Financial Implications of Extreme and Rare Events

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August 11, 2010

Abstract

The recent financial crisis has reinforced the critical need for economic policy to anticipate and reduce the impact of unexpected, extreme, events. In this paper, we develop a framework based on latent regime shifts to analyze why investors may undertake overly risky investment strategies and how policymakers may attempt to constrain such behavior. This framework explains, in particular, why banks, investors, and policymakers may decide not to hedge against extreme events, even when those events are exogenously determined and have well understood probabilities and consequences. We also examine cases in which the extreme events are endogenously created by the investment strategies themselves. Our most striking finding is that the private costs of sustained suboptimal investment may be bounded and small. Thus, investors may knowingly ignore or exacerbate the likelihood of extreme events, especially if they face informational costs to learning the structure of the financial environment in which the events are created. These results obtain both in the theoretical model and upon calibration to the last half-century of US economic experience. The results provide a strong motivation for policymakers to ensure the full disclosure and dissemination of information regarding the probabilities and consequences of extreme outcomes.

Keywords: Endogenous Probability; Extreme Event; Financial Crisis; Information Cost; Regime Shift

JEL Classification: E44 G01 G11 G32

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1 Introduction and motivation

"We all learn by experience, and your lesson this time is that you should never lose sight of the alternative." Sherlock Holmes: The Adventure of Black Peter.

Unexpected economic events can have massive, disruptive effects on a nation.¹ The experience of multiple crises during the 1990s and 2000s has stimulated researchers’ interest in understanding extreme events in the US economy. When such events occur, propagation mechanisms may amplify their impact.² For example, the collapse of a major lending institution of course affects its customers, but it may also have macroeconomic implications for aggregate consumption, investment, and unemployment.³ Such magnified and correlated outcomes are interesting not only for their practical relevance, but also economically, since they resemble the results from a broad class of theoretical studies on herding and strategic complementarities.⁴

The main goal of this paper is to develop a simple economic framework for analyzing rare extreme events, in order to understand their impact and ramifications.⁵ Our model delivers insights into how individuals respond to extreme events in terms of hedging and asset demands. Interestingly, we find that agents may rationally choose to ignore information about extreme events, if this information is costly. Such a finding ties our work closely to research on rational inattention, including Wilson (1975), Sims (2003), and Veldkamp and Van Nieuwerburgh (2009).

There are two other important areas of research intersection. First there is the recent work on extreme events, largely in response to the economic crisis.⁶ Much of this research analyzes systemic instability.⁷ Second, historically there is a long literature examining financial crises and bubbles, in both rational and behavioral frameworks.⁸ The quantitative models in most of these studies focus on a stationary environment, although the economic climate is subject to sudden shifts.⁹ In particular, little existing research examines the

¹ For evidence on the welfare costs of extreme events, see Chatterjee and Corbae (2007), Barro (2009), and the references therein.
² See Barro (2006) and Barro (2009). Also, see Horst and Scheinkman (2006), and Krishnamurthy (2009) for economic explanations of such amplifications.
³ For details on insurance during periods of economic disruptions, see Jaffee and Russell (1997); Jaffee (2006); and Ibragimov et al. (2009).
⁴ See Wilson (1975); Bikhchandani et al. (1992); Cooper (1999); and Vives (2008), chapter 6.
⁵ By extreme, we refer to events that have a high impact on the particular system. This impact can be in terms of financial or social cost, or in terms of disruption of equilibrium. By rare, we refer to events that are not observed frequently, as in Table 1.
⁶ For overviews of the crisis, see Acharya and Richardson (2009); Brunnermeier (2009); Reinhart (2008); and Reinhart and Rogoff (2009).
⁷ See Caballero and Krishnamurthy (2008); Ibragimov, Jafee and Walden (2009a, 2009b); and Shin (2009).
⁸ See Fisher (1933); Keynes (1936); Blanchard (1979); Minsky (1982); Friedman and Laibson (1989); Shleifer and Vishny (1997); Kindleberger (2000); Abreu and Brunnermeier (2003); Allen and Gale (2007); and Hong et al. (2008).
⁹ Brunnermeier and Sannikov (2010) present a very recent exception with a dynamic model that allows episodes of extreme volatility and instability.
economic impact of regime shifts in the probability of encountering extreme events. Our research starts to rectify this problem by incorporating a simple model of regime shifts in extreme events. We find that the existence of such shifts may help explain the experience of unhedged extreme events in the US economy, both theoretically and empirically.

The remainder of the paper is as follows. In Section 2 we review the theoretical and empirical literature on extreme events. In Section 3 we develop and calibrate a simple model of risky choice, where extreme events undergo exogenous regime shifts. Section 4 extends this model to endogenous extreme events, and Section 5 concludes.

2 Background and related literature

The paper builds on three strands of related research: extreme events and crises, information choice, and regime shifts. Regarding extreme events and crises, previous research includes behavioral work such as Kunreuther and Pauly (2006), who focus on the role of individual myopia in precipitating catastrophes. It also includes research on bubbles by Abreu and Brunnermeier (2003), and Blanchard (1979), among others. There is still no consensus modeling approach for the analysis of extremes. A major challenge is that it is unclear how individuals behave towards extreme or low probability events. Initial evidence by Allais (1953) and Kahneman and Tversky (1979) suggested that agents overweight low-probability events. However, more recent research has uncovered three additional results. First, there is evidence that agents underweight low probability events in realistic situations where they must estimate probabilities based on experience, as documented by Barron and Erev (2003), Hertwig et al. (2005), and Rabin (2002). Second, econometrically, there is a bias to underestimate rare events, examined by King and Zeng (2001), and de Haan and Sinha (1999). Third, expected utility may not accurately predict responses to low probability events, a phenomenon studied by Bhide (2000) and Chichilnisky (2000). The finding that agents may systematically underestimate low probability events is particularly interesting, and suggests a systematic lack of knowledge that is not possible to address in current economic frameworks such as robust control and the theory of ambiguity aversion. These frameworks typically presume that agents are aware of their lack of knowledge. By contrast, the most devastating types of rare events involve situations where agents are unaware of their lack of knowledge, which we may term meta-ignorance.

For empirical research on regime shifts in the economy, see Hamilton (1989); Hamilton and Lin (1996); Ang and Bekaert (2002); Ang and Bekaert (2004); and Ang and Bekaert (2005). For theoretical modelling of regime shifts, see Reitz (1988); Evans (1996); Bekaert et al. (2001); and Angeletos et al. (2007).

Other relevant research includes Jaffee (2006); Ibragimov et al. (2009); and Lorenzoni (2008).

Regarding information choice, work by Morris and Shin (2002), Sims (2003), Veldkamp and Wolfers (2007), Skreta and Veldkamp (2009), and Veldkamp and Van Nieuwerburgh (2009) shows that agents do not always use all available information. This approach appeals to costs of information processing, so that agents choose to ignore potentially valuable, available information. However, these papers generally do not specify the form and size of costs. The information choice approach has been able to explain a number of anomalies in economics, including the home bias puzzle, asymmetric business cycles, portfolio under-diversification, and ratings inflation. Recent economic experience suggests, moreover, that an important impediment to market performance is lack of knowledge about how to forecast and hedge extreme events. This lack of knowledge reflects non-stationarity of the economic environment, which we embed in our model with the device of regime shifts.

Regarding regime shifts, there is ample evidence that the economic structure of major economic and financial variables is subject to sharp breaks. Hamilton (1989) develops the modern methodology of regime shifts, and shows its applicability to the macroeconomy. In financial markets, evidence of regime shifts is documented by Hamilton and Lin (1996), Ang and Bekaert (2002), Ang and Chen (2002), Ang and Bekaert (2004), and Ang and Bekaert (2005). Recent theoretical research has also examined economic foundations for regime changes, such as Angeletos et al. (2007).

2.1 Contribution of our paper

Our paper contributes to the literature in several important ways. First, we examine extreme events using the well-understood portfolio choice framework based on constant relative risk aversion and lognormal returns. We therefore can exhibit behavior concerning the impact of extreme events in a transparent, rational setting. Second, based on theoretical and empirical considerations, we incorporate latent regime switches in the likelihood of extreme events, which may be exogenous or endogenous. Our paper appears to be the first to analyze the economic impact of extreme events using this framework. Finally, we provide support for the information choice literature of Sims (2003) and Veldkamp and Van Nieuwerburgh (2009), since we give evidence on the size of costs needed to make agents ignore information about important extreme events.

3 Risky choice with exogenous extremes

In this section, we describe the risky choices of an individual faced with rare extreme events. There are three basic ingredients in our setup. First, the base model features a lognormal distribution with constant relative aversion (CRRA) utility. This CRRA-lognormal approach is very tractable and replicates key features of
financial data. Therefore it is commonly used for macroeconomics, portfolio choice and asset pricing, as in the work of Campbell (1994), Campbell (1996), and Campbell and Viceira (2002).\textsuperscript{13}

Second, our framework consists of a single representative agent. This framework allows us to analyze the average behavior when large numbers of similar investors are engaged in risky borrowing.\textsuperscript{14} The representative agent approach is typical of modern finance research in the tradition of Lucas (1978). Third, the case of rare events is handled by a regime switch approach. Regime switches have been shown to characterize both economic and financial data, by Hamilton (1989) and Hamilton and Lin (1996). Regime switches are also empirically significant in modelling stock market correlations and variances, as shown by Ang and Chen (2002), Dueker (1997), and Haas et al. (2004). Regime switches have also been utilized to model rare events in finance---by Evans (1996)) and Gourieroux and Monfort (2004). Finally, the subprime lending innovation starting after 2000 may be considered a regime shift in the risk distribution of the associated mortgage securities.

