Unit Roots and Cointegration

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UNIT ROOTS AND COINTEGRATION:
AN INTRODUCTION SEMINAR

References:


UNIT ROOTS AND COINTEGRATION: AN INTRODUCTION

I. STOCHASTIC PROCESS

\( X_t \) - Random Variable

\( f(X_t) \) - Density Function

\( t = 0, 1, 2, 3, ..., T \)

Mean: \( \mu_t = E(X_t) \)

Variance: \( \sigma_t^2 = Var(X_t) \)

Autocovariance: \( \gamma_{i,j} = Cov(X_i, X_j) \quad i \neq j \)

A. PROBLEMS:

1. TOO MANY PARAMETERS TO BE ESTIMATED.

2. ONLY A SINGLE REALIZATION IN ANY GIVEN TIME PERIOD.

3. REDUCING THE NUMBER OF PARAMETERS WOULD REQUIRE EITHER OF TWO RESTRICTIONS:

   A) STATIONARITY: RESTRICTIONS ON THE TIME HETEROGENEITY

   B) ASYMPTOTIC INDEPENDENCE: RESTRICTIONS ON THE MEMORY OF THE PROCESS
B. STRICT STATIONARITY:

\[ f(X_{t_0}, X_{t_1}, \ldots, X_{t_T}) = f(X_{t_0+\tau}, X_{t_1+\tau}, \ldots, X_{t_T+\tau}) \]

1. PARAMETERS THAT CHARACTERIZE THE DISTRIBUTION DEPEND ONLY UPON THE LAG, \( \tau \).

2. DIFFICULT TO VERIFY IN PRACTICE BECAUSE IT IS DEFINED IN TERMS OF THE DISTRIBUTION FUNCTION.

C. WEAK STATIONARITY:

\[
\begin{align*}
E(X_t) &= \mu \quad \text{(a constant)} \\
\text{Var}(X_t) &= \sigma^2 \quad \text{(a constant)} \\
\text{Cov}(X_i, X_j) &= \text{Cov}(X_t, X_{t-\tau}) = \gamma_\tau \\
i - j &= \tau
\end{align*}
\]

1. MOST COMMONLY USED FORM OF STATIONARITY.

2. FOR A GAUSSIAN OR NORMAL PROCESS, WEAK STATIONARITY IMPLIES STRICT STATIONARITY, SINCE THE MULTIVARIATE NORMAL DISTRIBUTION IS COMPLETELY CHARACTERIZED BY ITS FIRST AND SECOND MOMENTS.
3. NUMBER OF PARAMETERS BEFORE STATIONARITY:

$$\frac{3T + T^2}{2}$$

4. NUMBER OF PARAMETERS AFTER STATIONARITY:

$$T + 1$$

D. ARMA( p , q ):

$$\phi(B)X_t = \theta(B)\varepsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$$

FOR STATIONARITY:

CHARACTERISTIC ROOTS OF

$$\phi(B) = 0$$ MUST BE OUTSIDE UNIT CIRCLE.

FOR INVERTIBILITY:

CHARACTERISTIC ROOTS OF

$$\theta(B) = 0$$ MUST BE OUTSIDE UNIT CIRCLE.
1. \textbf{ARMA}(1,0):

\[ X_t = \alpha + \phi X_{t-1} + \epsilon_t \]

Solution:

\[ X_t = \alpha \sum_{i=0}^{t-1} \phi^i + \sum_{i=0}^{t-1} \phi^i \epsilon_{t-i} + X_0 \phi^t \]

\[ E(X_t) = \alpha \sum_{i=0}^{t-1} \phi^i + X_0 \phi^t \]

\[ \text{Var}(X_t) = \sigma^2 \sum_{i=0}^{t-1} \phi^{2i} \]

\[ \text{Cov}(X_t, X_{t-\tau}) = \sigma^2 \phi^\tau \left( \sum_{i=0}^{t-1-\tau} \phi^{2i} \right) \]

