How You Estimate the Yield Curve Matters!

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Abstract

We evaluate a two-factor Cox et al. (1985a,b) model using Euribor zero-coupon yields. We estimate this model using a state-space framework, where we sum a log-likelihood function of the state vector dynamics to a log-likelihood function of cross-section pricing errors. We introduce a likelihood-scaling weight in the joint log-likelihood function and show that there is a tradeoff in how one estimates a yield curve. Giving more weight to the cross-section of pricing errors improves the fitting and forecasting of Euribor yields, while giving more weight to the log-likelihood function of the state vector dynamics improves interest rate derivative pricing at the expense of the first. The common practice of giving the same weight to both measures serves neither purpose and helps to explain why the literature has so far found that the Cox et al. (1985a,b) model prices interest rate derivatives poorly. Our cap pricing exercise is able to price cap volatilities within the bid-ask spread bounds albeit at the cost of worsening the cross-section fit of the yield curve.

1 Introduction

In the past few decades a special class of term structure models termed "affine" has received a lot of attention in finance. Affine term structure (AFTS) models are based on the risk-neutral dynamics of the instantaneous short rate process. These models allow all fundamental interest rate assets (bonds and derivatives) to be priced using no-arbitrage as terms of expectations of functionals of the short rate process. Assuming no-arbitrage seems natural for bond markets since they are usually very liquid, and arbitrage opportunities are traded away immediately by
investment banks. Thoroughly characterised by Duffie and Kan (1996) and Dai and Singleton (2000), this class of models encompasses the Vasicek (1977) and Cox et al. (1985a,b) seminal dynamic term structure models. It also generalises easily towards a multifactor specification of the short rate without losing its analytical tractability. Closed-form solutions for derivative prices are known for many models, adding to the desired analytical properties of this class of models.

Although these properties prove very convenient, empirical evidence against AFTS models is substantial. Backus et al. (2001) show that term premiums generated by affine models may be too low when compared to the data. Bansal and Zhou (2002) find that affine specifications are rejected by the data and propose a model that allows for regime shifts in order to account for conditional volatility and the conditional correlation across yields. Orphanides and Kim (2005) report the existence of numerous model likelihood maxima that have essentially identical fit to the data but very different implications for economic behavior. Duffee (2002) shows that AFTS models produce poor out-of-sample forecasts.¹ AFTS models have also been dismissed to price the two main interest rate derivative products: caps and swaptions. Instead, models known as "market models" are used to price these derivatives using Black’s (1976) formula (Brace et al. (1997), Jamshidian (1997), Miltersen et al. (1997), Longstaff et al. (2001a,b)).²

In this paper, we show that the way you estimate the model matters as much as the choice of specification. We estimate a two-factor Cox et al. (1985a,b) model (CIR) on a dataset of weekly zero-coupon Euribor yields from Datastream, for the period from April 3, 2002, to October 26, 2011. This model is well known and has been extensively studied in the literature (Longstaff and Schwartz (1992), Chen and Scott (1992, 1993), Pearson and Sun (1994), Ball and Torous (1996), Duffie and Singleton (1997), Dai and Singleton (2000), Lamoureux and Witte (2002), Jagannathan et al. (2003), Duffee and Stanton (2004), Phillips and Yu (2005)). It is particularly useful because closed form expressions for the transition and marginal densities

¹There is only one exception. Christensen et al. (2011) develop an AFTS model based on Diebold and Li (2006). They show that the arbitrage-free restriction improves forecasts. However, little is known of this model other than its pricing and forecasting ability. Interest rate derivative pricing has not yet been developed for this model.

²There are only few empirical studies of AFTS models using derivative price data. Jagannathan et al. (2003) apply the CIR model for pricing caps and swaptions and find pricing errors that are too large relative to the typical bid-ask spread.
are known. This makes the model convenient to estimate using maximum likelihood and to compute derivative prices using closed form solutions. We study three basic applications of term structure models: the fitting of the yield curve, yield forecasting, and derivative pricing. For the latter, we compute cap prices using closed form solutions from Chen and Scott (1992), and then invert the cap prices and compute implied volatilities using Black’s (1976) formula.\(^3\)

We then compare the implied cap volatilities from the two-factor CIR model with Euribor cap volatilities from Datastream, for the period from March 2, 2005 to October 26, 2011.

We follow an estimation method that is standard in this literature.\(^4\) We use a state-space framework where cross-section pricing errors link observable yields to the unobservable state vector of short rate factors. We maximize a joint log-likelihood that is the sum of the log-likelihood of the short rate factor dynamics and the log-likelihood of cross-section pricing errors. This framework makes it possible for the model to be identified under both physical (\(\mathbb{P}\)) and risk-neutral (\(\mathbb{Q}\)) measures. We approximate the log-likelihood of the short rate factor dynamics using Ait-Sahalia (1999, 2008) closed-form approximations based on Hermite expansions. Additionally, we follow a market price of risk specification as in Cox et al. (1985b), which allows the drift of the state vector to be affine under both the physical and risk-neutral measures.\(^5\)

