^T B)$. The set of all bi-matrix games is $\mathcal{L}^2 := \mathcal{L} \times \mathcal{L}$ and sometimes we will view a bi-matrix game (A, B) as a $(l_r + l_c) \times (l_r + l_c)$ matrix given by

$$(A, B) := \begin{pmatrix} O_r & A \\ B^T & O_c \end{pmatrix}$$

where O_r and O_c are $l_r \times l_r$ and $l_c \times l_c$ zero matrices, respectively. The space $\mathcal{L}^2$ is a linear subspace of the set of all $(l_r+l_c)\times(l_r+l_c)$ matrices of dimension $2l_rl_c$. We endow $\mathcal{L}^2$ with the inner product $< \cdot, \cdot >_{\mathcal{L}^2}$, where $\langle (A,B),(C,D)\rangle_{\mathcal{L}^2} := tr((A,B)^T(C,D))$. The elementary properties of this scalar product are summarized in the Appendix.

The set of all bi-matrix *passive games* $\bar{\mathcal{I}}$ is given by

$$\bar{\mathcal{I}} := \text{span}(\{(E_\gamma^{(j)}, O)\}_j \cup \{(O, E_\gamma^{(i)})\}_i).$$

and we say that the games (A,B) and(C,D) are equivalent if $(A,B)-(C,D) \in \bar{\mathcal{I}}$. In this case we write $(A,B) \sim (C,D)$. The set of Nash equilibria for a bi-matrix game is invariant under this equivalence relation.

Note that $(E_\kappa^{(ij)}, -E_\kappa^{(ij)})$ is a game whose restriction on the strategy set $\{i, i+1\} \times \{j, j+1\}$ is the Matching Pennies game and we call it an extended Matching Pennies game.

From Monderer and Shapley (1996) we recall that (A,B) is a *potential game* if there exist a matrix S and $\{\gamma_j\}_j$, $\{\eta_i\}_i$ such that

$$(A,B) = (S,S) + \sum_j \gamma_j (E_\gamma^{(j)}, O) + \sum_i \eta_i (O, E_\eta^{(i)})\,.$$

Denoting by $\bar{\mathcal{M}}$ the subspace of all potential games, we have the orthogonal decomposition $\mathcal{L}^2 = \bar{\mathcal{M}} \oplus \bar{\mathcal{M}}^\perp$. The dimension of the subspace of all exact potential games is $l_r \times l_c$ and the dimension of the subspace of all passive games is $l_r + l_c$. Arguing as for symmetric games, one finds that the dimension of $\bar{\mathcal{M}}$ is given by

$$\dim(\bar{\mathcal{M}}) = l_rl_c + l_r + l_c - 1 = 2l_rl_c - (l_r-1)(l_c-1). \tag{10}$$

Note also that $(E_\kappa{}^{(ij)}, -E_\kappa{}^{(ij)})$ is an anti-symmetric matrix as an element in $\mathcal{L}^2$ whose column sum and row sum are all 0's, thus we have $\left\langle (A,B), (E_\kappa{}^{(i,j)}, -E_\kappa{}^{(i,j)})\right\rangle_{\mathcal{L}^2} = 0$ for all $(A,B) \in \bar{\mathcal{M}}$. In other words, $(E_\kappa{}^{(ij)}, -E_\kappa{}^{(ij)}) \in \bar{\mathcal{M}}^\perp$ for all i,j and the number of such $(E_\kappa{}^{(i,j)}, -E_\kappa{}^{(i,j)})$ is $(l_r-1)(l_c-1)$. Hence we have

Proposition 2.8 (Anti-potential games). *The set* $\{(E_\kappa{}^{(i,j)}, -E_\kappa{}^{(i,j)})\}_{1\le i<l_r, 1\le j<l_c}$ *is a basis for* $\bar{\mathcal{M}}^\perp$.

Proof. From the discussion above, it is enough to show the linear independence

among $(E_\kappa^{(ij)}, -E_\kappa^{(ij)})$. To do this, we consider the following linear combination:

$$\sum_{i,j} \kappa^{(ij)} E_\kappa^{(ij)} = 0.$$

Then, it is easy to see that $\kappa^{(11)} = 0$. This implies $\kappa^{(1,j)} = 0$ for all j which, in turn, implies $\kappa^{(i,j)} = 0$ for all i. ■

Proposition 2.8 shows that a basis for $\mathcal{M}^\perp$ can be obtained from the Matching Pennies games and its extensions. From this, we say that (A, B) is an bi-matrix *anti-potential* game whenever $(A, B) \in \mathcal{M}^\perp$. Proposition 2.8 provides an alternative and simple proof for the well-known criterion for the potential game by Monderer and Shapley (1996):

Corollary 2.9 (Potential games). *(A, B) is a potential game if and only if for all $i, i' \in S_r$, $j, j' \in S_c$,*

$$a(i', j) - a(i, j) + b(i', j') - b(i', j) + a(i, j') - a(i', j') + b(i, j) - b(i, j') = 0$$

Proof. It is enough to notice that

$$\begin{aligned} & a(i', j) - a(i, j) + b(i', j') - b(i', j) + a(i, j') - a(i', j') + b(i, j) - b(i, j') \\ = \; & \left\langle (A, B), (K^{(i,i')(j,j')}, -K^{(i,i')(j,j')}) \right\rangle_{\mathcal{L}^2} \end{aligned}$$

where $(K^{(i,i')(j,j')}, -K^{(i,i')(j,j')})$ is an extended Matching Pennies game whose restriction on $\{i, i'\} \times \{j, j'\}$ is a Matching Pennies game. ■

Next we consider a decomposition using zero-sum games as in Section 2.2. We call a game of the form $(A, -A)$ an exact zero-sum game and say that a game is a *zero-sum* game if it can be written as the sum of an exact zero-sum game and a passive game. We denote by $\bar{\mathcal{N}}$ the subspace of all bi-matrix zero-sum games and have $\dim(\bar{\mathcal{N}}) = 2l_r l_c - (l_r - 1)(l_c - 1)$. A similar argument as in Section 2.2 yields

Proposition 2.10 (Anti-zero-sum games). *The set $\{(E_\kappa^{(ij)}, E_\kappa^{(ij)})\}_{1 \le i < l_r, 1 \le j < l_c}$ is a basis for $\bar{\mathcal{N}}^\perp$.*

Again the following corollary is an immediate consequence of orthogonality (See Exercise 11.2.9 in Hofbauer and Sigmund, 1998).

Corollary 2.11 (Zero-sum games). *(A, B) is a zero-sum game if and only if for all $i, i' \in S_r$, $j, j' \in S_c$,*

$$a(i', j) - a(i, j) - b(i', j') + b(i', j) + a(i, j') - a(i', j') - b(i, j) + b(i, j') = 0.$$

Finally to consider the decomposition in terms of the projection mapping onto the tangent space as in Section 2.3, we modify the definition of Γ :

$$\Gamma : \mathcal{L} \to \mathcal{L},\ A \mapsto P_r A P_c,\ P_{l_r} = I_r - \frac{1}{l_r}\mathbf{1}_r\mathbf{1}_r^T,\ P_c = I_c - \frac{1}{l_c}\mathbf{1}_c\mathbf{1}_c^T.$$

and define $\mathbf{\Gamma} : \mathcal{L}^2 \to \mathcal{L}^2$ by

$$(A, B) \mapsto \mathbb{P}(A, B)\mathbb{P} := \begin{pmatrix} P_r & O \\ O & P_c \end{pmatrix} \begin{pmatrix} O & A \\ B^T & O \end{pmatrix} \begin{pmatrix} P_r & O \\ O & P_c \end{pmatrix}.$$

As in symmetric games (Proposition 2.5), we obtain the following characterizations for $\ker(\mathbf{\Gamma})$ and range $(\mathbf{\Gamma})$:

Proposition 2.12. *We have*

$$\begin{aligned} \ker(\mathbf{\Gamma}) &= \operatorname{span}\left(\{(E_\eta^{(i)}, O)\}_{i\neq 1} \cup \{(E_\gamma^{(i)}, O)\}_i \cup \{(O, E_\eta^{(i)})\}_i \cup \{(O, E_\gamma^{(i)})\}_{i\neq 1}\right) \\ \operatorname{range}(\mathbf{\Gamma}) &= \operatorname{span}\left(\{(E_\kappa^{(ij)}, O)\}_{i\geq 1, j\geq 1} \cup \{O, E_\kappa^{(ij)}\}_{i\geq 1, j\geq 1}\right) \end{aligned}$$

Clearly results similar to Proposition 2.6, and Theorem 2.7 hold for $\mathcal{L}^2$ and the subspaces $\bar{\mathcal{M}}$, $\bar{\mathcal{M}}^\perp$, $\bar{\mathcal{N}}$,$\bar{\mathcal{N}}^\perp$, $\ker(\mathbf{\Gamma})$, and range$(\mathbf{\Gamma})$. To understand the relationship between the decompositions of symmetric games and bi-matrix games, note that the set of two player symmetric games corresponds to the set of all bi-matrix games with $l = l_r = l_c$ satisfying $A = B^T$. Thus in this case,

$$(A, B) \text{ is a symmetric game if } A = B^T.$$

To avoid confusion, we denote by $\mathcal{L}_{sym}$ the set of all symmetric games as a subspace

of $\mathcal{L}^2$ and write $[A] = (A, A^T)$. Consider the following example:

$$[E_\kappa^{(12)} - E_\kappa^{(21)}] = (E_\kappa^{(12)}, -E_\kappa^{(12)}) - (E_\kappa^{(21)}, -E_\kappa^{(21)})$$

$$= \begin{array}{|c|c|c|}\hline 0,0 & -1,1 & 1,-1 \\ \hline 0,0 & 1,-1 & -1,1 \\ \hline 0,0 & 0,0 & 0,0 \\ \hline \end{array} - \begin{array}{|c|c|c|}\hline 0,0 & 0,0 & 0,0 \\ \hline -1,1 & 1,-1 & 0,0 \\ \hline 1,-1 & -1,1 & 0,0 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0,0 & -1,1 & 1,-1 \\ \hline 1,-1 & 0,0 & -1,1 \\ \hline -1,1 & 1,-1 & 0,0 \\ \hline \end{array}.$$

Thus $[E_\kappa^{(12)} - E_\kappa^{(21)}]$ is the Rock-Paper-Scissors game; this example shows how one can "symmetrize" the bi-matrix games to obtain the symmetric version of them. More generally, we obtain the orthonormal bases of anti-potential games and anti-zero-sum symmetric games in symmetric games by restricting the bases of subspaces of bi-matrix games using the following lemma.

Lemma 2.13. *Suppose that* $\{(A^{(ij)}, A^{(ij)})\}_{i,j\in\mathcal{I}_1} \cup \{(B^{(ij)}, -B^{(ij)})\}_{i,j\in\mathcal{I}_2} \cup \{(C^{(i)}, O)\}_{i\in\mathcal{I}_3} \cup \{(O, (C^{(i)})^T)\}_{i\in\mathcal{I}_3}$ *form a basis for* K*, a subspace of* $\mathcal{L}^2$ *and* $\{A^{(ij)}\}_{i,j} \cup \{B^{(ij)}\}_{i,j} \cup \{C^{(i)}\}_i$ *are linearly independent. Then* $\{[A^{(ij)}+A^{(ji)}]\}_{i,j\in\mathcal{I}_1\cap\{j\geq i\}} \cup \{[B^{(ij)}-B^{(ji)}]\}_{i,j\in\mathcal{I}_2\cap\{j>i\}} \cup \{[C^{(i)}]\}_{i\in\mathcal{I}_3}$ *form a basis for* $K \cap \mathcal{L}_{sym}$.

As an immediate consequence of the decomposition we obtain the alternative proof for the following well-known characterization for potential and zero-sum games (Hofbauer and Sigmund, 1998; Sandholm, 2010b). Notice that a similar characterization for the symmetric potential and zero-sum games is also readily available.