As \( t \to \infty \) and \( |\phi| < 1 \) and \( X_0 = 0 \):

\[ X_t = \frac{\alpha}{1-\phi} + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \]

\[ E(X_t) = \frac{\alpha}{1-\phi} = \mu \]

\[ \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2} = \sigma^2 \]

\[ \text{Cov}(X_t, X_{t-\tau}) = \frac{\sigma^2 \phi^\tau}{1-\phi^2} = \gamma_\tau \]
\[ X_t = 0.8X_{t-1} + \varepsilon_t \]
\[ \varepsilon_t \sim N(0, 9) \]
E. SUMMARY:

1. IN ORDER TO HAVE STATIONARITY:

A) THE TIME SERIES MUST HAVE STARTED INFINITELY FAR IN THE PAST OR THE PROCESS MUST ALWAYS BE IN EQUILIBRIUM.

B) THE CHARACTERISTIC ROOTS MUST ALWAYS BE LESS THAN UNITY IN ABSOLUTE VALUE.

II. NON-STATIONARITY:

A. TWO MODELS, NO CONSTANT:

\[ X_t = \phi X_{t-1} + u_t \quad \text{- AR(1) Model} \]
\[ Y_t = Y_{t-1} + v_t \quad \text{- Random Walk Model} \]
\[ | \phi | < 1 \]
\[ u_t, v_t \sim \text{i.i.d. } N(0,1) \]

1. SOLUTION:

\[ X_t = \sum_{i=0}^{t-1} \phi^i u_{t-i} + X_0 \phi^t \]
\[ Y_t = \sum_{i=0}^{t-1} v_{t-i} + Y_0 \]
IF BOTH $X_0$ AND $Y_0$ ARE EQUAL TO ZERO, THEN:

\begin{align*}
E(X_t) &= E(Y_t) = 0 \\
Var(X_t) &= \frac{1}{1-\phi} \\
Var(Y_t) &= t \\
\gamma^X_{\tau} &= \frac{\phi^\tau}{1-\phi^2} \\
\gamma^Y_{\tau} &= t - \tau
\end{align*}

B. TWO MODELS, WITH A CONSTANT:

\begin{align*}
X_t &= \alpha + \phi X_{t-1} + u_t & \text{- AR(1) Model} \\
Y_t &= \alpha + Y_{t-1} + v_t & \text{- Random Walk with Drift Model} \\
|\phi| &< 1 \\
u_t, v_t &\sim \mathcal{N}(0, 1)
\end{align*}
1. SOLUTION:

\[ X_t = \alpha \sum_{i=0}^{t-1} \phi^i + \sum_{i=0}^{t-1} \phi^i \epsilon_{t-i} + X_0 \phi^t \]

\[ Y_t = \alpha t + \sum_{i=0}^{t-1} \epsilon_{t-i} + Y_0 \]

\[ E(X_t) = \frac{\alpha}{1-\phi} \]

\[ E(Y_t) = \alpha t \]

\[ \alpha = .5, \phi = .9 \]
A) AUTOCORRELATION FUNCTION:

\[ \rho_t^X = \phi^\tau \]

\[ \lim_{T \to \infty} \rho_t^X = 0 \]

\[ \rho_t^Y = \frac{t - \tau}{\bar{t}} \]
C. DIFFERENCING:

\[ Y_t = Y_{t-1} + \nu_t \]
\[ \Delta Y_t = Y_t - Y_{t-1} = (1 - B)Y_t = \nu_t \]
\[ Y_t \sim I(1) \]
\[ X_t \sim I(0) \]

\( I(d) \) - "d" DIFFERENCE STATIONARY PROCESS
D. TREND STATIONARY PROCESS

\[ X_t = \alpha + \lambda t + \varepsilon_t \]

\[ E(X_t) = \alpha + \lambda t \]

\[ Var(X_t) = \sigma^2 \]