**Notation and Calibration.** The core notation used in the paper is as follows:

- The quantity $d$ denotes agents’ demand for risky investment, relative to available wealth;
- Superscript $*$ denotes an optimum;
- Superscript $T$ denotes a decision or wealth level during typical periods;
- Superscript $E$ denotes a decision or wealth level during extreme periods;
- Subscript $n$ denotes an endogenous investment or wealth level;
- Subscript $x$ denotes an exogenous investment or wealth level.

### 3.1 Excessive investment in a risky asset: A general case

Economic research concerning crises often focuses on the aggregate effects of excess borrowing for investment, as discussed by researchers from Fisher (1933) to Allen and Gale (2007). Such excessive borrowing is sometimes motivated as irrational. While irrationality can certainly drive excess behavior in many settings, it is valuable to determine whether such behavior may arise in a simple, rational framework. We start by showing that such excessive investment may be consistent with rational behavior in a very general setting.

\textsuperscript{13}For further details on the rationale and implementation of the CRRA-lognormal model, see Campbell (1994) page 469; Campbell (1996) page 304; and Campbell and Viceira (2002) Chapter 2. Other textbooks that use this approach are Huang and Litzenberger (1988) and Lyons (2001).

\textsuperscript{14}The analysis of large numbers of similar investors is also examined by the literature on strategic complementarities, see Cooper (1999).
Consider a general neoclassical utility function $U(W)$ that depends on wealth $W$. Among other qualities, this utility function is strictly increasing, bounded, continuous and concave. Following the approach of Campbell and Viceira (2002), the agent is endowed with initial wealth $W_0$, and invests a proportion $d$ in a risky asset with returns $r = r^f + \varepsilon$. The remainder is invested in a riskfree asset with returns $r^f$. Thus, $W = dW_0(1 + r) + (1 - d)W_0(1 + r^f)$, or

$$
W = dW_0(1 + r^f + \varepsilon) + (1 - d)W_0(1 + r^f)
$$

We will use the expression for the objective function in (1) for proving the propositions below. The agent maximizes utility subject to the wealth constraint, which is a strictly convex program—that yields a unique solution $d^*$—and unique expected wealth $W^*(d^*)$. We have the following proposition and corollary.

**Proposition 1** If the investor deviates from the optimal investment strategy $d^*$ by choosing a suboptimal investment strategy $\hat{d}$ during a small proportion $\alpha$ of the time, her expected utility loss is bounded above.

**Proof.** Intuitively, if the investor deviates from the optimum only a discrete proportion of the time and the utility function is bounded, then the expected utility loss is bounded. See the Appendix for a formal proof.

**Corollary 1** If there are high enough costs to learning whether she is behaving suboptimally a small proportion $\alpha$ of the time, the investor will rationally choose to continue behaving suboptimally.

**Proof.** See Appendix.

Theorem 1 and Corollary 1 show that for standard expected utility functions, if agents are suboptimal some of the time and there are costs to detecting extremes, then agents can rationally choose to be suboptimal.$^{16}$ While this insight is valuable, it is important to be able to quantify the results with observable economic parameters. We provide such calibrations below based on standard parametric utility functions and return processes, to which we now turn.

$^{15}$ By neoclassical utility function, we mean one that is strictly increasing and differentiably continuous, as in Allen and Gale (2007), chapter 2.

$^{16}$ Similar results appear in the literature on satisficing behavior and procrastination, see Akerlof and Yellen (1985); Akerlof and Yellen (1991); Rabin (2002); and O’Donoghue and Rabin (2008).
3.2 Base model

We first consider a base model of 'typical' events, where asset returns obey a simple stochastic law. The decision environment consists of a single individual with initial wealth $W_0$, choosing a fraction of wealth $d$ to invest in a risky asset. For these typical economic environments, the investor’s problem is straightforward: she maximizes expected utility by choosing the fraction $d$ to invest in the risky assets. In order to develop the intuition of the previous subsection more concretely, we utilize an important class of preferences and return processes. In particular, we suppose that the investor’s preferences exhibit constant relative risk aversion over wealth $W$, $U(W) = W^{1-\gamma}$, where $\gamma$ is the coefficient of relative risk aversion. We also assume that the random terms in risky asset returns are lognormally distributed,

$$\tilde{r} \equiv \log(1 + \tilde{R}) \sim N(\mu, \sigma^2).$$

These classes of preferences and returns are widely used in financial economics, for example Campbell (1996), and Campbell and Viceira (2002). To solve the investor’s problem, observe that the expectation of a lognormal variable $z$ satisfies $\log E[z] = E[\log z] + \frac{1}{2} V[\log z]$. Then, ignoring the constant $1 - \gamma$, and exchanging logs and expectations, we can write the investor’s maximization problem as

$$\max_d \log EW^{1-\gamma} = (1 - \gamma)E[w] + \frac{1}{2}(1 - \gamma)^2 V[w],$$

subject to $w = r + w_0$, where $w = \log W$, $r = \log(1 + R)$, and $w_0 = \log W_0$. To evaluate the above objective function, we therefore must compute the mean and variance of portfolio returns. The mean excess return is $E[r - r_f] = d[E(r) - r_f] + \frac{1}{2}d(1 - d)V[r]$. The variance of the portfolio return is $d^2V[r]$. Using equation (2), and standard algebraic manipulation as in Chapter 2 of Campbell and Viceira (2002), we can rewrite the investor’s problem as

$$\max_d d[E(r) - r_f] + \frac{1}{2}d(1 - d)V[r] + \frac{1}{2}(1 - \gamma)d^2V[r] = d\mu + \frac{1}{2}d(1 - d)\sigma^2 + \frac{1}{2}(1 - \gamma)d^2\sigma^2,$$

where $\mu = [E(r) - r_f]$. Taking derivatives yields first order conditions $\dot{\mu} + \frac{1}{2}(1 - 2d)\sigma^2 + (1 - \gamma)d\sigma^2 = 0$, or $d[\sigma^2 - (1 - \gamma)\sigma^2] = \dot{\mu} + \frac{1}{2}\sigma^2$. The optimal solution is therefore

$$d^* = \frac{\dot{\mu} + \sigma^2}{\gamma\sigma^2} = \frac{2\dot{\mu} + \sigma^2}{2\gamma\sigma^2}.$$

Equations (3) and (4) represent the basic form of objective function and optimum, which we shall use throughout the remainder of this paper. Intuitively, the optimal risky investment is increasing in expected returns and decreasing in risk aversion and variance. Our model assumes that the investor can borrow at
the riskfree rate, and abstracts from the possible deadweight costs of default that might arise when the agent chooses to invest with leverage. If we included the possibility of default costs, this would qualify our results.

### 3.3 A model of exogenous extremes

Now we consider the case of rare extreme events. Following the literature on peso problems, we model this situation as a small-probability regime switch in risky asset returns. Specifically, the structure of the problem is unchanged from above, except that the risky return now obeys (2) most of the time, but a small fraction $\alpha$ of the time, there is a regime shift to a period of larger tail events:

$$\tilde{r} \sim N(\mu, \sigma^2), \text{ with probability } 1 - \alpha \text{ (Typical regime)}$$

$$\tilde{r} \sim N(\mu, \frac{\sigma^2}{\alpha}), \text{ with probability } \alpha \text{ (Extreme regime)},$$

where $\alpha$ is small.\(^{17}\) We next examine three levels of investor awareness about the stochastic environment: full knowledge of the environment and the specific regime, full knowledge of the environment but stochastic determination of the specific regime, and complete ignorance that an extreme regime may occur at all.

**Agent Knows the Environment and the Specific Regime**

First, consider a situation where the individual knows the stochastic environment as well as which regime holds at each moment. At the beginning of each period, based on the regime that prevails, she solves for the optimal demand in each regime.\(^{18}\) Using the same optimization approach as before, the optimal demand will now depend on the regime, and is a vector $d^* = (d^T, d^E)$, with $T$ referring to the typical regime and $E$ to the extreme regime. The typical regime occurs with probability $1 - \alpha$ and extreme periods occur with probability $\alpha$. Therefore the optimal demand vector is

$$d^T = \frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma \sigma^2} = \frac{2\hat{\mu} + \frac{\sigma^2}{2}}{2\gamma \sigma^2}, \text{ with probability } 1 - \alpha$$

$$d^E = \frac{\hat{\mu} + \frac{\sigma^2}{\alpha}}{\gamma \frac{\sigma^2}{\alpha}} = \frac{2\alpha \hat{\mu} + \frac{\sigma^2}{2}}{2\gamma \sigma^2}$$

$$= \alpha d^T + \frac{1 - \alpha}{2\gamma}, \text{ with probability } \alpha.$$

\(^{17}\) In this simple specification, the probability of the rare event is inversely proportional to its impact: the lower the probability, the higher the impact on variance. Therefore it is an easy way to deliver a low probability, high impact event. This specification is similar to that of Gourieroux and Jasiak (2001).

\(^{18}\) The individual does not know the value of risky returns, just the distribution from which they come. Observe that the mixture of log-normals is not restrictive on the unconditional distribution. Conditionally, each regime satisfies log-normality, but unconditionally, a mixture of normals can approximate most empirically observed return distributions arbitrarily closely. For more details on normal mixtures, see McLachlan and Peel (2000).
This is the basic form of investment demand with exogenous extremes in our framework.\textsuperscript{19}

Properties of the Solution The solution in (6) has two distinct properties. First, for positive excess returns $\hat{\mu}$, it is clear by inspection that investment in the typical regime exceeds the investment in the extreme regime, i.e. $d^T > d^E$; it is intuitive that more would be invested in the safer typical regime $d^T$ than in the riskier extreme regime $d^E$.\textsuperscript{20} Second, extreme regime investment $d^E$ depends positively and linearly on the probability of the extreme regime. This is intuitive since as $\alpha$ rises the extreme regime becomes less extreme.