The impact of the estimation approach in economic applications has not been studied before. We add an intermediate step before the optimisation procedure. We introduce likelihood-scaling weights, that sum up to one unit, in the joint log-likelihood. By varying these weights, we implicitly give more or less importance to fitting the term structure versus capturing the dynamics of interest rates. We find that these weights have great impact in the results. We show that giving more weight to likelihood of cross-section pricing errors improves the cross-section fit and forecasting performance of the medium and long end of the Euribor yield curve. The fitting and forecasting root-mean-square errors (RMSE) for the model estimated with 90\% of the weight allocated to likelihood of cross-section pricing errors are almost double compared

\(^3\)The model in Chen and Scott (1992) is a special case of a two-factor CIR model analysed in Longstaff and Schwartz (1992). The advantage of Chen and Scott (1992) is that it reduces bond option expressions to univariate integrals.

\(^4\)Ait-Sahalia and Kimmel (2010) provide a thorough four step description of the estimation procedure.

\(^5\)This market price of risk specification is also used in most empirical studies of the CIR term structure model (Chen and Scott (1993), Pearson and Sun (1994), Lamoureux and Witte (2002), Jagannathan et al. (2003), Duffee and Stanton (2004), Phillips and Yu (2005))
to those of the model estimated with only 10% of the weight allocated to likelihood of cross-section pricing errors. Forecasting RMSE on the 10-year yield are 0.4333% and 0.9568% for the model with 90% and 10% of weight allocated to likelihood of cross-section pricing errors, respectively. On the other hand, giving more weight to likelihood of the short-rate dynamics slightly improves pricing and forecasting performance on the short end of the Euribor yield curve, but greatly improves the pricing of cap volatilities. The 10-year cap volatility RMSE are 11.7909% and 3.0918% for the model with 10% and 70% of weight allocated to likelihood of the short-rate dynamics, respectively. However, allocating too much weight on the dynamics likelihood worsens cap volatility pricing performance. The 10-year cap volatility RMSE for the model with 90% of weight allocated on the dynamics likelihood is 7.5296%.

This tradeoff is striking. A small deterioration in fitting the term structure results in a significant gain in the derivative pricing performance. This result is consistent with the results from Phillips and Yu (2005). They show that changes in CIR model parameters have little impact in bond pricing compared to pricing of European options.

Our paper proceeds as follows. In section 2 we describe the CIR model and the pricing, forecasting and derivative pricing applications, as well as the estimation procedure. Section 3 describes the data and presents the results for the model estimation using different measure-specific weights. Section 4 concludes.

2 Methodology

The central goal of this paper is to assess the performance of the two-factor CIR model, applied to Euribor rates, under different estimation approaches. We identify three main direct implementations of term structure models that give rise to numerous applications: fitting of the yield curve, yield forecasting, and derivative pricing. In practice, discount rates at exactly the desired maturities are not observed. Instead, they must be estimated from observed Libor, Swap and Futures quotes. If our model fits well a Euribor yield curve of bootstrapped rates, then it also fits well the original Euribor, Swaps and Futures quotes from which it was bootstrapped. Sec-
ond, we test the forecasting performance of the model by forecasting 3-month ahead Euribor yields. Third, we test how our model prices interest rate caps of different maturities.

We begin this section by describing the CIR two-factor model for Euribor rates. We also describe how the model forecasts yields and prices caps. Lastly, we explain the estimation methodology.

2.1 A two-factor CIR model for the Euribor

Under the assumption of no arbitrage, the value process of a contingent claim $P(t, T)$, with terminal payoff $P(T, T)$, in the event of no default can be expressed in terms of the risk-free pricing kernel $k_t$ as a martingale under the equivalent measure as

$$P(t, T) = E^Q \left[ e^{\int_t^T k_s ds} P(T, T) \right].$$

We assume no default. In this case, we can replace the risk-free pricing kernel $k_t$ with the default-adjusted pricing kernel $R_t$. Let $P(t, T)$ be the price of a bond that pays one currency unit at maturity, without paying any intermediate coupons. $R_t$ is the instantaneous short rate that drives the dynamics of the term structure. We assume the short rate to be the sum of two independent square root processes plus a constant,

$$R_t = r_{1t} + r_{2t} + \tau.$$

The constant is added to help guarantee that interest rates are bounded away from zero. The

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6In our study we use Euribor rates which reflect the credit risk of lending to commercial banks in the Eurozone. Duffie and Singleton (1999) show that we can use the same models with different interpretations of $R_t$. They argue that discounting at the adjusted short rate $R_t$ accounts for both the probability and timing of a default event, as well as for the effect of losses on default.