E. SPURIOUS REGRESSION

\[ Y_t = Y_{t-1} + u_t \quad u_t \sim iiN(0, \sigma^2_u) \]

\[ X_t = X_{t-1} + v_t \quad v_t \sim iiN(0, \sigma^2_v) \]

\[ Y_t = \hat{\beta}_0 + \hat{\beta}_1 X_t + \hat{\varepsilon}_t \]

Analysis of Variance

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
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<tr>
<td>Model</td>
<td>1</td>
<td>794.61482</td>
<td>794.61482</td>
<td>168.27</td>
<td>&lt;.0001</td>
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<tr>
<td>Error</td>
<td>148</td>
<td>698.90458</td>
<td>4.72233</td>
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<tr>
<td>Corrected Total</td>
<td>149</td>
<td>1493.51939</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Root MSE 2.17309, R-Square 0.5320, Adj R-Sq 0.5289, Coeff Var -265.22599

Parameter Estimates

| Variable | DF | Parameter Estimate | Standard Error | t Value | Pr > |t| |
|----------|----|--------------------|----------------|---------|-------|
| Intercept| 1  | 2.85388            | 0.33417        | 8.54    | <.0001|
| X        | 1  | -0.38843           | 0.0299        | -12.97  | <.0001|
WITHOUT DRIFT:

\[ p \lim_{T \to \infty} \hat{\beta}_1 = \text{A Random Variable} \]

WITH DRIFT:

\[ p \lim_{T \to \infty} \hat{\beta}_1 = \text{Ratio of Drift Terms} \]

\[ \hat{\beta}_0 \text{ has a divergent distribution in both cases} \]

<table>
<thead>
<tr>
<th>( X_t / Y_t )</th>
<th>Deterministic</th>
<th>Non-Stationary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic</td>
<td>Regression Valid</td>
<td>Spurious Regression</td>
</tr>
<tr>
<td>Non-Stationary</td>
<td>Spurious Regression</td>
<td>Spurious Regression, unless variables are cointegrated.</td>
</tr>
</tbody>
</table>
F. DETERMINISTIC AND STOCHASTIC TRENDS

1. INTEGRATED VARIABLES EXHIBIT A SYSTEMATIC, BUT UNPREDICTABLE VARIATION KNOWN AS A STOCHASTIC TREND.

2. TRENDS THAT ARE PREDICTABLE ARE KNOWN AS DETERMINISTIC.

ARIMA (0, 1, 0)

\[
\Delta Y_t = \alpha + \varepsilon_t \\
Y_0 = \varepsilon_0 = 0 \\
Y_t = \alpha t + \sum_{i=0}^{t-1} \varepsilon_{t-i} \\
DT = \alpha t \\
ST = \sum_{i=0}^{t-1} \varepsilon_{t-i}
\]
III. UNIT ROOTS

A. IMPORTANT DIFFERENCES BETWEEN STATIONARY AND NONSTATIONARY SERIES.

1. STATIONARY SERIES:

A) SHOCKS TO A STATIONARY SERIES ARE TEMPORARY; OVER TIME, THE EFFECTS OF THE SHOCKS WILL DISSIPATE AND THE SERIES WILL REVERT TO ITS LONG-RUN MEAN LEVEL (MEAN REVERSION).

B) A STATIONARY SERIES HAS A FINITE VARIANCE THAT IS TIME-INVARIANT.

C) A STATIONARY SERIES HAS A THEORETICAL CORRELOGRAM THAT DIMINISHES AS LAG LENGTH INCREASES.
D) LONG-TERM FORECASTS OF A STATIONARY SERIES WILL CONVERGE TO THE UNCONDITIONAL MEAN OF THE SERIES.

\[ X_{T+k} = \alpha + \phi X_{T+k-1} + \varepsilon_{T+k} \]

\[ f_{T+k}^X = \alpha \sum_{i=0}^{k-1} \phi^i + \phi^k X_T \]

\[ \lim_{k \to \infty} f_{T+k}^X = \frac{\alpha}{1-\phi} = E(X_t) \]

2. NONSTATIONARY SERIES:

A) A NONSTATIONARY SERIES HAS NO LONG-RUN MEAN TO WHICH THE SERIES RETURNS.