Proposition 2.14. *The following conditions are equivalent:*

(1) (A, B) *is a potential game (a zero-sum game, respectively)*

(2) $\mathbb{P}(A, B)\mathbb{P}$ *is a symmetric* $(l_r + l_c) \times (l_r + l_c)$ *matrix (an antisymmetric* $(l_r + l_c) \times (l_r + l_c)$ *matrix, respectively)*

(3) $(A, B) - (A, B)^T \in \ker(\mathbf{\Gamma})$ $((A, B) + (A, B)^T \in \ker(\mathbf{\Gamma})$*, respectively.)*

Proof. For a given (A, B), using $\text{range}(\mathbf{\Gamma}) = \mathcal{M}^\perp \oplus \mathcal{N}^\perp$ (Proposition 2.6) we have

$$\mathbb{P}(A, B)\mathbb{P} = (V, V) + (N, -N) \text{ for some } V \text{ and } N \in \mathcal{L}.$$

Since (V, V) is a $(l_r+l_c)\times(l_r+l_c)$ symmetric matrix and $(N, -N)$ is a $(l_r+l_c)\times(l_r+l_c)$ anti-symmetric, so (1) $\Leftrightarrow$ (2). For (2) $\Leftrightarrow$ (3), we first note that $(A, B)^T = (B, A)$.

Thus $(A \pm B, B \pm A) \in \ker(\mathbf{\Gamma})$, if and only if $\mathbb{P}(A \pm B, B \pm A)\mathbb{P} = O$, if and only if $\mathbb{P}(A, B)\mathbb{P} = \pm\mathbb{P}(B, A)\mathbb{P}$, if and only if $\mathbb{P}(A, B)\mathbb{P} = \pm(\mathbb{P}(A, B)\mathbb{P})^T$. ■

2.4. Decompositions of n-player normal form games.

In this section we will briefly discuss how to generalize the decomposition to the case of $n-$player normal form games. We provide more detailed discussion in the Appendix. For the simplicity of exposition, we suppose that all $n-$players have the same strategy set S. We denote by $\mathcal{L}_n$ the set of all n player games, by $\mathcal{S}$ the set of all strategy profiles and by $\mathcal{P}$ the set of all players. First note that we have $\dim(\mathcal{L}_n) = nl^n$. We use a l^n dimensional tensor A to denote a player's payoffs and thus a normal form game is given by $(A_{p_1}, A_{p_2}, \cdots, A_{p_n})$ for $p_l \in \mathcal{P}$. We introduce an inner product $\langle\rangle_{\mathcal{L}_n}$ in $\mathcal{L}_n$:

$$\langle (A_{p_1}, \cdots, A_{p_n}), (B_{p_1}, \cdots, B_{p_n})\rangle_{\mathcal{L}_n} = \sum_{i=1,\cdots n} \langle A_{p_i}, B_{p_i}\rangle_{\mathcal{L}} ,$$

where

$$\langle A, B\rangle_{\mathcal{L}} = \sum_{(i_{p_1},\cdots,i_{p_n})\in S} a_{i_{p_1},\cdots,i_{p_n}} b_{i_{p_1},\cdots,i_{p_n}}.$$

Similarly we denote by $\mathcal{M}_n$ the subspace of all potential games. We have the following recursive formula for the dimension of $\mathcal{M}_n$.

Proposition 2.15. *We have* $\dim(\mathcal{M}_{n+1})^\perp = (l-1)^2 nl^{n-1} + \dim(\mathcal{M}_n)^\perp$.

Proof. First note that $\dim(\mathcal{M}_n) = l^n - 1 + nl^{n-1}$. and thus

$$\begin{aligned}
\dim(\mathcal{M}_{n+1})^\perp &= (n+1)l^{n+1} - l^{n+1} - (n+1)l^n + 1 \\
&= (l-1)^2(nl^{n-1} + (n-1)l^{n-2} + \cdots + 2l + 1) \\
&= (l-1)^2 nl^{n-1} + \dim(\mathcal{M}_n)^\perp.
\end{aligned}$$

■

The recursive relation in Proposition 2.15 shows that a basis for $(\mathcal{M}_{n+1})^\perp$ can be obtained from the existing basis of $(\mathcal{M}_n)^\perp$ by adding $(l-1)^2 nl^{n-1}$ additional elements. To illustrate this, we consider two strategy three player games. From

$\mathcal{M}_2 = \text{span}($

-1,1	1,-1
1,-1	-1,1

$)$,

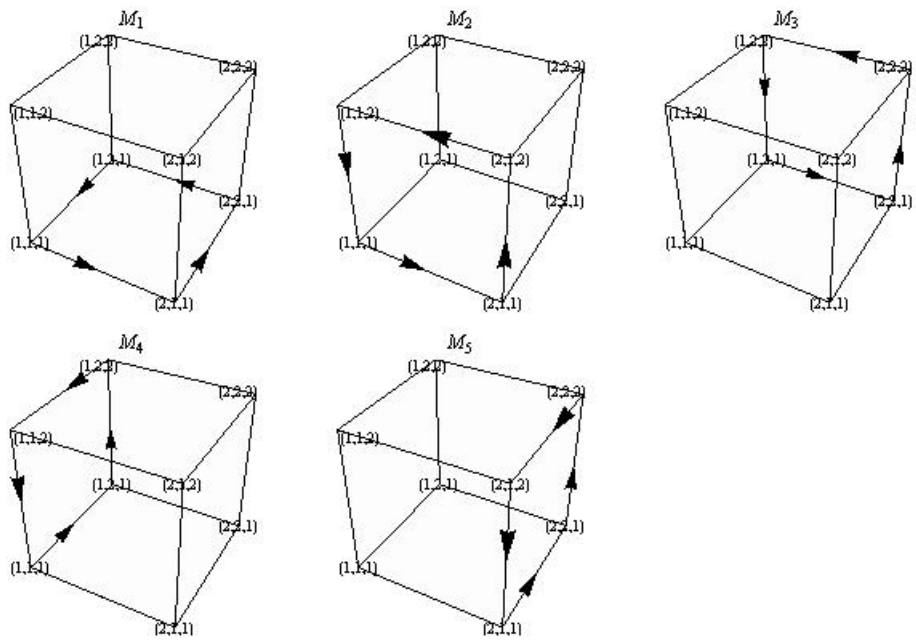


Figure 3: **A basis for three-player anti-potential games.** Each vertex in each cube represents the strategy profile and the arrows show the deviation motivations based on the payoffs from the game.

we expand this basis bi-matrix to obtain an element of the basis set for $\mathcal{M}_3$ by making player 3 as a null player (See the first cubic in Figure 3) . That is,

$$M_1 = \begin{array}{|c|c||c|c|} \hline \text{-1,1,0} & \text{1,-1,0} & \text{0,0,0} & \text{0,0,0} \\ \hline \text{1,-1,0} & \text{-1,1,0} & \text{0,0,0} & \text{0,0,0} \\ \hline \end{array}.$$

Now we imagine that one of existing players, player 1 and player 2, is matched with player 3 to play the Matching Pennies game. Then since the null player, either player 1 or player 2, can choose one strategy from the two strategies, there are four possible situations in which two players play the Matching Pennies game and one player plays the null player (See Figure 3). Thus we obtain the following basis games.

$$M_2 = \begin{array}{|c|c||c|c|} \hline \text{-1,0,1} & \text{0,0,0} & \text{1,0,-1} & \text{0,0,0} \\ \hline \text{1,0,-1} & \text{0,0,0} & \text{-1,0,1} & \text{0,0,0} \\ \hline \end{array}, \quad M_3 = \begin{array}{|c|c||c|c|} \hline \text{0,0,0} & \text{-1,0,1} & \text{0,0,0} & \text{1,0,-1} \\ \hline \text{0,0,0} & \text{1,0,-1} & \text{0,0,0} & \text{-1,0,1} \\ \hline \end{array}$$

$$M_4 = \begin{array}{|c|c||c|c|} \hline \text{0,-1,1} & \text{0,1,-1} & \text{0,1,-1} & \text{0,-1,1} \\ \hline \text{0,0,0} & \text{0,0,0} & \text{0,0,0} & \text{0,0,0} \\ \hline \end{array}, \quad M_5 = \begin{array}{|c|c||c|c|} \hline \text{0,0,0} & \text{0,0,0} & \text{0,0,0} & \text{0,0,0} \\ \hline \text{0,-1,1} & \text{0,1,-1} & \text{0,1,-1} & \text{0,-1,1} \\ \hline \end{array}$$

It is easy to see that $M_1, \cdots, M_5$ are independent and belong to $\mathcal{M}_3$. Thus $\{M_1, \cdots, M_5\}$ form a basis for $\mathcal{M}_3$. Here we verify Proposition 2.15 as follows:

$$\dim(\mathcal{M}_3)^{\perp} = (2-1)^2 2 \times 2^{2-1} + \dim(\mathcal{M}_2)^{\perp}.$$

Note that

$M_6 =$

0,0,0	0,0,0	-1,1,0	1,-1,0
0,0,0	0,0,0	1,-1,0	-1,1,0

can be obtained by taking $M_1 - (M_2 - M_3 - M_4 + M_5)$. Next we characterize the subspace of all zero-sum games. We call a game $(A_{p_1}, A_{p_2}, \cdots, A_{p_n})$ is an exact zero-sum game if

$$(A_{p_1})_{(i_{p_1}, \cdots, i_{p_n})} + \cdots + (A_{p_n})_{(i_{p_1}, \cdots, i_{p_n})} = 0 \text{ for all } (i_{p_1}, \cdots, i_{p_n}) \in \mathcal{S}.$$

The following lemma reveals the structure of the subspace of all zero-sum games.

Lemma 2.16. *$A = (A_{p_1}, A_{p_2}, \cdots, A_{p_n})$ is an exact zero-sum game if and only if A can be written as a finite sum of tensors Z's of the form:*

$$Z = (O, \cdots, O, Z_{p_i}, O, \cdots, O, -Z_{p_i}, O, \cdots).$$

Proof. "If part" is trivial. For "only if part", we decompose A first into l^n tensors whose $(i_{p_1}, \cdots, i_{p_n})$th element is the same as $((A_{p_1})_{(i_{p_1}, \cdots, i_{p_n})}, \cdots, (A_{p_n})_{(i_{p_1}, \cdots, i_{p_n})})$ and other elements are all 0's. Then since $((A_{p_1})_{(i_{p_1}, \cdots, i_{p_n})}, \cdots, (A_{p_n})_{(i_{p_1}, \cdots, i_{p_n})}) \in T\Delta_n$ and $\{(1, -1, 0, \cdots, 0), (1, 0, -1, \cdots, 0), \cdots, (1, 0, 0, \cdots, -1)\}$ form a basis for $T\Delta_n$, we have the desired representation. ■

Then we can define the subspace of zero-sum games and obtain the following proposition.

Proposition 2.17. *We have* $\dim(\mathcal{N}^{\perp}) = (l-1)^p$

From this discussion, we obtain the decompositions of potential games and anti-potential games and the decompositions of zero-sum games and anti-zero-sum games as in Section 2.1-2.3.

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \Leftrightarrow$$

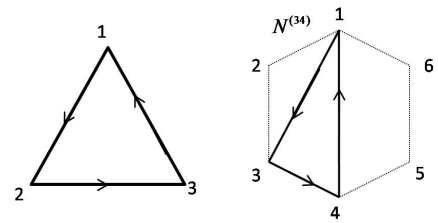


Figure 4: **Representation of the Rock-paper-scissors games**

2.5. Examples of Decompositions

Because of the simple structure of basis games in the subspace of anti-potential games we can associate a class of anti-potential games with a set of graphs. To explain this we focus on symmetric games. First observe that all basis elements in $\mathcal{M}^{\perp}$, $N^{(ij)}$ have payoffs consisting 0, 1, and -1. Thus we can assign a binary relation to (i, j): for given A, $i \succ j$ if $a(i, j) = 1$ (i is better than j), $i \prec j$ if $a(i, j) = -1$ (i is worse than j), and $i \sim j$ if $a(i, j) = 0$ (i is as good as j). Since every anti-potential game is anti-symmetric, the relation is symmetric; i.e., $i \succ j$ if and only if $j \prec i$. Therefore we can represent a given basis element of anti-potential games in a diagram as in Figure 4.