7The short rate positivity matter has been solved in the case of the single-factor CIR model by Feller (1951). The multi-dimensional case is much less understood. Duffie and Kan (1996) and Dai and Singleton (2000) generalize vanishing conditions for multi-factor models. The usual empirical fix to this problem is to introduce a constant to the short rate (see Pearson and Sun (1994)), Duffie and Singleton (1997), Lamoureux and Witte (2002), Jagannathan et al. (2003)).
standard two-factor CIR model can be seen as a special case when \( \tau = 0 \). The square root process under the physical measure is

\[
dr_{it} = k_i (\theta_i - r_{it}) dt + \sigma_i \sqrt{r_{it}} dW_{it}, \quad \text{for } i = 1, 2.
\]

Where \( W_{it} \) are independent Brownian motions. It can be shown that under the risk-neutral probability measure it maintains a square root structure, with linear market prices of risk \( \lambda_i \) associated with each state variable (Cox et al. (1985b)),

\[
dr_{it} = \bar{k}_i (\bar{\theta}_i - r_{it}) dt + \sigma_i \sqrt{r_{it}} dW_{it}^Q, \quad \bar{k}_i = k_i + \lambda_i, \quad \bar{\theta}_i = \frac{k_i \theta_i}{k_i + \lambda_i}. \tag{1}
\]

We refer to the physical probability measure as \( \mathbb{P} \), and the risk-neutral measure as \( \mathbb{Q} \). The price of a discount bond is

\[
P(t, T) = A_1(t, T) A_2(t, T) e^{-B_1(t, T) r_{it} - B_2(t, T) r_{2t} - r} \tag{2}
\]

where

\[
A_i(t, T) = \left[ \frac{2r_i e^{\left[(\bar{k}_i + \gamma_i)(T-t)\right]/2}}{(\bar{k}_i + \gamma_i) \left(e^{(T-t)\gamma_i} - 1\right) + 2\gamma_i} \right]^{\frac{2\lambda_i \theta_i}{\sigma_i^2}}, \tag{3}
\]

\[
B_i(t, T) = \frac{2 \left( e^{(T-t)\gamma_i} - 1 \right)}{(\bar{k}_i + \gamma_i) \left(e^{(T-t)\gamma_i} - 1\right) + 2\gamma_i}. \tag{4}
\]

and \( \gamma_i = \left[ \bar{k}_i^2 + 2\sigma_i^2 \right]^{1/2} \). The instantaneous expected return on any default-free bond in the CIR model is

\[
r_{it} + \frac{\lambda_i}{P(t, T)} \frac{\partial P(t, T)}{\partial r_{it}} = r_{it} - \lambda_i B_i(t, T) r_{it}.
\]

Therefore the risk premium is positive whenever \( \lambda_i < 0 \).
2.2 Interest rate forecasts

The conditional mean and variance of \( r_{it} \) conditional on \( r_{is} \) are given by

\[
E[r_{it} | r_{is}] = r_{is} e^{-k_i (t-s)} + \theta_i \left( 1 - e^{-k_i (t-s)} \right),
\]

\[
Var[r_{it} | r_{is}] = r_{is} \frac{\sigma_i^2}{k_i} \left( e^{-k_i (t-s)} - e^{-2k_i (t-s)} \right) + \theta_i \frac{\sigma_i^2}{2k_i} \left( 1 - e^{-k_i (t-s)} \right)^2.
\]

3-month Euribor zero-coupon yield forecasts can be computed using (5) and (2) and the definition of bond yield. We assess the forecasting performance through the root mean-squared error of Euribor yield forecasts.

2.3 Interest rate caps

A cap can be viewed as a payer interest rate swap contract where each payment is made only if it has positive value. The interest rate caps that we examine are written on Euribor with payments made at the end of each period and settlement periods of 3 months.

Euribor rates are rates at which deposits between banks are exchanged in the European Union interbank market. They can be seen as a simple forward rate on a defaultable bond. For the period \([T, S]\), the Euribor is defined as

\[
Euribor(t, T, S) = \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right).
\]

The forward swap rate is the rate at the fixed leg of the swap contract that makes the receiver forward swap receive zero net present value. At fixed year fractions \( \tau \) (usually 3 or 6 months), the forward swap rate for the period \([\alpha, \beta]\) is

\[
R_{\alpha, \beta}^{\text{swap}}(t) = \frac{P(t, T_{\alpha}) - P(t, T_{\beta})}{\sum_{i=\alpha+1}^{\beta} \tau P(t, T_i)}.
\]
Cap contracts can be decomposed additively. For each period, the potential payment is the face value times \( \tau [Euribor_t - R_K]^+ \). The call option on the rate being capped is referred as a caplet. It is market practice is to price a caplet using Black (1976) (see Hull (2008)), which assumes a lognormal process for the Euribor. The cap contract is said to be ATM if \( R_K \) equals the forward swap rate at the relevant period.

Hull (2008) shows that a cap can be transformed into a portfolio of European puts on discount bonds. Let \( R_i \) be the value of the rate being capped. The value at time \( i \) of the payoff from the caplet that occurs at time \((i + 1)\) is

\[
\frac{\tau}{1 + R_i} \max [R_i - R_K, 0] = (1 + \tau R_K) \max \left[ \frac{1}{1 - \tau R_K} - \frac{1}{1 - \tau R_i}, 0 \right],
\]

which is \( 1 + \tau R_K \) times the payoff on a put option on a par zero-coupon bond with strike price \( 1/(1 + \tau R_K) \). Therefore, a cap can also be considered a portfolio of put options on zero-coupon bonds. We use the second interpretation, a portfolio of put options on zero-coupon bonds, to compute the cap price for a two-factor CIR model following Chen and Scott (1992). We first compute the price of a put option on a discount bond. The integration region is given by \( P(T, S) \leq K \) where \( P(T, S) \) is the discount bond price. This generates a linear boundary

\[
\sum_{i=1}^{2} \frac{r_{iT}}{r^*_i} \geq 1,
\]

where

\[
r^*_i = \frac{1}{B_i} \left( \ln \left( \prod_{i=1}^{2} \frac{A_i}{K} \right) - \bar{y} \right).
\]