A quantitative sense of this differential is achieved by calibrating expression (6) to US data. We use a standard set of calibration values, as displayed in Table 2. Table 3 shows the results of applying these values to expression (6). We find $d^T$ always greatly exceeds $d^E$, as expected. For example, with risk aversion $\gamma = 2$, we find that $d^T = 1.73$ and $d^E = 0.28$. The ratio of $d^T$ to $d^E$ is 6.18. Thus the demand in a known extreme regime involves no leverage (i.e. $d^E < 1$), and is around 6 times smaller than in the typical regime. This result is qualitatively intuitive, if we think of the extreme regimes as high volatility, disaster periods, where most investors hold small amounts of risky assets, and typical regimes as good or boom periods, when it is relatively more attractive to hold a large position in risky assets.

We also examine another perspective on investors’ risk positions, based on the common propensity of individuals to expand their consumption through borrowing.\textsuperscript{21} To measure the actual situation for the US economy, we calculate an empirical version of the investment ratio $d$ by computing the ratio of total assets to net worth based on the data for households and nonprofit corporations in the US. The results are illustrated in figure 1. Evidently, this ratio has tended to increase slightly relative to the first observations in the 1940s and 1950s, and has exceeded unity in every year. Thus, the historical experience of the US economy indicates that $d$ is consistent with a leveraged average investor throughout the last half century.\textsuperscript{22}

\textsuperscript{19}To see the third row, note that

\[
\begin{align*}
\hat{\mu} + \frac{\mu^2}{\gamma \sigma^2} & = \frac{\alpha \hat{\mu} + \frac{\sigma^2}{\gamma} - \frac{\alpha \sigma^2}{\gamma}}{\gamma \sigma^2} = \alpha d^T + \frac{\sigma^2}{\gamma \sigma^2} \\
\hat{\mu} + \frac{\mu^2}{\gamma \sigma^2} & = \alpha d^T + \frac{\sigma^2(1 - \alpha)}{\gamma \sigma^2} = \alpha d^T + \frac{1 - \alpha}{2 \gamma}.
\end{align*}
\]

\textsuperscript{20} To see this, observe that the condition for $d^T > d^E$ can be written, using expression (6), as $d^T > \alpha d^T + \frac{\alpha - \gamma}{2 \gamma}$, or $d^T > \frac{1}{2 \gamma}$. Substituting in the definition of $d^T$ yields $\frac{\hat{\mu} + \frac{\mu^2}{\gamma \sigma^2}}{\gamma \sigma^2} > \frac{1}{2 \gamma}$, which simplifies to $2 \hat{\mu} > 0$.

\textsuperscript{21} This propensity is related to the concept of “over-borrowing,” used by Fisher (1933) in the context of financial crises. For related research on excessive expansion of credit, see Abreu and Brunnermeier (2003); Lorenzoni (2008); and Shin (2009).

\textsuperscript{22} We can derive the parameter ranges over which $d^E$ involves leverage. We need to show that $d^T > 1$, or using definition (6), this means $\frac{\hat{\mu} + \frac{\mu^2}{\gamma \sigma^2}}{\gamma \sigma^2} > 1$. By positivity of $\gamma$ and $\sigma^2$, we can write this as $2 \hat{\mu} + \frac{\sigma^2}{\gamma \sigma^2} > 2 \gamma \sigma^2$, or

\[
\frac{\hat{\mu}}{\sigma^2} > \frac{2 \gamma - 1}{2}.
\]

(7)
Agent Knows the Environment, but does Not Know the Regime

Now we consider a situation where the investor knows that there are regime shifts, but does not know, ex ante, which regime will obtain. The investor, as an expected utility maximizer, will choose an intermediate level of demand $\hat{d}$, as shown in the following proposition.\footnote{For background on expected utility, see Von Neumann and Morgenstern (1944); Gilboa and Schmeidler (2001) and Gilboa (2009). If the investor were a non-expected utility maximizer, she could have a different weighting scheme. For a survey of related research, see Gilboa (2004).}

**Proposition 2:** Consider an investor that is an expected utility maximizer with a neoclassical strict monotone increasing utility function $u(d)$, a convex budget constraint, and an environment of regime shifts in extreme events. Denote the optimal risky asset demands during known typical and extreme regimes as $d^T$ and $d^E$, respectively, where $d^E < d^T$. Denote the optimal risky demand when the investor does not know which regime obtains as $\hat{d}$. In this setting, $\hat{d}$ is bounded by $d^T$ and $d^E$, that is, $d^E \leq \hat{d} \leq d^T$.

\textit{Proof.} See Appendix. \hfill $\blacksquare$

This result is intuitive: if the investor is unsure whether an extreme regime prevails she will reduce her level of investment relative to the typical demand $d^T$, but not as low as if she were certain to be in an extreme regime, $d^E$.

Agent Unaware of the Extreme Regime

In the preceding examples, the investor was aware that the environment presented the possibility of extreme risks. By contrast, some of the most significant extreme events in history have been unknown and unforeseen by the public at large.\footnote{In addition to 2008’s financial crisis, other negative examples include the Black Death of 1348; the 1929 US stock market crash; the set of events leading up to the creation of the atomic bomb; global warming; and the devastation of 2005’s Hurricane Katrina. Positive examples include the invention of the wheel; signing of the first US copyright law in 1790; the Wright brothers’ 1903 flight; and the record-breaking US stock market levels of the 1990s.} One way to model such ex ante ignorance about extreme regimes is to use a hidden regime shift.\footnote{The best of our knowledge, this formulation of hidden extreme events is novel to the current paper. A parallel framework is used by Gourieroux and Jasiak (2001), who provide an asset demand application, although they do not consider hidden regimes, nor endogenous extremes.} Specifically, although the true risky return distribution features a regime shift as in (5), the investor believes that the typical regime always holds, i.e. $\tilde{\varepsilon} \sim N(\mu, \sigma^2)$ with probability 1. Accordingly, she demands $d = d^T$ with probability 1, instead of probability $1 - \alpha$ as in equation (6). The investor is therefore over-invested $\alpha \%$ of the time, investing $d^T$ instead of the much smaller and optimal $d^E$.

We may ask two important questions about the investor’s behavior. First, how much does this suboptimal investment hurt her? This question is natural in light of Proposition 1 because the suboptimality only occurs a small percentage of the time. Second, if the investor can learn about the extreme regime at a cost, what is...
the impact on her investment strategy? We summarize the answers to these questions in Proposition 3 and Corollary 2, below.

**Proposition 3** The cost to investors of suboptimal behavior during extremes is bounded above by a constant $K$, which is proportional to squared, standardized excess returns $\left( \frac{\mu}{\sigma} \right)^2$.

*Proof.* See Appendix.

**Corollary 2.** If the costs of learning about extreme events are above a finite threshold, the investor will prefer to over-invest during extreme periods.

*Proof.* See Appendix.

We can see from the Appendix that the constant $K$ is decreasing in risk aversion $\gamma$ and volatility $\sigma$, and increasing in expected excess returns $\hat{\mu}$. This is intuitive because investors who are more risk averse and face more volatility and lower returns will risk a smaller part of their wealth. Consequently, such investors will have less at stake during extreme periods and will therefore face lower costs of suboptimal behavior.

In sum, according to Proposition 3 and Corollary 2, if there are large enough costs to learning about extremes, the investor’s strategy is insensitive to rare extreme events. This is true even when extremes deliver a large effect on return volatility.\(^{26}\) To summarize this subsection, we have shown that in an environment of exogenous extremes, a knowledgeable investor may invest much more in normal times than in known extreme regimes. We have also provided a bound on the utility loss from suboptimal behavior by investors who do not understand the economic environment. The existence of this bound is consistent with the literature on global games, rational inattention and information choice.\(^{27}\) It suggests that even if agents were informed of the suboptimality of their investment strategy, a high enough level of costs associated with learning about extremes will prevent them from shifting their strategy.

### 3.4 Calibration to the US economy

We calibrate Proposition 3 to US data using equation (14) from the Appendix. The results are displayed in Figure 2. This figure shows that the costs of maintaining excessive risky investments is bounded from above by a range from around 1% to 6% of wealth, based on the calibrated probability of the extreme regime of 1.7% annually. These costs decrease with risk aversion, because more risk averse investors would always

\(^{26}\)Insensitivity of general expected utility functionals to rare events has been examined by Chichilnisky (2000); For related contexts involving biased perception of virgin risks and fearsome risks, see Chichilnisky (2007); Chichilnisky and Heal (2003); Pavlov and Wachter (2006); Sunstein and Zeckhauser (2008); and Weber (2006).

\(^{27}\)See Morris and Shin (2002); Sims (2003); Skreta and Veldkamp (2009); Veldkamp and Van Nieuwerburgh (2009).
maintain smaller investment amounts. Since U.S. wealth is approximately four times its annual GDP, this range also represents approximately 4% to 24% of GDP. The high values for this range indicate that extreme events represent a material cost for the U.S. economy, and thus suggest there could be significant social benefits from creating greater information to allow greater and more efficient hedging of such risks.

4 Risky Choice with Endogenous Extremes

The likelihood of extreme and rare events is often affected by the investment behavior of agents in the economy. Endogenous extreme events would include, for example, the effect of human activity on extreme climate changes, and the effect of risky borrowing on financial crises. Accordingly, in this section, we consider a situation where excessive risky borrowing raises the likelihood that the extreme regime will occur. This environment entails more complex information processing for investors, since their returns depend on the likelihood of extremes, which in turn depend on their investment strategies. Similar to the literature on information choice of Sims (2003), such processing costs may lead investors to ignore potentially important information. A further layer of complexity arises when there is complete lack of knowledge, such that individuals are unaware of their collective impact on the likelihood of unforeseen extremes. In light of these considerations, we formalize endogenous extremes by considering an investor who believes the risky return comes from a single distribution as in equation (2), while in truth, the distribution switches endogenously. Optimally the investor should use a cutoff level for risky investment, as we showed in (6). However, unaware of the consequences, she follows the approach of (4) and just chooses the more aggressive investment demand appropriate for the typical regime, namely $d^T$.