For games with cyclic symmetry (Hofbauer and Sigmund, 1998, p.173) we have the following decomposition.

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ a_4 & 0 & a_1 & a_2 & a_3 \\ a_3 & a_4 & 0 & a_1 & a_2 \\ a_2 & a_3 & a_4 & 0 & a_1 \\ a_1 & a_2 & a_3 & a_4 & 0 \end{pmatrix} \sim \underbrace{\frac{1}{2}\begin{pmatrix} 0 & a_1+a_4 & a_2+a_3 & a_2+a_3 & a_1+a_4 \\ a_1+a_4 & 0 & a_1+a_4 & a_2+a_3 & a_2+a_3 \\ a_2+a_3 & a_1+a_4 & 0 & a_1+a_4 & a_2+a_3 \\ a_2+a_3 & a_2+a_3 & a_1+a_4 & 0 & a_1+a_4 \\ a_1+a_4 & a_2+a_3 & a_2+a_3 & a_1+a_4 & 0 \end{pmatrix}}_{\mathcal{M}}$$

$$+ \underbrace{\frac{1}{2}\begin{pmatrix} 0 & a_1-a_4 & a_2-a_3 & -a_2+a_3 & -a_1+a_4 \\ -a_1+a_4 & 0 & a_1-a_4 & a_2-a_3 & -a_2+a_3 \\ -a_2+a_3 & -a_1+a_4 & 0 & a_1-a_4 & a_2-a_3 \\ a_2-a_3 & -a_2+a_3 & -a_1+a_4 & 0 & a_1-a_4 \\ a_1-a_4 & a_2-a_3 & -a_2+a_3 & -a_1+a_4 & 0 \end{pmatrix}}_{\mathcal{M}^{\perp}}$$

If $a_1 - a_4 = a_2 - a_3 - 1$, then the anti-potential part of game can be represented in Figure 5.

In case of two-strategy bi-matrix coordination games, we have the following de-

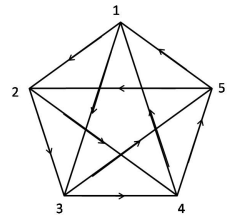


Figure 5: **Games with cyclic symmetry.**

composition of a two-strategy.

$$\begin{array}{|c|c|}\hline a,b & 0,0 \\ \hline 0,0 & c,d \\ \hline\end{array} \sim \frac{1}{2}\underbrace{\begin{array}{|c|c|}\hline 0,0 & 0,-b+d \\ \hline -a+c,0 & -a+c,-b+d \\ \hline\end{array}}_{\ker(\mathbf{\Gamma})} + \frac{1}{8}(a+b+c+d)\underbrace{\begin{array}{|c|c|}\hline 1,1 & -1,-1 \\ \hline -1,-1 & 1,1 \\ \hline\end{array}}_{\bar{\mathcal{N}}^{\perp}}$$

$$+\frac{1}{8}(-a+b-c+d)\underbrace{\begin{array}{|c|c|}\hline -1,1 & 1,-1 \\ \hline 1,-1 & -1,1 \\ \hline\end{array}}_{\bar{\mathcal{M}}^{\perp}}$$

Therefore, a two-strategy coordination game is a potential game if and only if $-a+b-c+d=0$ and a zero-sum game if and only if $a+b+c+d=0$. In other words, the coefficients of the anti-potential game and the anti-zero-sum game corresponds to the condition for payoffs in four-cycle criteria as in Corollary 2.9 and 2.11.

3. Applications of Decompositions

3.1. Decompositions and Stable Games

In this section, we provide a characterization of stable games (for properties of stable games see Hofbauer and Sandholm, 2009). A symmetric game $[A]$ is a stable game if $\langle y-x, A(y-x)\rangle_{\mathbb{R}_l} \leq 0$ for all $x,y \in \Delta_l$. A bi-matrix game (A,B) is a stable game if $\langle y-x, (A,B)(y-x)\rangle_{\mathbb{R}_{l_r+l_c}} \leq 0$ for all $x,y \in \Delta_{l_r} \times \Delta_{l_c}$. A null-stable game is a stable for which equality holds instead of an inequality.

Note that since $[A] = (A, A^T)$, the condition for a symmetric game to be stable can be written as

$$\left\langle y-x, (A,A^T)(y-x)\right\rangle_{\mathbb{R}_{l+l}} \leq 0 \text{ for all } x,y \in \{(p,q) \in \Delta_l \times \Delta_l : p=q\}. \tag{11}$$

By comparing this to the condition for bi-matrix games to be stable (with $l_r = l_c = l$)

we see that the inequality in (11) holds for a smaller subset of $\mathbb{R}^{2l}$. This opens the possibility that more stable games arise in symmetric games. Using the projection operator P defined in Section 2 we see that a symmetric game A is a stable game if and only if $\langle x, PAPx\rangle \leq 0$ for all $x \in \mathbb{R}_l$ and a bi-matrix game (A, B) is a stable game if and only if $\langle x, \mathbb{P}(A, B)\mathbb{P}x\rangle \leq 0$ for all $x \in \mathbb{R}_{l_r+l_c}$ (Hofbauer and Sandholm, 2009, Theorem 2.1).

We first characterize stable symmetric matrix games. To do this we define a function V_A for a given symmetric game A, which will play an important role in characterizing stable games: $V_A(x) := \frac{1}{2}\langle x, Ax\rangle$. Then using the decomposition, we obtain the following representation of V_A.

Proposition 3.1. *Suppose that $A \in \mathcal{L}$. Then there exists a symmetric matrix S with $S\mathbf{1} = 0$ and a column vector a vector c such that, for any $x \in \Delta$ and any $z \in T\Delta$*

$$V_A(x) = \frac{1}{2}\langle x, Sx\rangle + \langle x, c\rangle, \quad V_A(z) = \frac{1}{2}\langle z, Sz\rangle.$$

Moreover there exists an orthonormal basis $\{v_1, \cdots, v_{l-1}\}$ of $T\Delta$ (in particular $\langle v_i, \mathbf{1}\rangle = \mathbf{0}$) such that $S = \sum_{i=1}^{l-1} \lambda_i S_i$ where S_i is the orthogonal projection onto the eigenspace spanned by v_i.

Proof. Let $A \in \mathcal{L} = \mathcal{N}^\perp \oplus \ker(\Gamma) \oplus \mathcal{M}^\perp$ and thus we can write

$$A = S + c_1\mathbf{1}^T + \mathbf{1}c_2^T + N$$

where S is symmetric with $S\mathbf{1} = 0$ and N is anti-symmetric with $N\mathbf{1} = 0$. Thus for any $x \in \mathbb{R}^l$ we have

$$V_A(x) = \frac{1}{2}\langle x, Sx\rangle + \frac{1}{2}\left\langle x, (c_1\mathbf{1}^T + \mathbf{1}c_2^T)x\right\rangle$$

For $x \in \Delta$ we have $\left\langle x, c_1\mathbf{1}^T x\right\rangle = \sum_i x_i \langle x, c_1\rangle = \langle x, c_1\rangle$ and $\left\langle x, \mathbf{1}c_2^T x\right\rangle = \sum_i x_i \langle x, c_2\rangle = \langle x, c_2\rangle$, and thus

$$\frac{1}{2}\left\langle x, (c_1\mathbf{1}^T + \mathbf{1}c_2^T)x\right\rangle = \langle x, c\rangle \qquad \text{where } c = c_1 + c_2.$$

Note further that $S = PSP$ and $\left\langle z, c_1\mathbf{1}^T z\right\rangle = \left\langle z, \mathbf{1}c_2^T z\right\rangle = 0$ for $z \in T\Delta$. Since

S is symmetric, all eigenvectors are orthogonal and since $\mathbf{1}$ is an eigenvector with the corresponding eigenvalue $\lambda = 0$, all other eigenvectors belong to $T\Delta$ and the representation of S follows from the spectral theorem. ■

To characterize the stable games using Proposition 3.1, we let $A \in \mathcal{L}$ and $z \in T\Delta$. Then

$$V_A(z) = \frac{1}{2}\langle z, Sz\rangle = \frac{1}{2}\left\langle \sum_i \xi_i v_i, S\sum_i \xi_i v_i \right\rangle = \frac{1}{2}\sum_i \xi_i^2 \lambda_i$$

where v_i is orthonormal basis for $T\Delta$ consisting of eigenvectors of S. Thus A is null-stable iff $\lambda_i = 0$ for all i. Therefore A is a null-stable game if and only if $A \in \mathcal{N}$. We put this fact as Proposition 3.2 of which another direct proof is presented in the Appendix. Similarly note that $V_A(z) < 0$ for all $z \neq 0$ if and only if $\lambda_i < 0$ for all i. Thus a game is a strict stable game if and only if the eigenvalues for S, except the one corresponding to $\mathbf{1}$, are all negative.

Proposition 3.2. *$\langle x, PAPx\rangle = 0$ for all $x \in \mathbb{R}^l$ if and only if $A \in \mathcal{N}$.*

As is well-known, *the Hawk-Dove game* provides the simplest possible strictly stable game, and from the equivalence we find

$$(z_1, z_2)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -(z_1 - z_2)^2 < 0, \text{ for } (z_1, z_2) \neq (0, 0).$$

This observation can be generalized via the basis of the subspace of anti-zero-sum games $\mathcal{N}^\perp$.

Corollary 3.3 ($l-$strategy strictly stable games). *Suppose that*

$$A \in \{\sum_{j>i} \alpha^{(ij)} K^{(ij)} : \alpha^{(ij)} > 0\} + \ker(\mathbf{\Gamma}) + \mathcal{M}^\perp.$$

Then A is a strict stable game.

Proof. Recall that $[A]$ is a strict stable game if $\langle z, Az\rangle < 0$ for all $z \in T\Delta$ such that $z \neq \mathbf{0}$.Let $A \in S$. Then we have $\langle z, Az\rangle = -\sum_{j>i} \alpha^{(ij)}(z_i - z_j)^2 \leq 0$.Now suppose that $-\sum_{j>i} \alpha^{(ij)}(z_i - z_j)^2 = 0$. Then we have $z_i - z_j = 0$ for all $j > i$. Since $z \in T\Delta$, this implies that $z = \mathbf{0}$. ■

In case of three-strategy games, we can strengthen Corollary 3.3 so as to characterize three-strategy strict stable games completely, since the computation in three-strategy case is less demanding.