The price of a put option on a discount bond is given by

\[
P_{\text{put}} (t, T, S, K) = KP (r_1, r_2, t, T) \left( 1 - \chi^2 (L_1, L_2, \nu_1, \nu_2, \lambda_1, \lambda_2) \right) - P (r_1, r_2, t, S) \left( 1 - \chi^2 (L_1^*, L_2^*, \nu_1, \nu_2, \lambda_1^*, \lambda_2^*) \right)
\]
where

\[
\chi^2 (L_1, L_2, \nu_1, \nu_2, \lambda^*_1, \lambda^*_2) = \int_0^{L_2} F^* \left( L_1 - \frac{L_1}{L_2} x_2, \nu_1, \lambda^*_1 \right) f \left( x_2, \nu_2, \lambda^*_2 \right) dx_2,
\]

and

\[
L_i = 2\psi_i r_i^*, \quad L_i^* = 2\psi_i^* r_i^*,
\]

\[
\delta_i = r_i \phi_i^2 \exp (\gamma_i (T-t)) / \psi_i, \quad \delta_i^* = r_i \phi_i^2 \exp (\gamma_i (T-t)) / \psi_i^*,
\]

\[
\psi_i = 2 \left( \phi_i + \frac{\gamma_i k_i^*}{\sigma_i^2} \right), \quad \psi_i^* = 2 \left( \phi_i + \frac{\gamma_i^* k_i^*}{\sigma_i^*} + B_i (T, S) \right),
\]

\[
\phi_i = \frac{2\gamma_i}{\sigma_i \exp(\gamma_i(T-t))-1}, \quad \nu_i = \frac{4k_i \theta}{\sigma_i^2}.
\]

The \( \chi^2 \) denotes a multidimensional cumulative noncentral chi-square distribution function. \( F \) and \( f \) are the distribution and density functions, respectively, of a univariate noncentral chi-square distribution. Numerical approximations to the function above can be found in Chen and Scott (1992, 1995).

### 2.4 Econometric method

We follow Ait-Sahalia and Kimmel (2010) and estimate the two-factor CIR model in four steps. These estimation steps are similar to those in Chen and Scott (1992, 1993), Duffie and Singleton (1997), Lamoureux and Witte (2002), and Jagannathan et al. (2003). First, we extract the value of the state vector \( R_t \) from a cross-section of zero-coupon yields. The state vector is not directly observable. Under the physical measure, bond prices follow the pricing equation in (2). It is possible to invert for the \( N \) state variables using \( N \) discount bonds at different maturities. It is usual in the literature, when using multi-factor models, to use a short and a long maturity in order to capture the different dynamics of the short end and the long end of the yield curve, and therefore better replicate the whole dynamics of the term structure. We choose the 9-month and 5-year Euribor zero-coupon yields to invert for the short rate factors. We invert for the two
using the system of equations of yields,

\[
\begin{pmatrix}
y_1(t, \tau_{9m}) \\ y_2(t, \tau_{5y})
\end{pmatrix}
= \begin{pmatrix}
B_1(t, \tau_{9m})r_{1t} + B_2(t, \tau_{9m})r_{2t} + \tau - \log A_1(t, \tau_{9m}) - \log A_2(t, \tau_{9m}) \\ B_1(t, \tau_{5y})r_{1t} + B_2(t, \tau_{5y})r_{2t} + \tau - \log A_1(t, \tau_{5y}) - \log A_2(t, \tau_{5y})
\end{pmatrix},
\]

(7)

where \( y_i(t, s) \) represents a zero-coupon bond yields with maturity \( s \), assumed to be observed without error. Zero-coupon yields are affine functions of the state vector, and thus the likelihood function of yields is readily determined from the likelihood function of the state vector.

Second, we compute the conditional density function for the square-root process \( r_i \) at time \( t + s \), conditional on its realisation on time \( t \),

\[
f_{r_i} (r_{i,t} | r_{i,t-1}) = \frac{2c_i f_{\text{ncx}2}^{r_i} (2c_i r_i, v_i, 2u_i)}{c_i e^{-u_i - v_i} (v_i / u_i)^{q_i / 2} I_{q_i} (2 (u_i v_i)^{1/2})},
\]

where \( f_{\text{ncx}2}^{r_i} \) is the conditional noncentral chi-square distribution and \( I_{q_i} \) is a modified Bessel function of the first kind and order \( q_i \), and

\[
c_i = \frac{2k_i}{\sigma_i^2 (1 - e^{-k_i s})}, \quad u_i = c_i r_{i,t} e^{-k_i s}, \quad v_i = c_i r_{i,t+s}, \quad q_i = \frac{2k_i \theta_i}{\sigma_i^2} - 1.
\]

The joint likelihood of the short rate factors is the product of the two transition functions. Instead of using the analytical solution for the conditional density of the short rate factor in the likelihood function, we use Ait-Sahalia (1999, 2008) closed-form approximations based on Hermite expansions to the CIR likelihood function.\(^8\) This procedure is faster and more accurate in computing the joint likelihood.