B) A NONSTATIONARY SERIES HAS A VARIANCE THAT IS TIME DEPENDENT AND GOES TO INFINITY AS TIME APPROACHES INFINITY.
C) A NONSTATIONARY SERIES HAS THEORETICAL AUTOCORRELATIONS THAT DO NOT DECAY AND IN FINITE SAMPLES, DIES OUT SLOWLY.

B. IT IS IMPRECISE TO DETECT A UNIT ROOT PROCESS BY USING THE SAMPLE CORRELOGRAM BECAUSE IT IS NEXT TO IMPOSSIBLE TO DISTINGUISH BETWEEN A UNIT ROOT AND A NEAR UNIT ROOT.

C. HYPOTHESIS TESTING

\[ X_t = \phi X_{t-1} + \varepsilon_t \]

\[ H_0 : \phi = 0 \]

Since \(|\phi| < 1\), \(X_t\) is stationary and the estimate of \(\phi\) is consistent. Standard "t" test is appropriate.

\[ H_0 : \phi = 1 \]

Since the \(\lim_{T \to \infty} \text{Var}(X_t) = \infty\), the estimate of \(\phi\) is biased and inconsistent. Standard "t" test is inappropriate.
Monte Carlo Simulation

Distribution of $\hat{\phi}$ Under $H_0: \phi = .8$
Monte Carlo Simulation

Distribution of $\hat{\phi}$ Under $H_0 : \phi = 1$
D. DICKEY-FULLER TESTS

\[ X_t = \phi X_{t-1} + \varepsilon_t \]

Subtract \( X_{t-1} \) from Each Side:

\[ \Delta X_t = \gamma X_{t-1} + \varepsilon_t \quad \gamma = \phi - 1 \]

\( H_0 : \gamma = 0 \rightarrow H_0 : \phi = 1 \)

<table>
<thead>
<tr>
<th>Model</th>
<th>Hypothesis</th>
<th>Critical Values (.05 and .01)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta X_t = \gamma X_{t-1} + \varepsilon_t )</td>
<td>( \gamma = 0 )</td>
<td>-1.95, -2.60</td>
</tr>
<tr>
<td>( \Delta X_t = \alpha + \gamma X_{t-1} + \varepsilon_t )</td>
<td>( \gamma = 0 )</td>
<td>-2.89, -3.51</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0 ) given ( \gamma = 0 )</td>
<td>2.54, 3.22</td>
</tr>
<tr>
<td></td>
<td>( \alpha = \gamma = 0 )</td>
<td>4.71, 6.70</td>
</tr>
<tr>
<td>( \Delta X_t = \alpha + \gamma X_{t-1} + \theta t + \varepsilon_t )</td>
<td>( \gamma = 0 )</td>
<td>-3.45, -4.04</td>
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<td></td>
<td>( \alpha = 0 ) given ( \gamma = 0 )</td>
<td>3.11, 3.78</td>
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<tr>
<td></td>
<td>( \theta = 0 ) given ( \gamma = 0 )</td>
<td>2.79, 3.53</td>
</tr>
<tr>
<td></td>
<td>( \gamma = \theta = 0 )</td>
<td>6.49, 8.73</td>
</tr>
<tr>
<td></td>
<td>( \alpha = \gamma = \theta = 0 )</td>
<td>4.88, 6.50</td>
</tr>
</tbody>
</table>
1. AUGMENTED DICKEY-FULLER TEST

$$A(B)X_t = C(B)\varepsilon_t$$

$$D(B)X_t = \varepsilon_t \quad \text{where} \quad D(B) = \frac{A(B)}{C(B)}$$

$$\Delta X_t = \alpha + \gamma X_{t-1} + \sum_{i=1}^{n} \beta_i \Delta X_{t-i+1} + \varepsilon_i$$