Corollary 3.4 (three-strategy strictly stable games). *A three−strategy symmetric game A is strictly stable if and only if*

$$A \in \left\{ \begin{pmatrix} -a-b & a & b \\ a & -a-c & c \\ b & c & -b-c \end{pmatrix} : 4a+b+c>0,\ ab+bc+ca>0 \right\} + \ker(\Gamma) + \mathcal{M}^{\perp}.$$

First we note that when $l = 3$ in Corollary 3.3 the condition for strictly stable games is a special case of Corollary 3.4 by the choices of $a, b > 0$ and $c = 0$. As another important special case of Corollary 3.4, consider game B given by

$$B = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ \beta_{12} & 0 & \beta_{23} \\ \beta_{13} & \beta_{23} & 0 \end{pmatrix}.$$

First note that B is a potential game, so there is no anti-potential part of B. Thus B can be decomposed into

$$\begin{aligned} B &= \begin{pmatrix} -a-b & a & b \\ a & -a-c & c \\ b & c & -b-c \end{pmatrix} + \underbrace{C}_{\in\ \ker(\Gamma)} \text{ and} \\ a &= \frac{1}{9}(5\beta_{12} - \beta_{13} - \beta_{23}), b = \frac{1}{9}(-\beta_{12} + 5\beta_{13} - 2\beta_{23}), c = \frac{1}{9}(-\beta_{12} - \beta_{13} + 5\beta_{23}). \end{aligned}$$

Then the conditions in Corollary 3.4 imply

$$\beta_{12} > 0 \quad \text{and } (\beta_{12} + \beta_{23} + \beta_{13})^2 > 2(\beta_{12}^2 + \beta_{23}^2 + \beta_{13}^2). \tag{12}$$

Recall that the generalized Rock-Paper-Scissors game can be decomposed as follows:

$$\begin{pmatrix} 0 & -l & w \\ w & 0 & -l \\ -l & w & 0 \end{pmatrix} \sim \frac{1}{2}(w-l)\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \frac{1}{2}(w+l)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

We see that the case when $\beta_{12} = \beta_{23} = \beta_{13}$, $\beta_{12} > 0$ satisfies conditions in (12), so using Corollary 3.4 we conclude that the generalized Rock-Paper-Scissors game is strictly stable if and only if $w > l$ (See the discussion in (See the discussion in Hofbauer and Sandholm, 2009)). In the next section we will provide another useful parametrization of three-strategy anti-zero-sum games.

Next we characterize the bi-matrix stable games. First we recall that for J given by

$$J := \begin{pmatrix} O & A \\ B^T & O \end{pmatrix},$$

the characteristic polynomial $p(\lambda) = \det(J - \lambda I)$ satisfies $p(\lambda) = (-1)^{l_r+l_c}p(-\lambda)$. Hence if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue. For a given bi-matrix game (A, B), we can write $(A, B) \sim (V, V) + (C, D) + (N, -N)$ where $(C, D) \in \ker(\mathbf{\Gamma})$. Thus, $\mathbb{P}(A, B)\mathbb{P} = \mathbb{P}(V, V)\mathbb{P}$. So if (A, B) is a stable game, all its eigenvalues must have the same sign and, thus, they must be all zeros. Hence every stable bi-matrix game is always null-stable (Hofbauer and Sandholm, 2009, Theorem2.1). Then, as the similar argument as Proposition 3.1 shows, every null-stable bi-matrix game is a zero-sum game. As a result, we provide the complete characterization of the set of all stable bi-matrix games; the set of all stable bi-matrix games is the set of all zero-sum games. Proposition 3.5 can be proved via either the straightforward extension of Proposition 3.2 or the direct use of the basis of decompositions. We provide the direct proof in the Appendix.

Proposition 3.5. $\langle w, \mathbb{P}(A, B)\mathbb{P}w\rangle = 0$ *for all* $w \in \mathbb{R}_{l_r+l_c}$ *if and only if* $(A, B) \in \bar{\mathcal{N}}$.

3.2. Decomposition and Deterministic Dynamics

Evolutionary dynamics based on the normal form games have been extensively examined and their important properties are closely related to the underlying games; for example, potential games yield the gradient like replicator dynamics (Hofbauer and Sigmund, 1998). Moreover the replicator dynamics are linear with respect to

the underlying game matrix (or matrices), so our decompositions naturally induce decompositions at the level of vector fields. We will consider the replicator dynamics given by

$$\text{One population:} \quad \dot{x}_i = x_i((Ax)_i - x^T Ax) \text{ for all } i \tag{13}$$

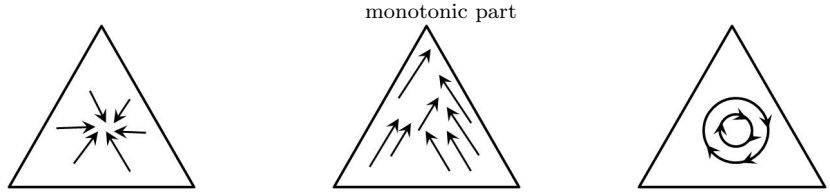
$$\text{Two population: } x_i = x_i((Ay)_i - x^T Ay), \quad \dot{y}_j = y_j((B^T x)_j - y^T B^T x)$$

When we have $A \sim S + G + N$, where $S \in \mathcal{N}^\perp$, $G \in \ker(\Gamma)$, $N \in \mathcal{M}^\perp$, the replicator dynamics can also be decomposed in three parts. First note that if $G = \sum_i \eta_i E_\eta^{(i)}$, then $(Gx)_i = \eta_i$ and $\langle x, Gx \rangle = \sum_{l \neq 1} \eta_l x_l$, so the vector field for the replicator dynamics induced by G is given by

$$x_i(\eta_i - \sum_{l \neq 1} \eta_l x_l)$$

and the system monotonically moves towards the dominating strategy state. Also when $x^T N x = 0$ for $N \in \mathcal{M}^\perp$. Thus, the replicator ordinary differential equation for the matrix A can be decomposed into

$$f_i(x) \sim \underbrace{x_i((Sx)_i - x^T Sx)}_{\text{potential part}} + \underbrace{x_i(\eta_i - \sum_{l \neq 1} \eta_l x_l)}_{\text{monotonic part}} + \underbrace{x_i N x}_{\text{conservative part}}$$



This decomposition of the vector field of the replicator ordinary differential equations coincides with the known Hodge decomposition which plays an important role in understanding the underlying dynamics (See Abraham et al. (1988) and equation (1) in Tong et al. (2003)).

We recall that a function $H\colon D \to \mathbb{R}$ is an integral of (13) on a region D if H is continuous differentiable and $H(x(t))$ is constant along the solution of (13);i.e., $LH(x(t)) := \langle \nabla H(x(t)), f(x(t)) \rangle = 0$ for a solution $x(t)$. The orbits of a conservative system must therefore lie on level curves of the integral H. A system (13) is said to be conservative if it has an integral H. We again recall that a function $V\colon D \to \mathbb{R}$ is a strict Lyapunov function for $C \subset D$ if V is continuous that achieves its minimum at

C is non-increasing along the solutions and is decreasing outside of C; i.e., $LV(x) := \langle \nabla V(x), f(x)) \rangle \leq 0$ for x in D and $LV(x) < 0$ for $x \notin C$.

It is well-known that the replicator dynamic for the Rock-Paper-Scissors games is conservative and volume-preserving, the dynamics of the Matching Pennies games can be transformed to Hamiltonian systems by change in velocity of solutions, and all the bi-matrix games preserve volume up to change in velocity of solutions (Hofbauer and Sigmund, 1998). As Proposition 3.6 shows, the class of anti-potential games provides the dynamics which are volume-preserving without involving the change of time.

Proposition 3.6. *(1) Suppose that $[A]$ is an anti-potential game. Then (13) is conservative and volume-preserving.*

(2) Suppose $(A, B) \in range(\mathbf{\Gamma})$. Then (A, B) is conservative.

(3) Suppose that (A, B) is an anti-potential game and $l_c = l_r$. Then (A, B) is volume preserving.

In the generalized Rock-Paper-Scissors game, it is easy to check that when $b > a$, $H(x) := \sum_i \log(x_i)$ is a strict Lyapunov function for $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Our decompositions show that this observation generalizes to the bigger class of games that have the similar structure to the generalized Rock-Paper-Scissors game.

Proposition 3.7. *Suppose*

$$A \in \{\sum_{j>i} \alpha^{(ij)} K : \alpha^{(ij)} > 0\} + \mathcal{M}^{\perp} \ .$$

Then, $H(x) := \sum_i \log(x_i)$ is a strict Lyapunov function for $\frac{1}{n}\mathbf{1}$. And thus a unique NE $\frac{1}{n}\mathbf{1}$ is evolutionarily stable.

Proof. Let $A = S + N$, where $S \in \{\sum_{j>i} \alpha^{(ij)} K : \alpha^{(ij)} > 0\}$ and $N \in \mathcal{M}^{\perp}$. Note that for $x \neq \frac{1}{n}\mathbf{1}$, we have

$$\begin{aligned} LH &= \sum_i ((Ax)_i - x^T Ax) = \sum_i (Sx)_i - \langle x, Sx \rangle + \sum_i (Nx)_i \\ &= -\langle x, Sx \rangle = -\langle x, PSPx \rangle = -\langle z, Sz \rangle > 0 \end{aligned}$$

■

Next we explain how to obtain the game which has a pure strategy ESS and an interior asymptotically stable NE (called the Zeeman game) using the decomposition. We consider a game $A \in \mathcal{N}^{\perp} \oplus \mathcal{M}^{\perp}$. Then since $A\mathbf{1} = \mathbf{0}$, $\frac{1}{n}\mathbf{1}$ is a Nash equilibrium. From the previous discussion, the anti-zero-sum part S of A is completely determined by its eigenvalues and (orthonormal) eigenvectors. Recall that S always has an eigenvector $\mathbf{1}$ with the corresponding eigenvalue 0 and other eigenvectors lie in the tangent space. Thus when the number of strategies is three, any two eigenvectors in the tangent space can be obtain by rotating given reference orthogonal eigenvectors around the axis $(1, 1, 1)$. First we denote the matrix for the Rock-Paper-Scissors game by N :

$$N := \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Next, to express this parameterization of S we define the rotation matrix R which rotates a given vector in $\mathbb{R}^3$ around the axis $(1, 1, 1)$, as follows:

$$R = I - P + (\cos\theta I + \sin\theta \frac{1}{\sqrt{3}} N)P. \tag{14}$$

To explain the meaning of R, we first recall that the rotation matrix in $\mathbb{R}^2$ acts as follows:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x = \cos\theta I x + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x.$$

Thus the rotation matrix map x to the linear combination of x itself and a vector orthogonal to x, and the coefficients of the combination are parameterized by an angle. Now note that $\langle Nx, x\rangle = 0$ for all x. Thus when $z \in T\Delta$,

$$Rz = \cos\theta I z + \sin\theta \frac{1}{\sqrt{3}} N z$$

and since Nz is orthogonal to z, R acts in the same way as the rotation in two-dimension. Also clearly $R\mathbf{1} = \mathbf{0}$. When $x \in \mathbb{R}^3$, x can be uniquely written as $x = (I - P)x + Px$ and R rotates the part belonging to $range(P)$. Thus, R has the representation in (14). Using the rotation matrix R, we can write a three-strategy

game A as follows:

$$A = R\begin{pmatrix} \alpha+\frac{\beta}{3} & -\frac{2\beta}{3} & -\alpha+\frac{\beta}{3} \\ -\frac{2\beta}{3} & \frac{4\beta}{3} & -\frac{2\beta}{3} \\ -\alpha+\frac{\beta}{3} & -\frac{2\beta}{3} & \alpha+\frac{\beta}{3} \end{pmatrix} R^{-1} + \eta \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$\text{where } \begin{pmatrix} \alpha+\frac{\beta}{3} & -\frac{2\beta}{3} & -\alpha+\frac{\beta}{3} \\ -\frac{2\beta}{3} & \frac{4\beta}{3} & -\frac{2\beta}{3} \\ -\alpha+\frac{\beta}{3} & -\frac{2\beta}{3} & \alpha+\frac{\beta}{3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & 0 \\ 0 & 0 & 2\beta \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}^{-1}$$

Then the matrix A has the characteristic polynomial $\phi(t) = t(t^2 - 2(\alpha+\beta)t + 4\alpha\beta + 3\eta^2)$, so it has the eigenvalues 0, $\alpha+\beta \pm \sqrt{(\alpha-\beta)^2 - 3\eta^2}$ and the eigenvector $\mathbf{1}$ corresponding to the eigenvalue 0. Note the eigenvalues for A do not depend on the choice of θ. We can also verify this as follows. From $RN = NR$, we have

$$A = REDE^{-1}R^{-1} + \eta N = REDE^{-1}R^{-1} + \eta RNR^{-1} = R(EDE^{-1} + \eta N)R^{-1},$$

where E denotes the matrix whose columns consist of orthogonal eigenvectors and D denotes the diagonal matrix which has 0, 2α,and 2β on the diagonal. Since $R(EDE^{-1} + \eta N)R^{-1}$ has the same eigenvalues as $EDE^{-1} + \eta N$, eigenvalues of A do not depend on the particular choice of θ.