Once more we may ask two questions. First, how much does this situation harm the investor? In order to answer this question, we compute the expected wealth from behaving optimally and suboptimally. Optimal investment involves a cutoff rule, with potentially non-constant $d$, while suboptimal investment involves a constant investment rule for $d$. We are able to compute a bound for the losses that will arise from the

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29It is possible to account for endogenous regimes in a less draconian way, for example if the probability of extremes lowers after a few periods of prudent behavior. The role of excess borrowing in precipitating extreme financial market behavior has been motivated in many ways. One approach emphasizes heightened investor and bank fragility due to lack of liquidity. Prominent examples are the cases of LTCM in 1998 and Lehman Brothers in 2008. Such firms and investors are especially susceptible to even small liquidity shocks and margin calls, see Shleifer and Vishny (1997). Another approach is taken by the research on bubbles and financial crises, see Allen and Gale (2000) and Blanchard (1979).

30Examples include climate change, or stock market bubbles. This class of extreme events is related to rare events of Taleb (2005), and oblivious ignorance of Bhide (2000). In geopolitics, an instance of unknown endogenous extremes could be the set of events in the early cold war that culminated in the Cuban missile crisis of 1962. This resembles a reverse peso problem: by failing to account for their own ignorance, rational individuals do not anticipate extreme events, which they themselves precipitated.
suboptimal strategy. Second, under what conditions will there be strong economic incentives for her to learn? If the costs of learning are high enough, a risk averse individual may well ignore endogenous, high-impact regime shifts.

4.1 A Two-period model

Substantial evidence indicates that excessive credit and risky borrowing help to create extreme financial events.\(^{31}\) We recognize this evidence by assuming that there are two periods in the economy, with the first-period investment choice affecting the return distribution in the second period. In particular, if the investor is too leveraged in period 1, then the likelihood of extremes is increased to \(\alpha_n\) in period 2. For simplicity, we set \(\alpha_n = 2\alpha\). Thus, in this endogenous extreme events model, if the investor chooses a leveraged position in period 1, then in period 2 the typical regime occurs with probability \(1 - 2\alpha\) and the extreme regime occurs with probability \(2\alpha\). Using the same approach as in Section 2, the optimal demand vector is similar to that of (6):

\[
\begin{align*}
    d^T &= \frac{2\hat{\mu} + \sigma^2}{2\gamma\sigma^2}, \text{ with probability } 1 - 2\alpha \\
    d^E &= \frac{\hat{\mu} + \frac{\sigma^2}{2\alpha}}{\gamma \frac{\sigma^2}{\alpha}} = \alpha d^T + \frac{1 - \alpha}{2\gamma}, \text{ with probability } 2\alpha.
\end{align*}
\]

We will use this expression to calculate the effect of endogenous extremes on risky behavior.

Period 1: We assume that period 1 is a typical regime. If the investor were unaware of the impact of her period 1 investment on period 2 volatility, she would then simply choose the optimal demand for a typical period, namely \(d^T\); we now assume that this decision exhibits leverage and refer to it as the leveraged (L) outcome, that is, \(d^L > 1\). On the other hand, if the investor is aware that her investment decision in the first period may have a negative impact on the second period distribution, then she may wish to temper the size of her first period investment. In particular, we will assume that as long as the period 1 investment does not exceed 1.0, that is, \(d^L \leq 1\). We will describe an investment decision as prudent when \(d = 1\), and denote it as \(d^P = 1\).

In this setting, the first period investment choice must consider the impact it will have on the second period volatility. She can invest \(d^P = 1\), which has the benefit of ensuring a lower probability of the extreme regime in the following period, but at the cost of foregone returns; or she can borrow to invest \(d^L > 1\), which has the benefit of higher expected returns in period 1, but at the cost of an increased danger of the extreme regime.

\(^{31}\)See Fisher (1933); Bernanke (1983); (Allen and Gale (2007); Lorenzoni (2008); and Shin (2009).
Period 2: In the second period, the probability of extremes is

\[
\Pr(\text{extremes}) = \begin{cases} 
\alpha, & \text{if investor choose } d^P \text{ in the first period,} \\
2\alpha, & \text{if investor choose } d^L \text{ in the first period.}
\end{cases}
\]

Since period 2 is the last period in the planning horizon, we assume the investor cannot take on a further leverage position with \( d^L > 1 \). Instead, the investor may either invest \( d^P = 1 \) for period 2—what we are calling the prudent level— or invest an even smaller amount \( d^E < 1 \) that would be optimal if the extreme regime occurs in period 2.

In analyzing behavior in the 2-period model, as earlier, we consider two levels of investor awareness of the economic environment, corresponding to complete understanding and complete misunderstanding.

**Agent Knows the Environment and the Specific Regime**

In this case, the representative investor understands that the environment features regime shifts in the likelihood of extreme events. Further, she knows that leverage raises the likelihood of the extreme regime. We summarize the result in Proposition 4.

**Proposition 4** The expected net benefit of leverage (\( d > 1 \)) in period 1 is bounded, for investors who know that the environment features regime-switching in extreme events. This effect depends on the values of \( \hat{\mu} \) and \( \sigma^2 \).

*Proof.* See Appendix.

**Agent Unaware of the Extreme Regimes**

In this case, the investor does not know that there are regime shifts and does not know that she can influence the likelihood of extremes. Therefore, in period 1 she always selects the optimal investment for a typical regime, namely \( d^L > 1 \), and in period 2 she invests as much as she can, namely \( d^P = 1 \). Nevertheless, even in this circumstance, the effect resulting from the suboptimal investment decision is bounded. We summarize this result in Proposition 5.

**Proposition 5** The expected net loss from suboptimal investment is bounded, for an investor who does not understand that the environment features regime-switching in extreme events.

*Proof.* See Appendix.

Proposition 4 above has shown that rational investors may knowingly increase the likelihood of extreme events in the second period. Proposition 5 now shows that investors who do not understand the environment face possible losses that are bounded. Therefore if the costs of learning about the environment are large enough, investors may choose to continue with a socially suboptimal strategy.
4.2 Calibration to the US economy

We calibrate Propositions 4 and 5 to the US data shown in Table 2—using the expressions for the net benefits and costs from the Appendix. The results are displayed in Figures 3 and 4. Figure 3 shows the effect of having leverage in the first period for an investor who understands that this will increase the likelihood of extreme events. In this case, there is a small net benefit of leverage, between 2% and 2.5% of wealth. Figure 4 shows the net effect for an investor who does not understand the extreme regimes and is levered \( d^L > 1 \) in period 1, and fully invested in period 2. We see that there are small costs from not reducing investment below wealth. These costs represent between 0.005% to 0.25% of available wealth. Thus, whether the investor was aware or unaware of endogenous regime shifts, for the calibrated US economy over the last 50 years it may not have been viable to suggest that she limit investment below available wealth. The reason is that the extreme event represents an externality that raises the cost of investing in the future, with little or no impact on the benefits of taking advantage of high returns in the present. Therefore it is necessary to provide incentives for investors to internalize the costs of future extreme events.

5 Conclusions

In this paper we construct a simple latent regime-switching model of portfolio choice, in order to assess the implications for over-investing. Motivated by theoretical and empirical considerations, we examine the benefits and costs of leverage, and of suboptimal investment. Our most striking finding is that in both one and two-period models, the benefits of sustained optimal investment are bounded. Thus, investors may knowingly ignore or exacerbate the likelihood of extreme events, especially if there are costs to learning the structure of the financial environment. We also document from our calibrated results that, in the typical regime, investors benefit from leverage, that is from an investment amount exceeding their wealth. Moreover, in the calibration for a two period model, the net costs of suboptimal investments are relatively small. Overall, we thus document that the costs of ignoring extreme events are small and the benefits of leverage are substantial.

Our paper therefore provides both a theoretical framework for examining extreme events, and empirical evidence on the scope of the benefits and costs related to learning about extremes. From a policy perspective, our results provide support for research that will provide more complete information on the likelihood and consequences of extreme and rare events. Indeed, if the losses from extreme outcomes create negative social externalities, there may even be a case for subsidizing the creation of information on the causes and consequences of the extreme events.

32The expressions for Propositions 4 and 5 are in equations (19) and (21).
References


### Table 1: Examples of Extreme and Rare Events

<table>
<thead>
<tr>
<th></th>
<th>Frequent</th>
<th>Rare</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Non-Extreme</strong></td>
<td>No war, post-1990 western Europe</td>
<td>↓ $CO_2$ pollution</td>
</tr>
<tr>
<td>(Small Impact)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Extreme</strong></td>
<td>↑ $CO_2$ Pollution</td>
<td>Multi-nation war, post-1990 western Europe</td>
</tr>
<tr>
<td>(Large Impact)</td>
<td></td>
<td>Multi-country stock market crash, post-Great Depression</td>
</tr>
</tbody>
</table>

### Table 2: Calibration to US data

<table>
<thead>
<tr>
<th>Variable</th>
<th>Calibrated Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualized excess stock return $\hat{\mu}$</td>
<td>$\hat{\mu} = 0.081 - 0.009 = 0.072$</td>
<td>Campbell (2003) p. 805</td>
</tr>
<tr>
<td>Annualized stock market volatility $\sigma$</td>
<td>$\sigma = 0.156$</td>
<td>Campbell (2003) p. 805</td>
</tr>
<tr>
<td>Average borrowing rate $r_b$</td>
<td>$r_b = 1 + R_b = 1.071$</td>
<td>Federal Reserve Bank of St. Louis</td>
</tr>
<tr>
<td>Discount factor $\beta$</td>
<td>$\beta = 0.99$</td>
<td>Mehra and Prescott (2003) p. 907</td>
</tr>
<tr>
<td>Risk aversion $\gamma$</td>
<td>$\gamma \in {1, 2, \ldots, 10}$</td>
<td>Lewis (1999) p. 576</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mehra and Prescott (1985) p. 154</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mehra and Prescott (2003) p. 907</td>
</tr>
<tr>
<td>Annualized likelihood of an extreme event, $\alpha$</td>
<td>$\alpha = 0.017$</td>
<td>Barro (2006) p. 837</td>
</tr>
</tbody>
</table>