Third, we multiply this joint likelihood by a Jacobian determinant to find the likelihood of the panel of observations of the benchmark yields. As we are working with Euribor zero-coupon

\(^8\)We find that these likelihood approximations produce better results than using the analytical density function. The whole algorithm and explicit expressions for the two-factor CIR likelihood approximations are described thoroughly in Ait-Sahalia (1999) and Ait-Sahalia (2008). The expressions are quite lengthy and take more than one page. Matlab codes are available at Ait-Sahalia’s website at http://www.princeton.edu/yacine/.
yields, the Jacobian $J$ is

$$J = \frac{1}{\tau_{9m}\tau_{5y}} \begin{vmatrix} B_1(t, \tau_{9m}) & B_2(t, \tau_{9m}) \\ B_1(t, \tau_{5y}) & B_2(t, \tau_{5y}) \end{vmatrix}.$$ 

The log-likelihood of the short rate dynamics is

$$\log Q = \log(J^{-1}) \sum_{t=1}^{T-1} \sum_{i=1}^{2} \log f_{r_i}(r_{i,t} | r_{i,t-1}).$$ \hspace{1cm} (8)$$

We follow Chen and Scott (1993), de Jong (2000), and Duffee (2002), and assume that a second set of yields is observed with error. It is common to assume that the errors are i.i.d Normal with zero mean. The log-likelihood of cross-section pricing errors is

$$\log \mathbb{P} = -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log(\det \Sigma_t) \frac{1}{2} \sum_{t=2}^{T} (\hat{y}_t - y_t)'\Sigma_t^{-1}(\hat{y}_t - y_t),$$

where $y_t$ is a vector of observed yields and $\hat{y}_t$ is a vector of yields estimated using (2). Different settings can be made on these measurement errors. Either all of the yields are observed with error or only a subset of yields are observed with error. The variance terms of $\Sigma_t$ is nonzero for all maturities we wish to add in the cross-section errors. In our estimation, we assume that the 6-month, 3- and 15-year Euribor zero-coupon yields are observed with error, and $\Sigma_t$ is a diagonal matrix.\textsuperscript{9}

As a fourth step we add the two log-likelihood functions to find the joint log-likelihood of the panel of all yields,

$$\log L = \log Q + \log \mathbb{P}.$$ 

We estimate the model by maximising $\log L$. This affine model can be seen as a state space system. The cross-section errors link observable yields to the state vector and the implied short rate factors describe the dynamics of the state vector.

\textsuperscript{9}Assuming a diagonal structure for the covariance matrix yielded better results in our estimation. The cross-covariance terms were close to zero and did not affect the results.
This framework is necessary to identify the parameters under the risk-neutral measure $Q$. Note that bond prices in (2) are written in terms of $k_i = k_i + \lambda_i$, and the conditional factor density function in (2.4) are written in terms of $k_i$ only. If we estimate the model using only $\log L^D$, we will not able to estimate $k_i$ and $\lambda_i$ separately. The market prices of risk, $\lambda_i$ identify the risk-neutral measure $Q$. Therefore, if we estimate the model in this way, we will estimate the parameters under the physical measure $P$. This will suffice to price and forecast bonds, since (2) and (5) are equivalent under $P$ and $Q$.

Conversely, if we estimate the model using only $\log L^P$ (this is equivalent to assume that all rates are observed with error), we can not invert the system of equations in (7) to compute the state vector. In this case we will estimate poorly the factor dynamics under the $Q$. Since interest rate derivatives are priced as expectations of functionals of the process short rate under $Q$, we will not be able to correctly price interest rate derivatives.

To estimate the model properly, we must use the joint log-likelihood. In other words, to estimate the model using $\log L^D$ and $\log L^P$, the weight of each log-likelihood in $\log L$ must be greater than zero. However, the magnitude of this weight has not subject of study. Previous studies usually assume that both measures enter with the same weight. The effects of changing the weights in the joint-likelihood estimation are unknown.

We introduce a likelihood-scaling weight alpha, in the joint log-likelihood function,

$$\log L = \alpha \log Q + (1 - \alpha) \log P.$$  \hfill (9)

We estimate the model using the joint log-likelihood above for different alphas in the open interval $(0, 1)$. We choose alphas equal 0.1, 0.3, 0.5, 0.7, and 0.9, and assess the CIR model performance in the three basic applications for term structure models described above.
3 Empirical Analysis

3.1 Data

Euro Interbank Offered Rates (Euribor) rates are based on the average interest rates at which a panel of more than 50 European banks borrow funds from one another. There are different maturities, ranging from one week to one year. The Euribor rates are considered to be the most important reference rates in the European money market. They provide the basis for the price of Euro interest rate swaps, interest rate futures, saving accounts and mortgages. We use a dataset of Euribor weekly zero-coupon yields bootstrapped from Euribor, swaps and futures quotes, obtained from Datastream for the period from April 3, 2002, to October 26, 2011. This dataset includes Euribor zero-coupon yields with maturities ranging from 3 months to 30 years. We need only a subset of yields for our estimation. We use the 9-month and 5-year yield to invert for the short rate, and the 6-month, 3 and 15-year yields for measurement errors. In addition, we use the 3-month, 1, 2, 7, 10 and 20-year yields for our out-of-sample pricing and forecasting exercises.