Since $\frac{\partial}{\partial x_i}(x^T Ax) = (Ax)_i + (A^T x)_i$, by differentiating (13), we find that

$$\frac{\partial f_i(x)}{\partial x_j} = x_i(a_{ij} - (Ax)_j - (A^T x)_j) \quad \text{for } j \neq i, \ \frac{\partial f_i(x)}{\partial x_i} = ((Ax)_i - xA^T x) + x_i(a_{ii} - (Ax)_i). \tag{15}$$

So if we evaluate the expressions in (15) at $x = \frac{1}{n}\mathbf{1}$, from $A\mathbf{1} = \mathbf{0}$, $A^T\mathbf{1} = \mathbf{0}$,and $(Ax)_i - xA^T x = 0$, we find the following Jacobian matrix

$$\left.\frac{\partial f_i(x)}{\partial x_j}\right|_{x=\frac{1}{n}\mathbf{1}} = \frac{1}{n}a_{ij}.$$

Thus the eigenvalues for the linearized system around $\frac{1}{n}\mathbf{1}$ are the same as the ones for A up to a scalar multiple $\frac{1}{n}$. Also we note that if $(\alpha-\beta)^2 < 3\eta^2$, then two non-zero eigenvalues are complex and in this case real parts of eigenvalues are negative (zero, positive, resp.) if and only if $\alpha+\beta < 0$ ($\alpha+\beta = 0, \alpha+\beta > 0$,resp.). Now we set

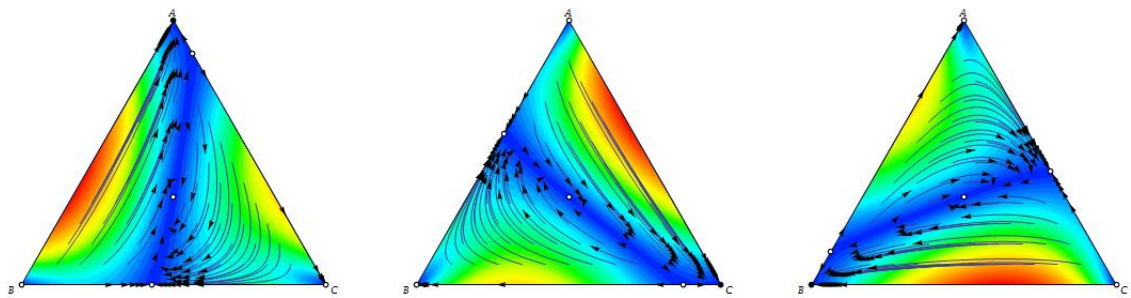


Figure 6: **Rotation of Eigenvectors in the Zeeman Games.** Figures are drawn using Dynamo: W. H. Sandholm, E. Dokumaci, and F. Franchetti (2010). Dynamo: Diagrams for Evolutionary Game Dynamics, version 0.2.5.

$\theta = 0$. Then

$$A = \frac{1}{3}\begin{pmatrix} 3\alpha+\beta & -2\beta+3\eta & -3\alpha+\beta-3\eta \\ -2\beta-3\eta & 4\beta & -2\beta+3\eta \\ -3\alpha+\beta+3\eta & -2\beta-3\eta & 3\alpha+\beta \end{pmatrix}$$

so it is easy to see that if $-(\alpha+\beta) < \eta < 2\alpha$, then strategy 1 is a strict Nash equilibrium, hence an evolutionary stable strategy. Thus we obtain the following characterization of Zeeman games.

Proposition 3.8. *Suppose that $-(\alpha+\beta) < \eta < 2\alpha$, and $(\alpha-\beta)^2 < 3\eta^2$. Then strategy 1 is an ESS and the interior fixed point is a sink (center, source, resp.) if $\alpha+\beta < 0$ ($\alpha+\beta = 0, \alpha+\beta > 0$,resp.).*

In Figure 6 we show how the vector field of the system changes when θ varies. To find a four-strategy Zeeman game we consider the following matrix using the similar idea:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -3 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -3 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}^{-1} + \eta\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

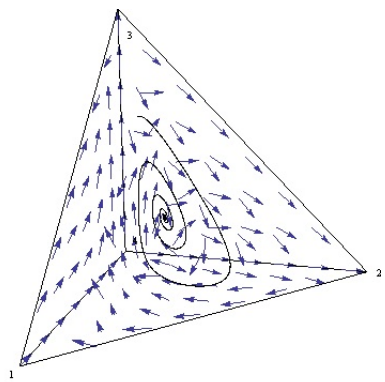


Figure 7: **4-strategy Zeeman game.** $\alpha : -2.5,\ \beta : -2.5,\ \gamma : 2,\ \eta : 1.9.$

Then, it is easy to see that if $-\gamma < \eta < \gamma$ and $\gamma > 0$, strategy 2 become a strict Nash equilibrium, so an ESS. The characteristic polynomial for A is

$$\phi(t) = t(t^3 - (\alpha+\beta+\gamma)t^2 + (\alpha\beta+\beta\gamma+\gamma\alpha+4\eta^2)t - \alpha\beta\gamma - \frac{1}{3}(6\alpha+2\beta+4\gamma)\eta^2).$$

Thus from the Routh-Hurwitz criterion(For example see Murray, 1989), we see that eigenvalues λ for A all have negative real parts (except 0 eigenvalue) if and only if

$$\begin{aligned}&\alpha+\beta+\gamma < 0,\ \alpha\beta+\beta\gamma+\gamma\alpha+4\eta^2 > 0,\ 3\alpha\beta\gamma + (6\alpha+2\beta+4\gamma)\eta^2 < 0,\\ &\alpha\beta\gamma + \frac{1}{3}(6\alpha+2\beta+4\gamma)\eta^2 > (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha+4\eta^2).\end{aligned}$$

Using these conditions we exhibit a four-strategy Zeeman game in Figure 7.

4. Conclusion

We have developed several decomposition methods for two player normal form games and discussed the extension to the general normal form games. Using decompositions, we characterize (1) the subspaces of potential games and their orthogonal complements, anti-potential games, (2) the subspaces of zero-sum games and their orthogonal complements, anti-zero-sum games, and (3) the subspaces of both potential and zero-sum games and their orthogonal complements. Notably, the subspaces of anti-potential games consist of special games, the Rock-Paper-Scissors games in the case of the symmetric games and the Matching Pennies games in the case of the bi-matrix games. We have explained how the previous known criterion for the poten-

tial games can be viewed from the perspective of decompositions and provided a new cycle criterion for symmetric zero-sum games.

We have discussed the various applications of the decompositions, including (1) the analysis of the generalized Rock-Paper-Scissors games, (2) the characterization of the stable games, (3) the decomposition of the vector field and the construction of Lyapunov functions in evolutionary dynamics, and so on. These decompositions turn out to be useful in the analysis of the stochastic dynamics of the evolutionary games; these applications will be discussed elsewhere.

Appendix A. Appendix

Appendix A.1. Properties of inner products

First we observe that

1. $(A, B)^T = (B, A)$
2. (A, B) is symmetric in $\mathcal{L}^2$ if $A = B$
3. (A, B) is anti-symmetric in $\mathcal{L}^2$ if $A = -B$
4. (A, B) is a symmetric game if $l_r = l_c$ and $A = B^T$

So, a bi-matrix symmetric game is not necessarily a symmetric matrix in $\mathcal{L}^2$. We endow $\mathcal{L}^2$ with an inner product $<,>_{\mathcal{L}^2}$ defined by:

$$\langle (A, B), (C, D) \rangle_{\mathcal{L}^2} := tr((A, B)^T (C, D))$$

We provide some properties of $\langle\ \rangle_{\mathcal{L}^2}$ and $\langle\ \rangle_{\mathcal{L}}$.

Lemma Appendix A.1. *For $(l_r \times l_c)$ matrices A, B, C, D , we have*

(1) $\langle (A, B), (C, D) \rangle_{\mathcal{L}^2} = \langle A, C \rangle_{\mathcal{L}} + \langle B, D \rangle_{\mathcal{L}}$

(2) $\langle SA, B \rangle_{\mathcal{L}} = \langle A, SB \rangle_{\mathcal{L}}$ *for a symmetric $(l_r \times l_r)$ matrix S*

(3) $\langle (A, A), (B, -B) \rangle_{\mathcal{L}^2} = 0$

(4) For $\mathbf{c} \in \mathbb{R}^{l_r}$ and A such that $A\mathbf{1}_{l_c} = \mathbf{0}$, $\left\langle A, \mathbf{c}\mathbf{1}_{l_c}^T \right\rangle_{\mathcal{L}} = 0$.

(5) For $\mathbf{c} \in \mathbb{R}^{l_c}$ and A such that $\mathbf{1}_{l_r}^T A = \mathbf{0}$, $\left\langle A, \mathbf{1}_{l_r}\mathbf{c}^T \right\rangle_{\mathcal{L}} = 0$.

Proof. (1) and (2) are obvious. (3) follows from

$$\langle (A, A), (B, -B) \rangle_{\mathcal{L}^2} = \langle A, B \rangle_{\mathcal{L}} - \langle A, B \rangle_{\mathcal{L}} = 0.$$

(4) follows from

$$\left\langle A, \mathbf{c}\mathbf{1}_{l_c}^T \right\rangle = tr(\mathbf{1}_{l_c}\mathbf{c}^T A) = tr(\mathbf{c}^T A \mathbf{1}_{l_c}) = 0$$

by the commutativity of trace and (5) follows from

$$\left\langle A, \mathbf{1}_{l_r}\mathbf{c}^T \right\rangle_{\mathcal{L}} = tr(\mathbf{c}\mathbf{1}_{l_r}^T A) = 0$$

■

Appendix A.2. Proof of Proposition 2.5

Proof. (1) We first show that

$$\ker \Gamma = \text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$$

Note that $PE_\gamma^{(j)} = O$ for all j. Then $E_\eta^{(i)} P = (P(E_\eta^{(i)})^T)^T = O$ for all i. Thus we have span $\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\} \subset \ker \Gamma$. Conversely, let A such that $\Gamma(A) = O$. Since

$$PAP = A - \frac{1}{l}\mathbf{1}\mathbf{1}^T A - \frac{1}{l} A \mathbf{1}\mathbf{1}^T + \frac{1}{l^2}\mathbf{1}\mathbf{1}^T A \mathbf{1}\mathbf{1}^T,$$

we have

$$A = \frac{1}{l}\mathbf{1}\mathbf{1}^T A + \frac{1}{l}A\mathbf{1}\mathbf{1}^T - \frac{1}{l^2}\mathbf{1}\mathbf{1}^T A\mathbf{1}\mathbf{1}^T$$

Then note the following properties of $\mathbf{1}\mathbf{1}^T$:

$$\mathbf{1}\mathbf{1}^T A = (\sum_k a_{k1}\mathbf{1} : \sum_k a_{k2}\mathbf{1} : \cdots : \sum_k a_{kl}\mathbf{1})$$

i.e., the left action of $\mathbf{1}\mathbf{1}^T$ on A turns A into a matrix with the same elements in each column. Since $A\mathbf{1}\mathbf{1}^T = (\mathbf{1}\mathbf{1}^T A^T)^T$, the right action of $\mathbf{1}\mathbf{1}^T$ on A turns A into a matrix with same elements in each column. Also it is easy to see that $\mathbf{1}\mathbf{1}^T A\mathbf{1}\mathbf{1}^T = \sum_k \sum_m a_{km}\mathbf{1}\mathbf{1}^T$. Thus A can be written as

$$A = \sum_i (\sum_k a_{ki})E_\gamma^{(i)} + \sum_j (\sum_k a_{jk})E_\eta^{(j)} + (\sum_k \sum_m a_{km}) \sum_j E_\gamma^{(j)}$$