We compute the borrowing rate $r_b$ as the average of the monthly Prime Bank Loan rate from January 1949 to December 2008.
Table 3: Risky Asset Demand in Extreme and Normal Times

The table presents risky demand $d^E$ and $d^L$ during extreme and normal times respectively, using equation (6). The calibration is as in Section 2. The parameter $\gamma$ denotes the coefficient of relative risk aversion.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$d^L$</th>
<th>$d^E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.4586</td>
<td>0.5503</td>
</tr>
<tr>
<td>2</td>
<td>1.7293</td>
<td>0.2751</td>
</tr>
<tr>
<td>3</td>
<td>1.1529</td>
<td>0.1834</td>
</tr>
<tr>
<td>4</td>
<td>0.8646</td>
<td>0.1376</td>
</tr>
<tr>
<td>5</td>
<td>0.6917</td>
<td>0.1101</td>
</tr>
<tr>
<td>6</td>
<td>0.5764</td>
<td>0.0917</td>
</tr>
<tr>
<td>7</td>
<td>0.4941</td>
<td>0.0786</td>
</tr>
<tr>
<td>8</td>
<td>0.4323</td>
<td>0.0688</td>
</tr>
<tr>
<td>9</td>
<td>0.3843</td>
<td>0.0611</td>
</tr>
<tr>
<td>10</td>
<td>0.3459</td>
<td>0.0550</td>
</tr>
</tbody>
</table>

Figure 1: Asset to Net Worth Ratio for US Households: 1945-2009

The figure shows the ratio of total US household assets to net worth. All variables are available from Flow of Funds accounts at the Board of Governors Federal Reserve Bank. The frequency is annual, and the time period is 1945 to 2009.
Figure 2: Net Cost of Excess Investment for a US Investor who faces Exogenous Extreme Events

The figure calibrates the net expected cost of over-investing during exogenous extreme events, as discussed in Section 3. The figure is computed from Proposition 3, equation (14), using US data and the calibration in Table 2. The data values displayed in the figure are given explicitly in Table 4. The bound shows how much an investor would have gained as a percentage of wealth, by investing a smaller, optimal amount during rare extreme events that occur with fixed probability $\alpha$ in a one-period model. According to Corollary 2 this bound may also be interpreted as the minimum cost to dissuade investors from learning about regime shifts.

![Figure 2](image)

Figure 3: Net Benefit from Leverage for US investor who Understands Endogenous Extreme Events

The figure calibrates the net benefit of leverage when the investor knows that excess leverage may raise the likelihood of extreme events in period 2, as discussed in Section 4. The figure is computed from Proposition 4, equation (19). We use US data and the calibration values from Table 2. The data values displayed in the figure are given explicitly in Table 4.

![Figure 3](image)
Figure 4: Net Cost of Suboptimal Investment for US investor who is Unaware of Endogenous Extreme Events

The figure calibrates the net cost of investing all her wealth \((d^P = 1)\) in period 2, when the investor is unaware of the possibility of endogenous extreme events in this period. The figure is computed from Proposition 5, equation (21). We use US data and the calibration values from Table 2. The data values displayed in the figure are given explicitly in Table 4. The red triangles show the effect for an investor who undertook leverage in the first period. This bound may be interpreted as the minimum cost to dissuade investors from learning about regime shifts in the likelihood of rare events, as discussed in Section 4.

Table 4: The Benefits and Costs of Leverage during Extreme Regimes

The table presents the benefits and costs of leverage and suboptimal demand as a percentage of investor wealth. The calibration corresponds to that in Propositions 3, 4 and 5, as displayed in Figures 2 to 4 above. The parameter \(\gamma\) denotes the coefficient of relative risk aversion. P3, P4, and P5 denote Propositions 3, 4, and 5, respectively.

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>Cost of Leverage (P3)</th>
<th>Benefit of Leverage (P4)</th>
<th>Loss from Suboptimal Investment (P5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.0540</td>
<td>2.3515</td>
<td>0.0049</td>
</tr>
<tr>
<td>2</td>
<td>3.0270</td>
<td>2.1897</td>
<td>0.0253</td>
</tr>
<tr>
<td>3</td>
<td>2.0180</td>
<td>2.1440</td>
<td>0.0482</td>
</tr>
<tr>
<td>4</td>
<td>1.5135</td>
<td>2.1273</td>
<td>0.0717</td>
</tr>
<tr>
<td>5</td>
<td>1.2108</td>
<td>2.1223</td>
<td>0.0954</td>
</tr>
<tr>
<td>6</td>
<td>1.0090</td>
<td>2.1230</td>
<td>0.1193</td>
</tr>
<tr>
<td>7</td>
<td>0.8649</td>
<td>2.1271</td>
<td>0.1432</td>
</tr>
<tr>
<td>8</td>
<td>0.7568</td>
<td>2.1333</td>
<td>0.1671</td>
</tr>
<tr>
<td>9</td>
<td>0.6727</td>
<td>2.1408</td>
<td>0.1911</td>
</tr>
<tr>
<td>10</td>
<td>0.6054</td>
<td>2.1493</td>
<td>0.2151</td>
</tr>
</tbody>
</table>
A Proofs of Propositions

Proposition 1. If the investor deviates from the optimal investment strategy $d^*$ by choosing a suboptimal investment strategy $\hat{d}$ during a small proportion $\alpha$ of the time, her expected utility loss is bounded above.

Proof. We need to show that the expected utility loss $\Delta EU$ satisfies $\Delta EU \leq K$, for some $K < \infty$. First, let us denote the suboptimal wealth level $\hat{W}(\hat{d})$. Now note that the expected utility loss is the difference between optimal utility and the suboptimal utility that occurs with probability $\alpha$. Thus $\Delta EU \equiv U(W^*) - [\alpha U(\hat{W}) + (1 - \alpha)U(W^*)]$, where we drop the argument in $W()$ for simplicity. Computing the expected utility loss, we obtain

$$\Delta EU \equiv U(W^*) - [\alpha U(\hat{W}) + (1 - \alpha)U(W^*)]$$

(9)

By boundedness of the utility function, the quantity in (9) is finite and bounded above, for example, by $\alpha U(W^*)$. Thus, for $K = \alpha U(W^*)$, we have that $\Delta EU$ satisfies $\Delta EU \leq K$, as was to be shown.

Corollary 1. If there are high enough costs to learning whether she is behaving suboptimally a small proportion $\alpha$ of the time, the investor will rationally choose to continue behaving suboptimally.

Proof. From Proposition 1, we know that the investor loses at most $K$ from investing suboptimally for a small portion of the time. If we set costs to $K$, it follows that the investor is better off using the suboptimal strategy.

Proposition 2. Consider an investor that is an expected utility maximizer with a neoclassical strict monotone increasing utility function $u(d)$, a convex budget constraint, and an environment of regime shifts in extreme events. Denote the optimal risky asset demands during known typical and extreme regimes as $d^T$ and $d^E$, respectively, where $d^E < d^T$. Denote the optimal risky demand when the investor does not know which regime obtains as $\hat{d}$. In this setting, $\hat{d}$ is bounded by $d^E$ and $d^T$, that is, $d^E \leq \hat{d} \leq d^T$.

Proof. We shall proceed by contradiction. First, the hypothesis of strict monotone increasing utility function $u$ implies that for any feasible demands $d^1$ and $d^2$, the following relation holds:

$$u(d^1) > u(d^2) \quad \text{if and only if } d^1 > d^2.$$
Second, the hypothesis of expected utility implies that if the investor faces a lottery between two alternative utilities in typical (T) and extreme (E) regimes \( u_T(d) \) and \( u_E(d) \) with probabilities \( 1 - \alpha \) and \( \alpha \), then the expected value of her utility is given by

\[
EU = (1 - \alpha)u_T(d) + \alpha u_E(d).
\]

As an expected utility maximizer, the investor will choose \( \hat{d} \) in the feasible set \( D \) to maximize \( EU \), that is

\[
\hat{d} = \arg \max_{d \in D} (1 - \alpha)u_T(d) + \alpha u_E(d),
\]

for all \( \alpha \in [0, 1] \). 

We now ascertain the value of \( \hat{d} \) and proceed by contradiction. There are two cases to consider.

**Case 1:** Suppose that \( \hat{d} > d^T \). Then it follows from (10) that \( u_T(\hat{d}) > u_T(d^T) \). Now set \( \alpha = 0 \), which implies from (11) that \( \hat{d} = \arg \max_{d \in D} u_T(d) \). But since \( \hat{d} \) is in the feasible set, this means that \( \hat{d} \) should have been chosen in the typical regime, as it gives higher utility. This contradicts that \( d^T \) is optimal for the typical regime.

**Case 2:** Suppose that \( \hat{d} < d^E \). Then it follows from (10) that \( u_E(\hat{d}) < u_E(d^E) \). Now set \( \alpha = 1 \), which implies from (11) that \( \hat{d} = \arg \max_{d \in D} u_E(d) \). But since \( d^E \) is feasible, \( d^E \) should have been chosen in (11), as it gives higher utility. This contradicts that \( \hat{d} \) maximizes expected utility. Thus, in both cases we obtain a contradiction. Therefore \( d^E \leq \hat{d} \leq d^T \), as was to be shown.

**Proposition 3.** The cost to investors of suboptimal behavior during extremes is bounded above by a constant \( K \), which is proportional to squared, standardized excess returns \( \left( \frac{\hat{\mu}}{\sigma} \right)^2 \).