Table 1 presents the summary statistics for the Euribor yield curve. The Euribor yield curve on average is upward sloping, long yields are less volatile than short yields, and yields of all maturities are very persistent (long yields are more persistent than short yields). The bottom three rows of the descriptive statistics show autocorrelations at displacements of 1, 3, and 12 months. For our 3-month forecast horizon, autocorrelations are around 0.80 across maturities.

We also collect weekly at-the-money cap volatilities based on the Euribor from Datastream, for the period from March 2, 2005 to October 26, 2011. Caps are said to be at the money if the strike rate equals the forward swap rate for the corresponding maturity. We choose cap volatilities with maturities of 3, 5, 7, 10, 15 and 20 years. The first two rows on Table 4 show the mean and standard deviations for cap volatilities.
3.2 Results

We estimate the model parameters for five choices of the measure weight alpha, 0.1, 0.3, 0.5, 0.7 and 0.9. A higher alpha means that the dynamics of interest rates have greater weight in the joint-likelihood than the pricing errors. Table 2 reports the parameter estimates and results for the estimation using a sample of Euribor zero-coupon yields. There is a clear pattern of results. Higher alphas yield higher dynamics likelihood values, and lower pricing likelihood values. The joint log-likelihood also increases with alpha, because the dynamics likelihood has greater magnitude than the pricing likelihood. It is also possible to observe a clear pattern in the model parameters. The speed of adjustment term of the first short rate factor, $k_1$, is of greater magnitude than $k_2$ and increases with alpha, which implies a faster mean reversion for $r_1$. Most of the variation in short-term rates is explained by the factor with higher mean reversion. This effect is the opposite for the second short rate factor. The speed of adjustment of this factor decreases with alpha. Factors with lower speed of adjustment parameters behave like a random walk and play the dominant role in the determination of long-term interest rates. The market prices of risk of the second short rate factor are negative for every value of alpha. The market prices of risk for the first short rate factor, on the other hand, change sign. They are positive for alphas up to 0.5, but become negative for higher alphas. The other observable pattern is for the short rate constant added to guarantee the positivity of the short rate factors. It changes sign from negative for the two lowest values of alpha to positive and increasing with alpha. It means that the higher the alpha, the model needs to compensate more to keep short rate factors bounded away from zero.

What is the effect of the different likelihood values in the model performance? Orphanides and Kim (2005) show that AFTS models can have numerous likelihood maxima with identical fit to the data, but different economic implications. We assess the model on three economic applications: fitting of the yield curve, yield forecasting, and derivative pricing. Table 3, Panel

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10 Duffee and Stanton (2004) and Phillips and Yu (2005) find estimation biases on the speed of adjustment and market price of risk parameters in one and two-factor square-root process models. Duffee and Stanton (2004) find these biases using different estimation techniques, such as the Kalman filter, maximum likelihood and efficient method of moments. These findings relate to bias in estimates of the speed of mean reversion of highly persistent processes (Ball and Torous (1996)) and of continuous time models (Chapman and Pearson (2000), Yu and Phillips (2001)). Biases persist even when large samples are used to estimate the models.
A. reports root mean-squared errors (RMSE) for the two-factor CIR model on a cross-section of Euribor zero-coupon yields. The 9-month and 5-year yields that are used to invert for the short rate factors are not reported, since they have zero pricing errors. In addition, we use the 6-month, 3 and 15-year yield pricing errors to identify the model parameters under $\mathbb{Q}$. The remaining maturities are priced out of sample. RMSE for yields of 1-year and higher maturities increases with alpha. The RMSE on the 2- and 20-year yields are 0.1344% and 0.3483%, using alpha equal to 0.1. Whereas the RMSE on the same maturities using alpha equal to 0.9 is almost double, at 0.2428% and 0.6775%. The models estimated with more weight to the likelihood of the short-rate dynamics have poorer performance fitting the mid and long end of the yield curve. Interestingly, model estimated with more weight to the likelihood of the short-rate dynamics, and thus put more weight in the short rate factor dynamics, are able to marginally price the short end of the yield curve (3- and 6-month yields) better. The RMSE on the 3-month yield is 0.2723% and 0.2331% using alpha equal to 0.1 and 0.7, respectively. The only exception is the model estimated with alpha equal to 0.9. In this setting, RMSE are higher for all yields. When alpha is too close to one, the model may be poorly estimated, as the cross-section pricing errors enter with little weight in the joint-likelihood.