So $A \in \text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$. Thus $\ker\Gamma = \text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$. Next note that

$$\sum_i E_\eta^{(i)} = \sum_j E_\gamma^{(j)}, \quad \text{so } E_\gamma^{(1)} = -\sum_{j\neq 1} E_\gamma^{(j)} - \sum_i E_\eta^{(i)},$$

thus

$$\text{span}\{E_\eta^{(1)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\} = \text{span}\{E_\eta^{(2)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}.$$

To show the linear independence among $\{E_\eta^{(2)}, \cdots, E_\eta^{(l)}, E_\gamma^{(1)}, \cdots, E_\gamma^{(l)}\}$, consider the linear combination of these matrices:

$$O = \sum_{i\neq 2} \eta_i E_\eta^{(2)} + \sum_j \gamma_j E_j^{(2)}.$$

Then since $E_\eta^{(1)}$ does not appear in the linear combination, we have $\gamma_j = 0$ for all j and this implies $\eta_i = 0$ for $i \neq 2$.
(2) Note because of $\mathbf{1}E_\kappa^{(ij)} = \mathbf{0}$ and $E_\kappa^{(ij)}\mathbf{1} = \mathbf{0}, \Gamma(E_\kappa^{(ij)}) = PE_\kappa^{(ij)}P = E_\kappa^{(ij)}$. So,

$$E_\kappa^{(ij)} \subset \text{range}(\Gamma) \quad \text{for all } i, j \geq 2$$

and it is easy to see that $E_\kappa^{(ij)}$ are linearly independent. Finally by $\left|\{E_\kappa^{(ij)}\}_{ij}\right| = (l-1)^2$ and since $\dim(\text{range}(\Gamma)) = (l-1)^2$, $\{E_\kappa^{(ij)}\}_{ij}$ is a basis for range(Γ) ■

Appendix A.3. Proof of Proposition 2.6

Proof. First we show that $\ker(\Gamma) = \mathcal{M} \cap \mathcal{N}$. Observe that for $i \geq 2$

$$(E_\eta^{(i)} + E_\gamma^{(i)})^T = (E_\eta^{(i)})^T + (E_\gamma^{(i)})^T = E_\gamma^{(i)} + E_\eta^{(i)}$$

Thus $(E_\eta^{(i)} + E_\gamma^{(i)})$ is symmetric and $(E_\eta^{(i)} + E_\gamma^{(i)})_{11} = 0$, so $E_\eta^{(i)} = (E_\eta^{(i)} + E_\gamma^{(i)})^T - E_\gamma^{(i)} \in \mathcal{M}$. Also

$$(E_\eta^{(i)} - E_\gamma^{(i)})^T = (E_\eta^{(i)})^T - (E_\gamma^{(i)})^T = -(E_\eta^{(i)} - E_\gamma^{(i)}),$$

so $(E_\eta^{(i)} - E_\gamma^{(i)})$ is anti-symmetric and $E_\eta^{(i)} = -(E_\eta^{(i)} - E_\gamma^{(i)})^T + E_\gamma^{(i)} \in \mathcal{N}$. Therefore $\ker\Gamma \subset \mathcal{M} \cap \mathcal{N}$. Conversely let $A \in \mathcal{M} \cap \mathcal{N}$. Then

$$A = S + \mathbf{1}\mathbf{c}_1^T \text{ and } A = B + \mathbf{1}\mathbf{c}_2^T \text{ for a symmetric } S \text{ and anti-symmetric } B.$$

Thus $B + \mathbf{1}\mathbf{c}_2^T - \mathbf{1}\mathbf{c}_1^T = B^T + \mathbf{c}_2\mathbf{1}^T - \mathbf{c}_1\mathbf{1}^T$ and using the anti-symmetry of B, we obtain

$$B = \frac{1}{2}(\mathbf{c}_2\mathbf{1}^T - \mathbf{c}_1\mathbf{1}^T + \mathbf{1}\mathbf{c}_2^T - \mathbf{1}\mathbf{c}_1^T)$$

and so

$$A = \frac{1}{2}(\mathbf{c}_2\mathbf{1}^T - \mathbf{c}_1\mathbf{1}^T + \mathbf{1}\mathbf{c}_2^T - \mathbf{1}\mathbf{c}_1^T) + \mathbf{1}\mathbf{c}_2^T \in \ker\Gamma.$$

Next we show that $\text{range}(\Gamma) = \text{span}(\mathcal{M}^\perp \cup \mathcal{N}^\perp)$. Then, we have

$$\begin{aligned}
\text{span}(\mathcal{M}^\perp \cup \mathcal{N}^\perp) &= \text{span}(\{N^{(ij)}\}_{j>i\geq 2} \cup \{H^{(ij)}\}_{j>i\geq 2} \cup \{K^{(ii)}\}_{i\geq 2}\} \\
&= \text{span}(\{K^{(ij)}\}_{j>i\geq 2} \cup \{K^{(ij)}\}_{i>j\geq 2} \cup \{K^{(ii)}\}_{i\geq 2}\} \\
&= \text{span}(\{K^{(ij)}\}_{i\geq 2, j\geq 2}\} = \text{range}(\Gamma)
\end{aligned}$$

■

Appendix A.4. Proof of Lemma 2.13

Proof. First we show that $\text{span}(\{[A^{(ij)}+A^{(ji)}]\}_{i,j\in\mathcal{I}_1\cap\{j\geq i\}} \cup \{[B^{(ij)}-B^{(ji)}]\}_{i,j\in\mathcal{I}_2\cap\{j>i\}} \cup \{[C^{(i)}]\}_{i\in\mathcal{I}_3}) = \mathcal{K}\cap\mathcal{L}_{sym}$. Obviously,

$$\begin{aligned}
[A^{(ij)} + A^{(ji)}] &= (A^{(ij)} + A^{(ji)}, (A^{(ij)})^T + (A^{(ji)})^T) = (A^{(ij)}, A^{(ji)}) + (A^{(ji)}, A^{(ij)}) \\
&= (A^{(ij)}, A^{(ij)}) + (A^{(ji)}, A^{(ji)}) \in \mathcal{K}\cap\mathcal{L}_{sym}
\end{aligned}$$

Similarly we have $\{(B^{(ij)}, -B^{(ij)})\}_{i,j} \in \mathcal{K}\cap\mathcal{L}_{sym}$. Also $[C^{(i)}] = (C^{(i)}, (C^{(i)})^T) = (C^{(i)}, O) + (O, (C^{(i)})^T) \in \mathcal{K}\cap\mathcal{L}_{sym}$. Conversely, let $(E,F) \in \mathcal{K}\cap\mathcal{L}_{sym}$. Then

$$\begin{aligned}
&(E,F) \\
=\ & \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)}(B^{(ij)}, -B^{(ij)}) + \sum_{i\in\mathcal{I}_3} \kappa_{(3)}^{(i)}(C^{(i)}, O) + \sum_{i\in\mathcal{I}_3} \kappa_{(4)}^{(i)}(O, (C^{(i)})^T) \\
=\ & (\sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ij)} + \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ij)} + \sum_{i\in\mathcal{I}_3} \kappa_{(3)}^{(i)} C^{(i)}, \\
& \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ij)} - \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ij)} + \sum_{i\in\mathcal{I}_3} \kappa_{(4)}^{(i)} (C^{(i)})^T)
\end{aligned}$$

Since $E = F^T$, we have

$$\sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ij)} + \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ij)} + \sum_{i\in\mathcal{I}_3} \kappa_{(3)}^{(i)} C^{(i)} = \sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)} A^{(ji)} - \sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)} B^{(ji)} + \sum_{i\in\mathcal{I}_3} \kappa_{(4)}^{(i)} C^{(i)}$$

Thus we obtain

$$\sum_{i,j\in\mathcal{I}_1} (\kappa_{(1)}^{(ij)} - \kappa_{(1)}^{(ji)}) A^{(ij)} + \sum_{i,j\in\mathcal{I}_2} (\kappa_{(2)}^{(ij)} + \kappa_{(2)}^{(ji)}) B^{(ij)} + \sum_{i\in\mathcal{I}_3} (\kappa_{(3)}^{(i)} - \kappa_{(4)}^{(i)}) C^{(i)} = O \tag{A.1}$$

Then from the linear independency of $\{A^{(ij)}\}_{ij} \cup \{B^{(ij)}\}_{ij} \cup \{C^{(i)}\}_i$ in $\mathcal{L}$, we conclude that

$$\kappa_{(1)}^{(ij)} = \kappa_{(1)}^{(ji)},\ \kappa_{(2)}^{(ij)} = -\kappa_{(2)}^{(ji)},\ \text{and } \kappa_{(3)}^{(i)} = \kappa_{(4)}^{(i)} \text{ for all } i,j$$

Note that $\kappa_{(2)}^{(ii)} = 0$ for all i. Thus we have

$$\begin{aligned}
&\sum_{i,j\in\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{\{j<i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{\{i=j\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}(A^{(ij)}, A^{(ij)}) + \sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ji)}(A^{(ji)}, A^{(ji)}) + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ii)}(A^{(ii)}, A^{(ii)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}((A^{(ij)} + A^{(ji)}, A^{(ij)} + A^{(ji)}) + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \frac{1}{2}\kappa_{(1)}^{(ii)}(A^{(ii)} + A^{(ii)}, A^{(ii)} + A^{(ii)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)}((A^{(ij)} + A^{(ji)}, A^{(ji)} + A^{(ij)}) + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \frac{1}{2}\kappa_{(1)}^{(ii)}(A^{(ii)} + A^{(ii)}, A^{(ii)} + A^{(ii)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_1} \kappa_{(1)}^{(ij)} \left[A^{(ij)} + A^{(ji)}\right] + \sum_{\{(i,i)\}\cap\mathcal{I}_1} \frac{1}{2}\kappa_{(1)}^{(ii)}[A^{(ii)} + A^{(ii)}]
\end{aligned}$$

Similar manipulation yields

$$\begin{aligned}
&\sum_{i,j\in\mathcal{I}_2} \kappa_{(2)}^{(ij)}(B^{(ij)}, -B^{(ij)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ij)}(B^{(ij)}, -B^{(ij)}) + \sum_{\{j<i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ij)}(B^{(ij)}, -B^{(ij)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ij)}(B^{(ij)}, -B^{(ij)}) + \sum_{\{j>i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ji)}(B^{(ji)}, -B^{(ji)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ij)}((B^{(ij)} - B^{(ji)}, -B^{(ij)} + B^{(ji)}) = \sum_{\{j>i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ij)}((B^{(ij)} - B^{(ji)}, B^{(ji)} - B^{(ij)}) \\
= &\sum_{\{j>i\}\cap\mathcal{I}_2} \kappa_{(2)}^{(ij)} \left[B^{(ij)} - B^{(ji)}\right]
\end{aligned}$$

and finally

$$\sum_i \kappa_{(3)}^{(i)}(C^{(i)}, O) + \sum_i \kappa_{(4)}^{(i)}(O, (C^{(i)})^T) = \sum_i \kappa_{(3)}^{(i)}(C^{(i)}, (C^{(i)})^T) = \sum_i \kappa_{(3)}^{(i)}[C^{(i)}].$$