$$ARIMA( p, 1, q ) \approx ARIMA( n, 1, 0 )$$

$$n \geq p$$

2. MULTIPLE ROOTS

$$\Delta^2 X_t = \alpha + \gamma \Delta X_{t-1} + \varepsilon_i$$

USE THE APPROPRIATE STATISTICS (DEPENDING UPON THE DETERMINISTIC ELEMENTS ACTUALLY INCLUDED IN THE REGRESSION) TO DETERMINE WHETHER $$\gamma = 0$$. IF YOU CANNOT REJECT THE NULL HYPOTHESIS THAT $$\gamma = 0$$, CONCLUDE THAT THE $$X_t$$ PROCESS IS $$I(2)$$. 
3. EXAMPLE:

PURCHASING-POWER PARITY (PPP):

THE RATE OF CURRENCY DEPRECIATION IS EQUAL TO THE DIFFERENCE BETWEEN THE DOMESTIC AND FOREIGN INFLATION RATES.

\[ E_t = P_t - F_t + \varepsilon_t \]

- \( E_t \) - Log of Dollar Price of Foreign Exchange
- \( P_t \) - Log of U.S. Price Level
- \( F_t \) - Log of Foreign Price Level
- \( \varepsilon_t \) - Deviation

TEST OF PPP: \( \varepsilon_t \sim I(0) \)

\[ \varepsilon_t = E_t + F_t - P_t \]

\[ \Delta \varepsilon_t = \alpha + \gamma \varepsilon_{t-1} + \sum_{i=1}^{n} \beta_i \Delta \varepsilon_{t-i+1} + \omega_t \]

Accept \( H_0 : \gamma = 0 \) → PPP Theory Rejected
4. PHILLIPS-PERRON TESTS

A GENERALIZATION OF THE DICKEY-FULLER PROCEDURE THAT ALLOWS FOR FAIRLY MILD ASSUMPTIONS CONCERNING THE ERROR DISTRIBUTIONS.

\[ X_t = \alpha^* + \phi^* X_{t-1} + \mu_t \]

\[ a(B)\mu_t = b(B)\varepsilon_t \]

THE PHILLIPS-PERRON TEST STATISTICS ARE MODIFICATIONS OF THE DICKEY-FULLER STATISTICS THAT TAKE INTO ACCOUNT THE LESS RESTRICTIVE NATURE OF THE ERROR PROCESS.

5. PROBLEMS IN TESTING FOR UNIT ROOTS

A) POWER

THE POWER OF A TEST IS THE PROBABILTY OF REJECTING THE NULL HYPOTHESIS WHEN IT IS FALSE.

UNIT ROOT TESTS DO NOT HAVE THE POWER TO DISTINGUISH BETWEEN A UNIT ROOT AND NEAR UNIT ROOT PROCESS.
UNIT Root tests do not have the power to distinguish between trend stationary and drifting processes.

B) Determination of the Deterministic Regressors

The key problem is that the tests for unit roots are conditional on the presence of the deterministic regressors and tests for the presence of the deterministic regressors are conditional on the presence of a unit root.
Estimate \( \Delta X_t = \alpha + \gamma X_{t-1} + \theta t + \sum_{i=1}^{n} \beta_i \Delta X_{t-i+1} + \varepsilon_t \)

**Yes, Test for Presence of Trend**

1. Is \( \gamma = 0 \)?
   - No
   - Yes, Test for the Presence of the Drift

2. Is \( \theta = 0 \) given \( \gamma = 0 \)?
   - No
   - Yes

3. Estimate \( \Delta X_t = \alpha + \gamma X_{t-1} + \sum_{i=1}^{n} \beta_i \Delta X_{t-i+1} + \varepsilon_t \)
   - Is \( \gamma = 0 \)?
     - No
     - Yes