**Proof.** We need to show that the utility loss \( \Delta EU \) from investing a proportion \( d^T \) instead of \( d^E \) during extreme periods is of the form \( \Delta EU \leq K \), where \( K = \theta \left( \frac{\hat{\mu}}{\sigma} \right)^2 \) for some positive, finite \( \theta \). In order to compute the utility loss, we just calculate the investor’s objective function (3) in both cases.

**Optimal:** The optimal strategy is to invest \( d^E \), yielding an objective function

\[
U(W(d^E)) = d^E \hat{\mu} + \frac{1}{2} d^E (1 - d^E) \frac{\hat{\sigma}^2}{\alpha} + \frac{1}{2} (1 - \gamma)(d^E)^2 \frac{\sigma^2}{\alpha}
\]

\[
= \left( \alpha d^T + \frac{1 - \alpha}{2\gamma} \right) \hat{\mu} + \frac{1}{2} \left( \alpha d^T + \frac{1 - \alpha}{2\gamma} \right) \left( 1 - \alpha d^T - \frac{1 - \alpha}{2\gamma} \right) \frac{\sigma^2}{\alpha}
\]

\[
+ \frac{1}{2} (1 - \gamma) \left( \alpha d^T + \frac{1 - \alpha}{2\gamma} \right)^2 \frac{\sigma^2}{\alpha},
\]

where the second line uses the fact that \( d^E = \alpha d^T + \frac{1 - \alpha}{2\gamma} \), from expression (6).
Suboptimal: In similar fashion, the suboptimal payoff can be calculated as

\[ U(W(d^T)) = d^T \hat{\mu} + \frac{1}{2} d^T (1 - d^T) \frac{\sigma^2}{\alpha} + \frac{1}{2} (1 - \gamma)(d^T)^2 \frac{\sigma^2}{\alpha}. \]  

(13)

Now the expected utility loss from suboptimal investment is just the difference between (12) and (13):

\[
\Delta EU = \hat{\mu} \left[ (1 - \alpha)(\frac{1}{2\gamma} - d^T) \right] + \frac{1}{2} \frac{\sigma^2}{\alpha} \left[ (d^T(\alpha - 1) + (d^T)^2(1 - \alpha^2) - \frac{2d^T(1 - \alpha)}{2\gamma} + (1 - \alpha) - \frac{1}{2\gamma} \right]^2
\]

\[
+ \frac{1}{2} (1 - \gamma) \frac{\sigma^2}{\alpha} \left[ (1 - \alpha)(\frac{1}{2\gamma} - d^T) \right] + \frac{1}{2} \frac{\sigma^2}{\alpha} \left[ (d^T(\alpha - 1) + (d^T)^2(1 - \alpha^2) - \frac{2d^T(1 - \alpha)}{2\gamma} + (1 - \alpha) - \frac{1}{2\gamma} \right]^2
\]

\[
+ \frac{1}{2} (1 - \gamma) \frac{\sigma^2}{\alpha} \left[ (d^T(\alpha - 1) + (d^T)^2(1 - \alpha^2) - \frac{2d^T(1 - \alpha)}{2\gamma} + (1 - \alpha) - \frac{1}{2\gamma} \right]^2
\]

\[
= \hat{\mu} \left[ (1 - \alpha)(\frac{1}{2\gamma} - d^T) \right] + \frac{1}{2} \frac{\sigma^2}{\alpha} \left[ (d^T(\alpha - 1) + (d^T)^2(1 - \alpha^2) - \frac{2d^T(1 - \alpha)}{2\gamma} + (1 - \alpha) - \frac{1}{2\gamma} \right]^2
\]

\[
+ \frac{1}{2} (1 - \gamma) \frac{\sigma^2}{\alpha} \left[ (d^T(\alpha - 1) + (d^T)^2(1 - \alpha^2) - \frac{2d^T(1 - \alpha)}{2\gamma} + (1 - \alpha) - \frac{1}{2\gamma} \right]^2
\]
We can now substitute the expression for $d^T$ from equation (6), to obtain

$$
\Delta EU = \hat{\mu} \left[ (1-\alpha) \left( \frac{1}{2\gamma} - \frac{\hat{\mu} + \sigma^2}{\gamma\sigma^2} \right) \right]
+ \frac{1}{2} \left( \frac{1}{2\alpha} \left( 1 - \alpha^2 \right) \left( \frac{1}{4\gamma} - \frac{\hat{\mu} + \sigma^2}{\gamma\sigma^2} + \frac{(\hat{\mu} + \sigma^2)^2}{\gamma^2\sigma^4} \right) \right)
= \hat{\mu} \left[ (1-\alpha) \frac{\sigma^2 - 2\hat{\mu} - \sigma^2}{2\gamma\sigma^2} \right] + \frac{1}{2} \left( \frac{1}{2\alpha} \left( 1 - \alpha^2 \right) \frac{4\hat{\mu}^2}{4\gamma\sigma^4} \right)
= \hat{\mu} \left[ (1-\alpha) \frac{\sigma^2 - 2\hat{\mu} - \sigma^2}{2\gamma\sigma^2} \right] + \frac{1}{2} \left( \frac{1}{2\alpha} \left( 1 - \alpha^2 \right) \frac{4\hat{\mu}^2}{4\gamma\sigma^4} \right)
= \hat{\mu}^2 \left[ 1 - \alpha^2 - 2\alpha + 2\alpha \right] \frac{1}{2\alpha\gamma\sigma^2}
= \hat{\mu}^2 \left[ 1 - \alpha^2 - 2\alpha + 2\alpha \right] \frac{1}{2\alpha\gamma\sigma^2}
= \hat{\mu}^2 \left[ 1 - \alpha^2 \right] \frac{1}{2\alpha\gamma\sigma^2}.
$$

(14)

The expression in (14) is of the form $K = \theta \left( \frac{\hat{\mu}}{\sigma} \right)^2$, where $\theta = \frac{(1-\alpha)^2}{2\alpha\gamma}$, as was to be shown.

**Corollary 2.** If the costs of learning about extreme events are above a threshold, the investor will prefer to over-invest during extreme periods.

**Proof.** From the previous proposition, it follows that if costs are above K, the investor will be better off by over-investing.

**Proposition 4.** The expected net benefit of leverage ($d > 1$) in period 1 is bounded, for an investor who knows that the environment features regime-switching in extreme events. This effect depends on the values of $\hat{\mu}$ and $\sigma^2$.

**Proof.** We need to show that the difference in the investor’s objective function from leveraged investment, $P^L$, minus that from prudent investment, $P^P$, is bounded. That is, we must show that $|P^L - P^P| < \infty$. It is enough to show that the difference comprises a sum or product of bounded finite terms. In order to do this, we calculate the investor’s expected payoff from choosing prudent and leveraged investment levels. We
denote $EU$ as the expected utility, from the objective function in equation (3). We assume that period 1 is a typical regime with variance $\sigma^2$. In period 1 the investor decides whether to borrow and invest $d^L$, or else invest the prudent amount $d^P = 1$. In period 2 the investor will choose optimally for that period: either $d^E$ if it is extreme, or else $d^P$ for typical economic climates. First we compute the payoff $P^P$ as follows:

$$ P^P = EU(d^P|\sigma^2) + \beta \left[ \alpha EU(d^E|\sigma^2) - (1 - \alpha)EU(d^P|\sigma^2) \right]. $$

(15)

Then we compute the payoff from leverage, $P^L$, as follows. In this case, the investor has to repay borrowing $r_b(d^L - 1)W_0$ in the second period, where we normalize $W_0 = 1$ to obtain $r_b(d^L - 1)$. Hence the payoff is

$$ P^L = EU(d^L|\sigma^2) + \beta \left[ 2\alpha \left( EU(d^E|\sigma^2) - EU(r_b(d^L - 1)|\sigma^2) \right) + \beta \left( 1 - 2\alpha \right) \left( EU(d^P|\sigma^2) - EU(r_b(d^L - 1)|\sigma^2) \right) \right]. $$

(16)

Now to calculate the expected net costs for leverage we compute $P^L - P^P$ from (15) and (16):

$$ P^L - P^P = EU(d^L|\sigma^2) - EU(d^P|\sigma^2)[1 + \beta(1 - \alpha) - \beta(1 - 2\alpha)] + \beta \left[ 2\alpha EU(d^E|\sigma^2) - \alpha EU(d^P|\sigma^2) \right] + \beta \left[ -2\alpha EU(r_b(d^L - 1)|\sigma^2) - (1 - 2\alpha)EU(r_b(d^L - 1)|\sigma^2) \right] = EU(d^L|\sigma^2) - (1 + \alpha\beta)EU(d^P|\sigma^2) + \alpha\beta EU(d^E|\sigma^2) - \beta \left[ 2\alpha EU(r_b(d^L - 1)|\sigma^2) + (1 - 2\alpha)EU(r_b(d^L - 1)|\sigma^2) \right]. $$

Now substitute in the definition of $EU(d|\sigma^2) = d\hat{\mu} + \frac{1}{2}d(1-d)\sigma^2 + \frac{1}{2}(1-\gamma)d^2\sigma^2$ from (3), to obtain

$$ P^L - P^P = d^L\hat{\mu} + \frac{1}{2}d^L(1-d^L)\sigma^2 + \frac{1}{2}(1-\gamma)(d^L)^2\sigma^2 - (1 + \alpha\beta) \left[ \hat{\mu} + \frac{1}{2}(1-\gamma)\sigma^2 \right] + \alpha\beta \left[ d^E\hat{\mu} + \frac{1}{2}d^E(1-d^E)\sigma^2 + \frac{1}{2}(1-\gamma)(d^E)^2\sigma^2 \right] - \beta \left[ r_b(d^L - 1)\hat{\mu} + (3 - 2\alpha)\frac{\sigma^2}{2}r_b(d^L - 1)(1 - \gamma r_b(d^L - 1)) \right]. $$

(18)

33 In the second period agents cannot borrow to invest $d^L$, since it is the end of economic activity. Therefore if it is a typical period, they just invest as much as they can, $d^P = 1$. 