Therefore, we have $\mathrm{span}(\{[A^{(ij)} + A^{(ji)}]\}_{i,j} \cup \{[B^{(ij)} - B^{(ji)}]\}_{i,j} \cup \{[C^{(i)}]\}_i) = \mathcal{K} \cap \mathcal{L}_{sym}$. Next we show that $\{[A^{(ij)} + A^{(ji)}]\}_{i,j} \cup \{[B^{(ij)} - B^{(ji)}]\}_{i,j} \cup \{[C^{(i)}]\}_i$ are linearly independent in $\mathcal{L}^2$. Suppose that

$$\sum_{\{j\geq i\}\cap\mathcal{I}_1} \alpha_{ij}[A^{(ij)} + A^{(ji)}] + \sum_{\{j>i\}\cap\mathcal{I}_2} \beta_{ij}[B^{(ij)} - B^{(ji)}] + \sum_{\mathcal{I}_3} \gamma_i[C^{(i)}] = O \text{ in } \mathcal{L}^2$$

Then we have

$$\sum_{\{j\geq i\}\cap\mathcal{I}_1} \alpha_{ij}(A^{(ij)} + A^{(ji)}) + \sum_{\{j>i\}\cap\mathcal{I}_2} \beta_{ij}(B^{(ij)} - B^{(ji)}) + \sum_{\mathcal{I}_3} \gamma_i(C^{(i)}) = O \text{ in } \mathcal{L}$$

and note that we have

$$\begin{aligned}\sum_{\{j\geq i\}\cap\mathcal{I}_1}\alpha_{ij}(A^{(ij)}+A^{(ji)}) &= \sum_{\{j>i\}\cap\mathcal{I}_1}\alpha_{ij}A^{(ij)}+\sum_{\{i>j\}\cap\mathcal{I}_1}\alpha_{ji}A^{(ij)}+\sum_{\{i=j\}\cap\mathcal{I}_3}\alpha_{ii}A^{(ii)}\\ \sum_{\{j>i\}\cap\mathcal{I}_2}\beta_{ij}(B^{(ij)}-B^{(ji)}) &= \sum_{\{j>i\}\cap\mathcal{I}_2}\beta_{ij}B^{(ij)}-\sum_{\{i>j\}\cap\mathcal{I}_2}\beta_{ji}B^{(ij)}\end{aligned}$$

and since $\{A^{(ij)}\}_{i,j}\cup\{B^{(ij)}\}_{i,j}\cup\{C^{(i)}\}_i$ are linearly independent in $\mathcal{L}$, we conclude that $\alpha_{ij}=0,\beta_{ij}=0$, and $\gamma_i=0$ for all i,j. ■

Appendix A.5. Decomposition of n−player games

We will denote by S, S_{-p}, and $S_{-p\cup q}$ the set of all strategy profiles, the set of all strategy profiles except player p,and the set of all strategy profiles except player p and q ; i.e.,

$$\begin{aligned}\mathcal{S} &:= \{(i_{p_1},\cdots,i_{p_n}):i_{p_1},\cdots,i_{p_n}\in S\}\\ \mathcal{S}_{-q} &:= \{(i_{p_1},\cdots,\hat{i}_q,\cdots,i_{p_n}):i_{p_1},\cdots,i_{p_n}\in S\}\\ \mathcal{S}_{-q\cup r} &:= \{(i_{p_1},\cdots,\hat{i}_q,\cdots,\hat{i}_r\cdots,i_{p_n}):i_{p_1},\cdots,i_{p_n}\in S\},\end{aligned}$$

where $\hat{i}_q$ means that we omit the qth element. Then it is easy to see that $|\mathcal{S}_{-p}|=l^{n-1}$. Also for $\vec{i}_{-q\cup r}\in S_{-q\cup\gamma}$,

$$(A_{p_1})_{\vec{i}_{-q\cup r}}:=(A_{p_1})_{(i_{p_1},\cdots,\hat{i}_q,\cdots,\hat{i}_r,\cdots,i_{p_n})}$$

can be written as an $l\times l$ matrix and for $\vec{i}_{-q}\in S_{-q},(A_{p_1})_{\vec{i}_{-q}}$ can be written as a $l\times 1$ vector. We also write

$$\vec{i}_{-q}\subset\vec{j}\ \text{ if } (i_{p_1},\cdots,k,\cdots,i_{p_n})=(j_{p_1},\cdots,j_q,\cdots,j_{p_n})\ \text{ for some } k\in S$$

for $\vec{i}_{-q}\in S_{-q}$ and $\vec{j}\in S$. To define passive games, we define a tensor $E_\gamma^{\vec{i}_{-q}}$ for $\vec{i}_{-q}\in S_{-q}$ as follows:

$$(E_\gamma^{\vec{i}_{-q}})_{\vec{i}_{-q}}=\mathbf{1}\text{ and 0's in other positions}\tag{A.2}$$

where 1 denotes a $l\times 1$ vector consisting of 1's. Then $E_\gamma^{\vec{i}_{-q}}$ in (A.2) is an tensor that describes the payoffs of player q and under this payoffs, given other players' strategy profile $(i_1,\cdots,\hat{i}_q,\cdots,i_n)$ for any choice of q player's strategy, q obtains payoff 1. Then similarly we set

$$\mathcal{I}=\text{span}(\{(E_\gamma^{\vec{i}_{-p_1}},O,\cdots,O)\}_{\vec{i}_{-p_1}\in\mathcal{S}_{-p_1}},\cdots,\{(O,\cdots,E_\gamma^{\vec{i}_{-p_n}})\}_{\vec{i}_{-p_n}\in\mathcal{S}_{-p_n}})$$

where O denotes a l^n- dimensional zero tensor. Then I is the set of all passive games. We also define the following tensors: for $\vec{i}\in S$,

$$(E_\beta^{\vec{i}})_{\vec{j}}=1\text{ if }\vec{i}=\vec{j}\ .$$

Then $E_\beta^{\vec{i}}$ is a tensor which has 1 at the position $\vec{i}$ and 0's at others. Then similarly we set $\mathcal{M}:=\text{span}(\{(E_\beta^{\vec{i}},\cdots,E_\beta^{\vec{i}})\}_{\vec{i}\in\mathcal{S}},I\}$. Then we obtain Proposition 2.15. Next, using Lemma 2.16, we define the subspace of all zero-sum games:

$$\mathcal{N}:=\text{span}(\{(O,\cdots,\underbrace{E_\beta^{\vec{i}}}_{i\text{th}},\cdots,\underbrace{-E_\beta^{\vec{i}}}_{j\text{th}},\cdots,O)\}_{(i_{p_1},\cdots,i_{p_n})\in\mathcal{S},p_i,p_j\in\mathcal{P}}\cup\mathcal{I}).$$

Then we have the following characterization for anti-zero-sum games. For the strategy profile $\vec{i} = (i_{p_1}, i_{p_2}, \cdots, i_{p_n})$ such that $i_p \geq 2$ for all p,we define

$$\begin{aligned}(E_\kappa^{\vec{i}})_{(\hat{i}_{p_1}, \hat{i}_{p_2}, 1, \cdots 1)} &= E_\kappa^{(i_1, i_2)}, (E_\kappa^{\vec{i}})_{(\hat{i}_{p_1}, \hat{i}_{p_2}, i_{p_3}, \cdots 1)} = -E_\kappa^{(i_1, i_2)}, \cdots, \\ (E_\kappa^{\vec{i}})_{(\hat{i}_{p_1}, \hat{i}_{p_2}, i_{p_3}, \cdots, i_{p_n})} &= (-1)^{2n-1} E_\kappa^{(i_1, i_2)}\end{aligned}$$

and all other entries are zeros. An example of such tensors for 4 player 2 strategy is given by

$$E_\kappa^{(2,2,2,2)} = \frac{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Bigg\| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Bigg\| \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}.$$

Then instead of Proposition 2.17, we will prove the following proposition.

Proposition Appendix A.2. *$\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i} \in \mathcal{S},\ i_p \geq 2 \text{ for all } p}$ form a basis for $N^\perp$. Thus* $\dim(\mathcal{N}^\perp) = (l-1)^p$

Proof. First since $(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})$ is a symmetric tensor, $\left\langle (E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}}), N \right\rangle_{\mathcal{L}^n} = 0$ for every exact zero-sum game N. Also

$$\left\langle (E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}}), (O, \cdots, E_\gamma^{\vec{i}-q}, \cdots O) \right\rangle_{\mathcal{L}^n} = \left\langle E_\kappa^{\vec{i}}, E_\gamma^{\vec{i}-q} \right\rangle_{\mathcal{L}} = 0.$$

Thus $\text{span}(\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i} \in \mathcal{S},\ i_p \geq 2 \text{ for all } p}) \subset \mathcal{N}^\perp$. Now we show $\mathcal{N}^\perp \subset \text{span}\ (\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i} \in \mathcal{S},\ i_p \geq 2 \text{ for all } p})$. If $(A_{p_1}, \cdots, A_{p_n}) \in \mathcal{N}^\perp$, then since all $(O, \cdots, Z, \cdots, -Z, \cdots, O) \in \mathcal{N}$, $(A_{p_1}, \cdots, A_{p_n}) = (V, \cdots, V)$. We now show how to express $(V, \cdots, V)$ in terms of $\{(E_\kappa^{\vec{i}}, \cdots, E_\kappa^{\vec{i}})\}_{\vec{i} \in \mathcal{S},\ i_p \geq 2 \text{ for all } p}$. To do this we use an induction. We suppose that $\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i} \in \mathcal{S},\ i_p \geq 2 \text{ for all } p}$ form a basis for the subspace of anti-zero-sum games for $n-1$ player games. Then for each $i_{p_n} \in S$ such that $i_{p_n} \geq 2$ (the strategy of n th player), $\{(V)_{(i_{p_1}, i_{p_2}, \cdots, i_{p_n})}\}_{i_{p_1}, \cdots, i_{p_{n-1}} \in S}$ can be viewed as l^{n-1} dimensional tensor and hence can be decomposed in terms of a basis of $\{(E_\kappa^{\vec{i}}, \cdots E_\kappa^{\vec{i}})\}_{\vec{i} \in \mathcal{S},\ i_p \geq 2 \text{ for all } p}$ of $n-1$ player games by the induction hypothesis. In this way we obtain $(l-1)^{n-1}$ coefficients of the basis elements for each $i_{p_n} \geq 2$ and, thus, in total $(l/-1)^n$ coefficients. We write this linear combination as follows:

$$B = \sum_{\vec{i}} \kappa^{\vec{i}} E_\kappa^{\vec{i}}$$

Then we have

$$(V)_{(i_{p_1}, i_{p_2}, \cdots, i_{p_n})} = (B)_{(i_{p_1}, i_{p_2}, \cdots, i_{p_n})} \quad \text{for } i_{p_n} \geq 2$$

by construction. Then it follows that $(V)_{(i_{p_1}, i_{p_2}, \cdots, 1)} = (B)_{(i_{p_1}, i_{p_2}, \cdots, 1)}$ since

$$(V)_{(i_{p_1}, i_{p_2}, \cdots, 1)} = -\sum_{j \geq 2} (V)_{(i_{p_1}, i_{p_2}, \cdots, j)} = -\sum_{j \geq 2} (B)_{(i_{p_1}, i_{p_2}, \cdots, j)} = (B)_{(i_{p_1}, i_{p_2}, \cdots, 1)}.$$

■

We illustrate the above proof by the following example. Suppose that $p = 2$ and $l = 3$. Suppose that a symmetric bi-matrix game (A, A) is given; $A = [a_1 : a_2 : a_3]$. Then we know that the basis

for $\mathcal{N}^{\perp}$ is given by

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

If $A \in \mathcal{N}^{\perp}$, A can be uniquely written as a linear combination of the above basis. On the other hand, if $A \in \mathcal{N}^{\perp}$, then a_2, $a_3 \in T\Delta$, so a_2, a_3 can be uniquely written as a linear combination of $(1,-1,0)^T, (1,0,-1)^T$. Clearly, the four coefficients that we obtain in the second way also are the same as the coefficients of the basis elements of $\mathcal{N}^{\perp}$.