With only the pricing information available above, a practitioner would be tempted to estimate the model using a lower alpha to ensure for a better cross-section fit, and then follow with forecasting and derivative pricing. Panel B. reports RMSE for the 3-month forecasting exercise. We observe a similar picture as for the pricing errors. The models estimated with higher alphas forecast the short end of the yield curve better. RMSE on a 6-month yield decreases from 0.3238% with alpha equal to 0.1 to 0.2967% with alpha equal to 0.9. The 5-year yield, used to invert for the short rate factors, is also forecasted marginally better with higher alpha, with exception of alpha equal to 0.9. On the other hand, the models estimated with lower alphas forecast longer maturities better. This fact is in accordance with pricing errors. The models estimated with lower alphas give more weight to cross-section pricing errors, and thus fit the average shape of the yield curve better, specially at the long end of the curve. This property is then transferred to the forecasting exercise. The 10 and 20 year forecast RMSE for the model estimated with alpha equal to 0.1 is 0.4333% and 0.4484%, respectively. The RMSE for the model estimated using alpha equal to 0.9 on the same maturities is 0.9568% and 0.8028%.
The forecasting result seems to validate the practice of choosing the estimation settings that yield the best cross-section fit. We now proceed to pricing interest caps. Table 4 reports Euribor cap volatilities for the market and the two-factor CIR model. Caps are usually quoted in Black’s volatilities, in units per 100. We price 3, 5, 7, 10, 15 and 20 year caps maturing in 6 months using the two-factor CIR model as in Chen and Scott (1993), and then compute volatilities inverting Black’s formula. This table shows that models estimated using alphas equal to 0.5 or lower underprice average cap volatilities. Underpricing decreases with alpha, and models estimated with alpha 0.7 and 0.9 slightly overprice average cap volatilities. Overall, cap volatility pricing errors are much larger than yield fitting errors. This result is in line with the results from Phillips and Yu (2005). They show that changes in a two-factor CIR model parameters have little impact in bond pricing compared to pricing of European options using this model. The model estimated using alpha equal to 0.7 has the best performance pricing caps, though both this and the model estimated using alpha equal to 0.5 produce cap RMSE that are within the 2 to 8% bid-ask volatility spread observed in the market. The average cap volatilities for the model estimated using alpha equal to 0.7 are 26.4853 and 21.1660 for the 5 and 10-year caps, whilst actual values are 24.7429 and 16.6586. The model estimated using alpha equal to 0.9 overprices cap volatilities for all maturities and has all RMSE values above 7.8%. Figures 1 to 5 show the historical fit of the model implied cap prices versus the actual prices computed using Black’s formula. From these figures it is clear that the model estimated with alpha equal to 0.7 prices has the best fit of cap prices. Cap prices are close to actual values for most of the time series. Before mid 2007 the model overprices caps, increasing average cap implied volatilities. Unlike the both applications above, the interest rate derivative pricing exercise benefits from giving greater weight to the likelihood of the short-rate dynamics.

There is an apparent tradeoff between applications. Lower alphas give more weight to measurement errors in the joint-likelihood, and thus improve the cross-section fit of the term structure. Forecasting results depend ambiguously on both measures. Higher alphas imply that the model forecasts the maturities used in inverting for the short rate factors better. However, yields of other maturities are forecasted with greater error since the model using this setting is less able to replicate the term structure fit. Lastly, the interest rate derivative pricing exercise yields the opposite result as the term structure fit. Higher alphas imply that the parameters under the Q
might be estimated more accurately.

Since interest rate derivative prices are computed as expectations of the short rate factors under the risk-neutral measure, this setting able to price more accurate risk neutral probabilities.

### 4 Conclusion

In this paper we assess the performance of a two-factor Cox et al. (1985a,b) model of the term structure. We estimate the model using a state-space framework, where cross-section errors link observable yields to the state vector, and the implied short rate processes describe the dynamics of the state vector. This framework is necessary to identify the model parameters under the risk-neutral measure, which in turn are necessary to accurately price interest rate derivatives. The usual way to achieve the affine term structure state-space framework in a maximum likelihood setting is to sum a log-likelihood function of pricing errors to the log-likelihood of the dynamics of short rate factors.

Most studies estimate affine term structure models in this manner, choosing the estimation settings that produce the best fit of the yield curve. On a second step they use the model for different applications, such as forecasting and derivative pricing. So far, empirical results have found that derivative prices computed by short rate models produce pricing errors are large relative to the bid-ask spread in the market. We add an intermediary step in the estimation procedure. We give different weights to each likelihood functions and maximise the joint log-likelihood. The model performance improves depending on the choice of weights. However, it comes with a cost. The results show a strong tradeoff in the performance of economic applications for this model. Giving more weight to the likelihood of cross-section of pricing errors improves the cross-section fit and forecasting performance. Conversely, giving more weight to the likelihood of interest rate dynamics improves pricing of interest rate caps. This tradeoff is asymmetric. A small deterioration of the pricing performance results in a significant gain in derivative pricing performance. We are able to price cap volatilities within the bid-ask spread bounds in the market.
Table 1: Euribor Zero Curve Summary Statistics

This table reports summary statistics for Euro Interbank Offered Rates zero-coupon yields obtained from Datastream, for the period from April 3, 2002 to October 26, 2011. Yields have maturities 3, 6 and 9 months, and 1, 2, 3, 5, 7, 10, 15 and 20 years. The last three rows contain sample autocorrelations at displacements of 1, 3, and 12 months.