29
Before proceeding further, note that the third line of (18) can be expressed as \( \beta \left[ \frac{(2\alpha \hat{\mu} + \sigma^2)^2}{2\gamma \sigma^2} \right] \), which follows from the definition of \( d^E = \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \) in equation (8). To see this, write the third line \( T \) (for 'third') as

\[
T = \alpha \beta \left[ d^E \hat{\mu} + \frac{1}{2} d^E (1 - d^E) \frac{\sigma^2}{\alpha} + \frac{1}{2} (1 - \gamma)(d^E)^2 \frac{\sigma^2}{\alpha} \right] \\
= \alpha \beta \left[ \hat{\mu} \left( \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right) + \frac{1}{2} \left( \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right) \left( \frac{2\gamma \sigma^2 - (2\alpha \hat{\mu} + \sigma^2)}{2\gamma \sigma^2} \right) \frac{\sigma^2}{\alpha} \right] \\
+ \alpha \beta \left[ \frac{1}{2} (1 - \gamma) \left( \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right)^2 \frac{\sigma^2}{\alpha} \right] \\
= \alpha \beta \hat{\mu} \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] + \beta \sigma^2 \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] \left( \frac{2\gamma \sigma^2 - (2\alpha \hat{\mu} + \sigma^2)}{2\gamma \sigma^2} \right) + \frac{\beta \sigma^2}{2} (1 - \gamma) \left( \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right)^2 \\
= \alpha \beta \hat{\mu} \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] + \beta \sigma^2 \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] \left( \frac{2\gamma \sigma^2 - (2\alpha \hat{\mu} + \sigma^2)}{2\gamma \sigma^2} \right) + (1 - \gamma) \left( \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right) \left( \gamma \sigma^2 - \frac{2\alpha \hat{\mu} \gamma}{\gamma \sigma^2} \right) \\
= \alpha \beta \hat{\mu} \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] + \beta \sigma^2 \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] \left( \frac{\gamma \sigma^2 - 2\alpha \hat{\mu} \gamma}{\gamma \sigma^2} \right) \\
= \alpha \beta \hat{\mu} \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] + \beta \sigma^2 \left[ \frac{2\alpha \hat{\mu} + \sigma^2}{2\gamma \sigma^2} \right] \left( \gamma \sigma^2 - \frac{2\alpha \hat{\mu} \gamma}{\gamma \sigma^2} \right)
\]

\[34 \text{To see this, note that application of (3) to the bracketed term on the last line } L \text{ (for 'last') of (17) yields}
\]

\[
L = 2\alpha EU \left( r_b(d^L - 1) \frac{\sigma^2}{\alpha} \right) + (1 - 2\alpha) EU \left( r_b(d^L - 1) \sigma^2 \right) \\
= 2\alpha \left[ r_b(d^L - 1) \hat{\mu} + \frac{1}{2} r_b(d^L - 1)(1 - r_b(d^L - 1)) \frac{\sigma^2}{\alpha} + \frac{1}{2} (1 - \gamma)(r_b(d^L - 1))^2 \frac{\sigma^2}{\alpha} \right] \\
+ (1 - 2\alpha) \left[ r_b(d^L - 1) \hat{\mu} + \frac{1}{2} r_b(d^L - 1)(1 - r_b(d^L - 1)) \sigma^2 + \frac{1}{2} (1 - \gamma)(r_b(d^L - 1))^2 \sigma^2 \right] \\
= r_b(d^L - 1) \hat{\mu} + (3 - 2\alpha) \left[ \frac{\sigma^2}{2} r_b(d^L - 1)(1 - r_b(d^L - 1)) \right] \\
+ (3 - 2\alpha) \left[ \frac{\sigma^2}{2} (1 - \gamma)(r_b(d^L - 1))^2 \right] \\
= r_b(d^L - 1) \hat{\mu} + (3 - 2\alpha) \frac{\sigma^2}{2} r_b(d^L - 1)(1 - r_b(d^L - 1) + (1 - \gamma)r_b(d^L - 1)) \\
= r_b(d^L - 1) \hat{\mu} + (3 - 2\alpha) \frac{\sigma^2}{2} r_b(d^L - 1)(1 - \gamma r_b(d^L - 1)).
\]
Thus, expression (18) may be written as

\[
\begin{align*}
P^L - P^P & = d^L \hat{\mu} + \frac{1}{2} d^L(1 - d^L)\sigma^2 + \frac{1}{2}(1 - \gamma)(d^L)^2 \sigma^2 \\
& \quad - (1 + \alpha\beta) \left[ \hat{\mu} + \frac{1}{2}(1 - \gamma)\sigma^2 \right] + \beta \left[ \frac{(2\alpha\hat{\mu} + \sigma^2)^2}{8\gamma\sigma^2} \right] \\
& \quad - \beta \left[ r_b(d^L - 1)\hat{\mu} + (3 - 2\alpha)\frac{\sigma^2}{2} r_b(d^L - 1)(1 - \gamma r_b(d^L - 1)) \right] \\
& = d^L \hat{\mu} + \frac{1}{2} d^L \sigma^2 - \frac{1}{2}(d^L)^2 \sigma^2 + \frac{1}{2}(d^L)^2 \sigma^2 - \frac{1}{2}\gamma(d^L)^2 \sigma^2 \\
& \quad - (1 + \alpha\beta) \left[ \hat{\mu} + \frac{1}{2}(1 - \gamma)\sigma^2 \right] + \beta \left[ \frac{(2\alpha\hat{\mu} + \sigma^2)^2}{8\gamma\sigma^2} \right] \\
& \quad - \beta \left[ d^L r_b\hat{\mu} - r_b\hat{\mu} + (3 - 2\alpha)\frac{\sigma^2}{2} r_b(d^L - 1) \right] \\
& \quad - \beta \left[ (3 - 2\alpha)\frac{\sigma^2}{2} r_b(d^L - 1)(-\gamma r_b(d^L - 1)) \right] \\
& = d^L \hat{\mu} + \frac{1}{2} d^L \sigma^2 - \frac{1}{2}\gamma(d^L)^2 \sigma^2 \\
& \quad - (1 + \alpha\beta) \left[ \hat{\mu} + \frac{1}{2}(1 - \gamma)\sigma^2 \right] + \beta \left[ \frac{(2\alpha\hat{\mu} + \sigma^2)^2}{8\gamma\sigma^2} \right] \\
& \quad - \beta \left[ d^L r_b\hat{\mu} - r_b\hat{\mu} + (3 - 2\alpha)\frac{\sigma^2}{2} r_b d^L - (3 - 2\alpha)\frac{\sigma^2}{2} r_b \right] \\
& \quad - \beta \left[ -(3 - 2\alpha)\frac{\sigma^2}{2} \gamma r_b^2(d^L)^2 - (3 - 2\alpha)\frac{\sigma^2}{2} r_b d^L \right]
\end{align*}
\]
\[-\beta \left( (3 - 2\alpha)\frac{\sigma^2}{2} \gamma r_b^2 d^L + (3 - 2\alpha)\frac{\sigma^2}{2} r_b \right) \]

\[= d^L \hat{\mu} + \frac{1}{2} d^L \sigma^2 - \frac{1}{2}\gamma (d^L)^2 \sigma^2 \]

\[-(1 + \alpha \beta) \left( \hat{\mu} + \frac{1}{2} (1 - \gamma) \sigma^2 \right) + \beta \left[ \frac{4\alpha^2 \hat{\mu}^2 + 4\alpha \hat{\mu} \sigma^2 + \sigma^4}{8\gamma \sigma^2} \right] \]

\[-\beta \left[ d^L r_b \hat{\mu} - r_b \hat{\mu} - (3 - 2\alpha)\frac{\sigma^2}{2} \gamma r_b^2 (d^L)^2 + (3 - 2\alpha)\frac{\sigma^2}{2} \gamma r_b^2 d^L \right]. \]

Now we can collect terms in \(d^L\), \((d^L)^2\), \(\hat{\mu}\) and \(\sigma^2\) to obtain

\[P^L - P^P = d^L \left[ \hat{\mu} + \frac{\sigma^2}{2} - \beta r_b \hat{\mu} - \beta (3 - 2\alpha)\frac{\sigma^2}{2} \gamma r_b^2 \right] \]

\[+ (d^L)^2 \left[ -\frac{1}{2}\gamma \sigma^2 + \beta (3 - 2\alpha)\frac{\sigma^2}{2} \gamma r_b^2 \right] \]

\[+ \hat{\mu} \left[ -(1 + \alpha \beta) + \beta \left( \frac{4\alpha^2 \hat{\mu}^2 + 4\alpha \hat{\mu} \sigma^2}{8\gamma \sigma^2} \right) + \beta r_b \right] \]

\[+ \frac{\sigma^2}{2} \left[ -(1 + \alpha \beta)(1 - \gamma) + \frac{\beta}{4\gamma} \right] \]

\[= d^L \left[ \hat{\mu} (1 - \beta r_b) + \frac{\sigma^2}{2} (1 - \beta (3 - 2\alpha) \gamma r_b^2) \right] \]

\[+ (d^L)^2 \left[ \frac{\gamma \sigma^2}{2} (-1 + \beta (3 - 2\alpha) r_b^2) \right] \]

\[+ \hat{\mu} \left[ \beta \left( \frac{4\alpha^2 \hat{\mu} + 4\alpha \sigma^2}{8\gamma \sigma^2} \right) + \frac{8\gamma \sigma^2 \beta r_b - 8\gamma \sigma^2 - 8\alpha \beta \gamma \sigma^2}{8\gamma \sigma^2} \right] \]