Appendix A.6. Proof of Proposition 3.2

Proof. "If part" is obvious, so we let $A \in \mathcal{L}$ such that $\langle x, PAPx\rangle = 0$ for all $x \in \mathbb{R}^l$. From the decomposition we can write A as the following:

$$A = \sum_{j \geq i \geq 2} \kappa^{(ij)}(E_{\kappa}^{(ij)} + E_{\kappa}^{(ji)}) + N + C,\ N \in (\mathcal{M}_{\mathcal{L}})^{\perp}, C \in \ker(\Gamma)$$

Since $\langle x, PAPx\rangle = 0$ for all $x \in \mathbb{R}^l$, we have

$$\sum_{j \geq i} \kappa^{(ij)} \left\langle x, (E_{\kappa}^{(ij)} + E_{\kappa}^{(ji)})x \right\rangle = 0 \text{ for all } x \in \mathbb{R}^l.$$

Let $K^{(ij)} := \frac{1}{2}(E_{\kappa}^{(ij)} + E_{\kappa}^{(ji)})$. Next by choosing appropriate x, we show that $\kappa^{(ij)} = 0$ for all $j \geq i$. Then it follows that $A \in \mathcal{N}_{\mathcal{L}}$. To do this, observe that

$$\left\langle x, K^{(ij)}x \right\rangle = -x_1^2 + x_1x_i + x_1x_j - x_ix_j,$$

so whenever $x_1 = x_i$ or $x_1 = x_j, \left\langle x, K^{(ij)}x \right\rangle = 0$.We first show that $\kappa^{(ii)} = 0$ for all i. For a given $\kappa^{(mm)}$, we choose

$$x = (1, 1, \cdots, 1, \overset{m\text{th}}{0}, 1, \cdots, 1)^T$$

i.e., x is a vector that has 0 in nth element and 1 otherwise. Then all $i < m$, $x_1 = x_i = 1$, so $\left\langle x, K^{(ij)}x \right\rangle = 0$. Similarly for all $i > m$, $x_1 = x_i = 1$, so $\left\langle x, K^{(ij)}x \right\rangle = 0$. When $i = m$, since $j > i$, $x_j = x_1 = 1$, thus $\left\langle x, K^{(ij)}x \right\rangle = 0$. For $i = j = m$, $x_i = x_j = 0$ so $\left\langle x, K^{(mm)}x \right\rangle = -1$. Therefore we have $-\kappa^{(mm)} = 0$,which implies $\kappa^{(mm)} = 0$. Next we show that $\kappa^{(ij)} = 0$ for all $i < j \leq l$ using induction. We start from the highest index, i.e., $\kappa^{(l-1,l)}$. For this case we set

$$x = (1, \cdots, 1, \overset{l-1\text{ th}}{0}, 0)^T.$$

where we assign an arbitrary value to x_n. For all $i < l-1$, $x_1 = x_i = 1$, $\left\langle x, K^{(ij)}x \right\rangle = 0$ and $\left\langle x, K^{(l-1,l)}x \right\rangle = -1$, so $\kappa^{(l-1,l)} = 0$. Next, we suppose that $\kappa^{(ij)} = 0$ for all $i > m$ and $j > n$ and show that $\kappa^{(mn)} = 0$. In this case, we set

$$x = (1, \cdots, 1, \overset{m\text{ th}}{0}, 1 \cdots, 1, \overset{n\text{ th}}{0}, x_{n+1}, \cdots, x_l)^T.$$

where we assign arbitrary values to elements over nth position. Since $n < l$, $x \in \mathbb{R}^l$. For all $i < m$, $x_i = x_1, \left\langle x, K^{(ij)}x \right\rangle = 0$. When $i = m$ and $j < n$, $x_j = 1$, so again $\left\langle x, K^{(ij)}x \right\rangle = 0$. When $i = m$, $j = n$, $x_i = x_j = 0$. Thus $\left\langle x, K^{(ij)}x \right\rangle = -1$ and we conclude $\kappa^{(ij)} = 0$. ■

Appendix A.7. Proof of Proposition 3.5

Proof. Again "If part" is obvious, so we let $(A,B) \in \mathcal{L}^2$ such that $\langle w, \mathbb{P}(A,B)\mathbb{P}w\rangle = 0$ for all $w \in \mathbb{R}_{l_r+l_c}$. From corollary 2.7 (3), we can write (A,B) as

$$(A,B) = \sum_{i\geq 2, j\geq 2} \kappa^{(ij)}(E_\kappa^{(ij)}, E_\kappa^{(ij)}) + (N,-N) + (C_1,C_2),$$

where $(N,-N) \in \mathcal{M}^\perp, (C_1,C_2) \in \ker(\mathbf{\Gamma})$. Since $\langle w, \mathbb{P}(A,B)\mathbb{P}w\rangle = 0$ for all $w \in \mathbb{R}_{l_r+l_c}$, we have

$$\sum_{i\geq 2, j\geq 2} \kappa^{(ij)}(\left\langle y, (E_\kappa^{(ij)})^T x\right\rangle_{\mathcal{L}} + \left\langle x, E_\kappa^{(ij)} y\right\rangle_{\mathcal{L}}) = 0 \text{ for all } x \in \mathbb{R}^{l_r},\ y \in \mathbb{R}^{l_c}.$$

Similarly to the previous section, by choosing appropriate x and y we show that $\kappa^{(ij)} = 0$ for all $i \geq 2, j \geq 2$. Then it follows that $(A,B) \in \mathcal{N}$. To do this, observe that

$$\frac{1}{2}(\left\langle y, E_\kappa^{(ji)} x\right\rangle_{\mathcal{L}} + \left\langle x, E_\kappa^{(ij)} y\right\rangle_{\mathcal{L}}) = -x_1y_1 + x_iy_1 + x_1y_j - x_iy_j, \tag{A.3}$$

so whenever $x_1 = x_i$ or $y_1 = y_j$,(A.3) becomes zero. We choose the following $(x^{(i)}, y^{(j)})$:

$$x^{(i)} = (1,1,\cdots,1,\overset{i\text{th}}{0},1,\cdots,1)^T,\ y^{(j)} = (1,1,\cdots,1,\overset{j\text{th}}{0},1,\cdots,1)^T.$$

Then for (k,m) such that $k \neq i$ or $m \neq j$, we have either $x_1^{(i)} = x_k^{(i)}$ or $y_j^{(j)} = y_1^{(j)}$. Thus for all (k,m) such that $k \neq i$ or $m \neq j$,

$$\left\langle y^{(j)}, E_\kappa^{(mk)} x^{(i)}\right\rangle_{\mathcal{L}} + \left\langle x^{(i)}, E_\kappa^{(km)} y^{(j)}\right\rangle_{\mathcal{L}} = 0$$

and

$$\left\langle y^{(j)}, E_\kappa^{(ji)} x^{(i)}\right\rangle_{\mathcal{L}} + \left\langle x^{(i)}, E_\kappa^{(ij)} y^{(j)}\right\rangle_{\mathcal{L}} = -2.$$

From this we conclude that $\kappa^{(ij)} = 0$. Thus, $(A,B) \in \mathcal{N}$. ■

Appendix A.8. Proof of Corollary 3.4

Proof. From proposition 3.1, we see that $[A]$ is strictly stable if and only if S, A's part belonging to $\mathcal{N}_{\mathcal{L}}^\perp$, is strictly stable and S has the following parameterization.

$$S = \begin{pmatrix} -a-b & a & b \\ a & -a-c & c \\ b & c & -b-c \end{pmatrix}$$

We recall that $\langle x, Sx\rangle$ satisfying $\sum_i x_i = 0$ is negative if and only if its bordered Hessians, given below, satisfies some sign condition as we will check below. In our case, these conditions are

$$\det\begin{pmatrix} -a-b & a & 1 \\ a & -a-c & 1 \\ 1 & 1 & 0 \end{pmatrix} > 0, \quad \det\begin{pmatrix} -a-b & a & b & 1 \\ a & -a-c & c & 1 \\ b & c & -b-c & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} < 0.$$

Then by computing determinants we find that

$$4a+b+c>0 \text{ and } ab+bc+ca>0$$

and obtain the desired result. ■

Appendix A.9. Proof of Proposition 3.6

Proof. (1) First we note that $x^TAx=0$ and $A\mathbf{1}=0$, so $x_0=(\frac{1}{n},\cdots,\frac{1}{n})$ is a rest point for (13). We consider $H(x):=\sum_i \log(x_i)$. Then $LH=\sum(Ax)_i=0$, thus H is an integral of (13). Thus (13) is conservative. To show the preservation of volume we first write $\hat{x}=(1-\sum_{i\neq 1}x_i,x_2,\cdots,x_n)$ and when $x\in\Delta, Ax=A\hat{x}$ and $\langle x,Ax\rangle=\langle\hat{x},A\hat{x}\rangle$. Also we note that for $k\geq 2$,

$$\begin{aligned}\frac{\partial}{\partial x_k}(A\hat{x})_k &= -a_{k1}+a_{kk}\\ \frac{\partial}{\partial x_k}\langle\hat{x},A\hat{x}\rangle &= -(Ax)_1-(A^Tx)_1+(A^Tx)_k+(Ax)_k\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{div}_\Delta f_A &= \sum_{k\neq 1}\frac{\partial f_k}{\partial x_k}(x)=\sum_{k\neq 1}(Ax)_k-(l-1)\langle x,Ax\rangle-\sum_{k\neq 1}x_ka_{k1}+\sum_{k\neq 1}x_ka_{kk}\\ &\quad +(1-x_1)(Ax)_1+(1-x_1)(A^Tx)_1-\sum_{k\neq 1}x_k(A^Tx)_k-\sum_{k\neq 1}x_k(Ax)_k\\ &= \sum_k(Ax)_k-l\langle x,Ax\rangle+\sum_k x_ka_{kk}-\left\langle x,A^Tx\right\rangle\end{aligned}$$

If A is anti-potential, then $\sum_k(Ax)_k=\langle\mathbf{1},Ax\rangle=\left\langle A^T\mathbf{1},x\right\rangle=0$ and all diagonal elements of A are zero. Thus $\operatorname{div}_\Delta f_A=0$
(2) Recall that $(A,B)\in range(\mathbf{\Gamma})$ if and only if $(\mathbf{1}_r^TA,\mathbf{1}_r^TB)=0$ and $(A\mathbf{1}_c,B\mathbf{1}_c)=0$.Thus (A,B) has an interior rest point, so from Hofbauer and Sigmund (1998) (p.130) the result follows.
(3) Similarly we have, for $l_r=:l\geq 2$

$$\frac{\partial}{\partial x_l}\langle\hat{x},Ay\rangle=(Ay)_l-(Ay)_1,\ \text{and}\ \ \frac{\partial}{\partial y_l}\left\langle\hat{y},B^Tx\right\rangle=(B^Tx)_l-(B^Tx)_1.$$

Thus

$$\begin{aligned}\operatorname{div}_\Delta f_{(A,B)} &= \sum_{i\neq 1}\frac{\partial f_i}{\partial x_i}(x,y)+\sum_{j\neq 1}\frac{\partial f_j}{\partial y_j}(x,y)=\sum_{i\neq 1}((Ay)_i-\langle x,Ay\rangle)-\sum_{i\neq 1}x_i((Ay)_i-(Ay)_1)\\ &\quad +\sum_{j\neq 1}((B^Tx)_i-\left\langle y,B^Tx\right\rangle)-\sum_{j\neq 1}y_j((B^Tx)_j-(B^Tx)_1)\\ &= \sum_i(Ay)_i-l_r\langle x,Ay\rangle+\sum_j(B^Tx)_j-l_c\left\langle y,B^Tx\right\rangle.\end{aligned}$$

Then since (A,B) is anti-symmetric in $\mathcal{L}^2$, which implies $\langle x,Ay\rangle+\left\langle y,B^Tx\right\rangle=0$,and $(A,B)\in range(\mathbf{\Gamma})$, the result follows. ■

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