Descriptive statistics, Euribor zero curve

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<tr>
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<th>3-month</th>
<th>6-month</th>
<th>9-month</th>
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<th>3-year</th>
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<th>10-year</th>
<th>15-year</th>
<th>20-year</th>
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<td>5.2411</td>
<td>5.2393</td>
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<td>5.3304</td>
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<td>0.9856</td>
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<td>0.9375</td>
<td>0.9272</td>
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Table 2: Estimation of two-factor CIR model

This table reports coefficient estimates and corresponding statistics for two-factor CIR models with different likelihood weights. All models are estimated using 6 and 9-month, 3, 5 and 15-year Euribor zero-coupon yields from Datastream, over the period from 03/04/2002 to 26/10/2011. Alpha is the likelihood weight of $\log L_P$ in $\log L$. The two-factor CIR model was estimated using quasi-maximum likelihood procedure, where the likelihood of the CIR factors was approximated by a closed-form likelihood expansion as in Ait-Sahalia and Kimmel (2010). The corresponding asymptotic covariance matrix is the inverse of the Hessian matrix, which consists of the second derivatives of the joint log-likelihood function with respect to the parameters. Standard errors are in parenthesis.

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<th>$k$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\lambda$</th>
<th>$\tau$</th>
<th>$\log L_P$ parameters</th>
<th>$\nu_{0m}$</th>
<th>$\nu_{3y}$</th>
<th>$\nu_{15y}$</th>
<th>Log Likelihood</th>
<th>$\log L_P$</th>
<th>$\log L$</th>
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<td>0.1819</td>
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<td>$r_1$</td>
<td>0.3523</td>
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Table 3: Pricing and Forecasting Errors

This table pricing and forecasting root mean-squared errors for the two-factor CIR model with different likelihood weights. All models are estimated using 6 and 9-month, 3, 5 and 15-year Euribor zero-coupon yields from Datastream, over the period from 03/04/2002 to 26/10/2011. The 9 month and 5 year yields were used to invert for the short rate factors. The forecasting window is 3 months (13 weeks).

Panel A. Pricing RMSE (%)  

<table>
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<tr>
<th>Alpha</th>
<th>3-month</th>
<th>6-month</th>
<th>9-month</th>
<th>1-year</th>
<th>2-year</th>
<th>3-year</th>
<th>5-year</th>
<th>7-year</th>
<th>10-year</th>
<th>15-year</th>
<th>20-year</th>
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<tbody>
<tr>
<td>0.1</td>
<td>0.2723</td>
<td>0.1388</td>
<td>-</td>
<td>0.0577</td>
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<td>0.1344</td>
<td>-</td>
<td>0.1965</td>
<td>0.3110</td>
<td>0.1898</td>
<td>0.3483</td>
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<tr>
<td>0.3</td>
<td>0.2411</td>
<td>0.1324</td>
<td>-</td>
<td>0.0607</td>
<td>0.0607</td>
<td>0.1449</td>
<td>-</td>
<td>0.2195</td>
<td>0.3474</td>
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<td>0.3645</td>
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<td>0.1254</td>
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<td>-</td>
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<td>0.4043</td>
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<tr>
<td>0.7</td>
<td>0.2331</td>
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<td>0.1894</td>
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<td>0.3379</td>
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<td>0.5576</td>
<td>0.9650</td>
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Panel B. 3-month Forecasting RMSE (%)  

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<th>Alpha</th>
<th>3-month Forecasting RMSE (%)</th>
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<td>0.3653 0.3238 0.3370 0.3440 0.3941 0.3954 0.3658 0.3942 0.4333 0.3482 0.4484</td>
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<td>0.5</td>
<td>0.3138 0.3117 0.3315 0.3405 0.3965 0.4010 0.3641 0.3641 0.4976 0.3611 0.4983</td>
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<td>0.7</td>
<td>0.3147 0.3026 0.3264 0.3383 0.4030 0.4114 0.3643 0.4716 0.6264 0.3991 0.6187</td>
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<td>0.9</td>
<td>0.4578 0.2967 0.3181 0.3387 0.4331 0.4528 0.3670 0.6237 0.9568 0.4620 0.8028</td>
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</table>
Table 4: Cap Volatilities

This table reports actual and implied volatility statistics for the market and the two-factor CIR model. The statistics are quoted in percents.

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<tr>
<th></th>
<th>3-year</th>
<th>5-year</th>
<th>7-year</th>
<th>10-year</th>
<th>15-year</th>
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<tr>
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Figure 1: CIR cap prices with Alpha = 0.1
Figure 2: CIR cap prices with Alpha = 0.3
Figure 3: CIR cap prices with Alpha = 0.5
Figure 4: CIR cap prices with Alpha = 0.7
Figure 5: CIR cap prices with Alpha = 0.9


References