\[+ \frac{\sigma^2}{2} \left[ -(1 - \gamma + \alpha \beta - \alpha \beta g) + \frac{\beta}{4\gamma} \right] \]

\[= d^L \left[ \hat{\mu} (1 - \beta r_b) + \frac{\sigma^2}{2} (1 - \beta (3 - 2\alpha) \gamma r_b^2) \right] \]

\[- (d^L)^2 \left[ \frac{\gamma \sigma^2}{2} (1 - \beta (3 - 2\alpha) r_b^2) \right] \]

\[+ \hat{\mu} \left[ \frac{4\alpha^2 \beta \hat{\mu} + 4\alpha \beta \sigma^2 + 8\gamma \sigma^2 \beta r_b - 8\gamma \sigma^2 - 8\alpha \beta \gamma \sigma^2}{8\gamma \sigma^2} \right] \]

\[+ \frac{\sigma^2}{2} \left[ \beta - 4\gamma (1 - \gamma + \alpha \beta (1 - \gamma)) \right] \]

\[= d^L \left[ \hat{\mu} (1 - \beta r_b) + \frac{\gamma \sigma^2}{2} (1 - \beta (3 - 2\alpha) r_b^2) \right] \]

\[- (d^L)^2 \left[ \frac{\gamma \sigma^2}{2} (1 - \beta (3 - 2\alpha) r_b^2) \right] \]
\[ + \hat{\mu} \left[ \frac{\alpha^2 \beta \hat{\mu} + \alpha \beta \sigma^2 + 2 \gamma \sigma^2 \beta r_b - 2 \gamma \sigma^2 - 2 \alpha \beta \gamma \sigma^2}{2 \gamma \sigma^2} \right] + \frac{\sigma^2}{2} \left[ \frac{\beta - 4 \gamma (1 - \gamma)(1 + \alpha \beta)}{4 \gamma} \right] = d^L \left[ \hat{\mu} (1 - \beta r_b) + \frac{\gamma \sigma^2}{2} (1 - \beta (3 - 2 \alpha) r_b^2) \right] - (d^L)^2 \left[ \frac{\gamma \sigma^2}{2} (1 - \beta (3 - 2 \alpha) r_b^2) \right] + \hat{\mu} \left[ \frac{\alpha^2 \beta \hat{\mu} + \sigma^2 (\alpha \beta + 2 \gamma \beta r_b - 2 \gamma - 2 \alpha \beta \gamma)}{2 \gamma \sigma^2} \right] + \frac{\sigma^2}{2} \left[ \frac{\beta - 4 \gamma (1 - \gamma)(1 + \alpha \beta)}{4 \gamma} \right]. \]

The expression in (19) comprises sums and products of terms in \( \hat{\mu} \) and \( \sigma^2 \) that are all finite and bounded. Therefore it is a bounded function of \( \hat{\mu} \) and \( \sigma^2 \), as was to be shown. The final expression for \( P^L - P^P \) in (19) is calibrated to US data (using the expression for \( d \) from equation (4)) in Figure 3.

\[ \square \]

**Proposition 5.** The expected net loss from suboptimal investment in period 2 is bounded, for an investor who does not understand that the environment features regime-switching in extreme events.

**Proof.** We have to show that \( \Delta EU \leq K \), for some positive constant \( K \). The investor assumes the world is always typical, and invests \( d^L > 1 \) in period 1 and the maximal \( d^P = 1 \) in period 2. The associated payoff is denoted \( \hat{P}^L \). To calculate the net expected cost we compute the difference between payoff to the optimal strategy \( P^L \) and its suboptimal counterpart \( \hat{P}^L \). That is, we compute \( P^L - \hat{P}^L \). Below we first present the optimal, then suboptimal payoffs.

**Optimal Payoff** \( P^L \). This is the same as above, in equation (16):

\[ P^L = EU(d^L | \sigma^2) + \beta \left[ 2 \alpha \left( EU(d^E | \frac{\sigma^2}{\alpha}) - EU(r_b(d^L - 1) | \frac{\sigma^2}{\alpha}) \right) \right] \]
\[+ \beta \left[ (1 - 2\alpha) \left( EU(d^P|\sigma^2) - EU(r_b(d^L - 1)|\sigma^2) \right) \right].\]

**Suboptimal Payoff** \(\hat{P}^L\). The strategy here involves investing \(d^L\) in period 1. Then in period 2 the investor repays any borrowing, and since she mistakenly believes the world is always in the typical regime, she always demands the most she can, \(d^P = 1\), regardless of whether the realized regime is extreme or typical. To compute the results, we proceed as follows. If she over-invests by choosing \(d^L\) in the first period, the likelihood of extremes raises from \(\alpha\) to \((1 - 2\alpha)\), and her payoff \(\hat{P}^L\) is

\[
\hat{P}^L = EU(d^L|\sigma^2) + \beta \left[ 2\alpha \left( EU(d^P|\sigma^2) - EU(r_b(d^L - 1)|\sigma^2) \right) \right] + \beta \left[ (1 - 2\alpha) \left( EU(d^P|\sigma^2) - EU(r_b(d^L - 1)|\sigma^2) \right) \right].
\]

**Utility Differential** \(P^L - \hat{P}^L\). Using equations (16) and (20), we obtain

\[
P^L - \hat{P}^L = 2\alpha\beta \left[ \left( EU(d^E|\sigma^2) - EU(d^P|\sigma^2) \right) \right],
\]

which from equations (3) and (6) yields

\[
P^L - \hat{P}^L = 2\alpha\beta \left[ d^E\hat{\mu} + \frac{1}{2}d^E(1 - d^E)\frac{\sigma^2}{\alpha} + \frac{1}{2}(1 - \gamma)(d^E)^2\frac{\sigma^2}{\alpha} \right] - 2\alpha\beta \left[ \hat{\mu} + \frac{1}{2}(1 - \gamma)\frac{\sigma^2}{\alpha} \right].
\]

We now factor this expression into terms involving \(\hat{\mu}\) and \(\sigma^2\), to obtain

\[
P^L - \hat{P}^L = \beta\sigma^2 [d^E(1 - d^E) + (1 - \gamma)(d^E)^2 - (1 - \gamma)] - 2\alpha\beta\hat{\mu} [1 - d^E]
\]

\[
= \beta\sigma^2 [d^E - (d^E)^2 + (d^E)^2 - (1 - \gamma)] - 2\alpha\beta\hat{\mu} [1 - d^E]
\]

\[
= \beta\sigma^2 [d^E - 1 - \gamma(d^E)^2] - 2\alpha\beta\hat{\mu} [1 - d^E]
\]

\[
= \beta\sigma^2 [d^E - 1 + \gamma(1 - (d^E)^2)] - 2\alpha\beta\hat{\mu} [1 - d^E]
\]

\[
= \beta\gamma\sigma^2 [1 - (d^E)^2] - \beta(1 - d^E)[2\alpha\hat{\mu} + \sigma^2].
\]

Now, substituting \(d^E = \frac{2\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\) from (6) or (8) yields

\[
P^L - \hat{P}^L = \beta\gamma\sigma^2 \left[ \frac{4\gamma^2\sigma^4 - 4\alpha^2\hat{\mu}^2 - 4\alpha\hat{\mu}\sigma^2 - \sigma^4}{4\gamma^2\sigma^4} \right]
\]

\[34\]
\[-\beta \left[ \frac{2\gamma \sigma^2 - 2\alpha \hat{\mu} - \sigma^2}{2\gamma \sigma^2} \right] (2\alpha \hat{\mu} + \sigma^2) \]

\[= \beta \left[ \frac{4\gamma^2 \sigma^4 - 4\alpha^2 \hat{\mu}^2 - 4\alpha \hat{\mu} \sigma^2 - \sigma^4}{4\gamma \sigma^4} \right] \]

\[-\beta \left[ \frac{4\alpha \gamma \sigma^2 \hat{\mu} - 4\alpha^2 \hat{\mu}^2 - 2\alpha \hat{\mu} \sigma^2 + 2\gamma \sigma^4 - 2\alpha \hat{\mu} \sigma^2 - \sigma^4}{2\gamma \sigma^2} \right] \]

\[= \frac{\beta}{4\gamma \sigma^2} \left[ \frac{4\gamma^2 \sigma^4 - 4\alpha^2 \hat{\mu}^2 - 4\alpha \hat{\mu} \sigma^2 - \sigma^4 - 8\alpha \gamma \sigma^2 \hat{\mu} + 8\alpha^2 \hat{\mu}^2 + 4\alpha \hat{\mu} \sigma^2 - 4\gamma \sigma^4 + 4\alpha \hat{\mu} \sigma^2 + 2\sigma^4}{4\gamma \sigma^2} \right] \]

\[= \frac{\beta}{4\gamma \sigma^2} \left[ \frac{4\gamma^2 \sigma^4 + 4\alpha^2 \hat{\mu}^2 + \sigma^4 - 8\alpha \gamma \sigma^2 \hat{\mu} - 4\gamma \sigma^4 + 4\alpha \hat{\mu} \sigma^2}{4\gamma \sigma^2} \right] \]

\[= \frac{\beta}{4\gamma \sigma^2} \left[ 4\alpha \hat{\mu} (\alpha \hat{\mu} + \sigma^2 - 2\gamma \sigma^2) + \sigma^4 (4\gamma^2 + 1 - 4\gamma) \right] \]

\[= \frac{\beta}{4\gamma \sigma^2} \left[ 4\alpha \hat{\mu} (\alpha \hat{\mu} - \sigma^2 (2\gamma - 1)) + \sigma^4 (4\gamma (\gamma - 1) + 1) \right] \] (21)

The expression in (21) represents the net utility gain from following the optimal versus the suboptimal strategy in the case of leverage, or equivalently, the net loss from the suboptimal strategy. Upon inspection this quantity can be confirmed as bounded.