**Yes, Test for the Presence of the Drift**

1. Is \( \alpha = 0 \) given \( \gamma = 0 \)?
   - No
   - Yes

2. Estimate \( \Delta X_t = \gamma X_{t-1} + \sum_{i=1}^{n} \beta_i \Delta X_{t-i+1} + \varepsilon_t \)
   - Is \( \gamma = 0 \)?
     - No
     - Yes

**Conclude** \( X_t \) **has a unit root**

**No Unit Root**
IV. COINTEGRATION

\[ m_t = \beta_0 + \beta_1 p_t + \beta_2 y_t + \beta_3 r_t + \varepsilon_t \]

- \( m_t \) - long-run money demand
- \( p_t \) - price level
- \( y_t \) - real income
- \( r_t \) - interest rate
- \( \varepsilon_t \) - stationary disturbance term

All variables are \( I(1) \).

\[ \varepsilon_t = m_t - \beta_0 - \beta_1 p_t - \beta_2 y_t - \beta_3 r_t \sim I(0) \]

EQUILIBRIUM THEORIES INVOLVING NONSTATIONARY VARIABLES REQUIRE THE EXISTENCE OF A COMBINATION OF THE VARIABLES THAT IS STATIONARY.

COINTEGRATION DOES NOT REQUIRE THAT THE LONG-RUN (EQUILIBRIUM) RELATIONSHIP BE GENERATED BY MARKET FORCES OR THE BEHAVIORAL RULES OF INDIVIDUALS.

IN ENGLE AND GRANGER'S USE OF THE TERM, THE EQUILIBRIUM RELATIONSHIP MAY BE CAUSAL, BEHAVIORAL, OR SIMPLY A REDUCED-FORM RELATIONSHIP AMONG SIMILARLY TRENDING VARIABLES.
A. EXAMPLE

1. UNBIASED FORWARD MARKET HYPOTHESIS

The forward or futures price of an asset should equal the expected value of that asset's spot price in the future.

\[ E(s_{t+1}) = f_t \]

- \( s_{t+1} \) - log of spot price in t+1 period
- \( f_t \) - log of one period price of forward exchange

If agent's expectations are rational, the forecast error for the spot rate in the t+1 period will have a conditional mean equal to zero, so that:

\[ \varepsilon_{t+1} = s_{t+1} - E(s_{t+1}) \quad E(\varepsilon_{t+1}) = 0 \]

Combining the two equations yields:

\[ s_{t+1} = f_t + \varepsilon_{t+1} \]

Since \( s_t, f_t \sim I(1) \), the unbiased forward market hypothesis necessitates that there be a linear combination of nonstationary spot and forward exchange rates that is stationary.
B. DEFINITION

A SET OF VARIABLES ARE IN LONG-RUN EQUILIBRIUM WHEN:

\[ \beta_1 X_{1t} + \beta_2 X_{2t} + \ldots + \beta_n X_{nt} = 0 \]

OR IN VECTOR FORM:

\[ \beta' X_t = 0 \]

THE DEVIATION FROM LONG-RUN EQUILIBRIUM:

\[ \varepsilon_t = \beta' X_t \]

IF THE EQUILIBRIUM IS MEANINGFUL, THE ERROR PROCESS MUST BE STATIONARY

THE COMPONENTS OF THE VECTOR \( X_t \) ARE SAID TO BE COINTEGRATED OF ORDER \( d, b \), DENOTED BY \( X_t \sim CI( d, b ) \), IF

1. ALL COMPONENTS OF \( X_t \) ARE INTEGRATED OF ORDER \( d \).

2. THERE EXISTS A VECTOR \( \beta \) SUCH THAT THE LINEAR COMBINATION \( \beta' X_t \) IS INTEGRATED OF ORDER \( (d - b) \), WHERE \( b > 0 \).

THE VECTOR \( \beta \) IS KNOWN AS THE COINTEGRATING VECTOR
EXAMPLE:

\( m_t \) - long-run money demand
\( p_t \) - price level
\( y_t \) - real income
\( r_t \) - interest rate

All variables are \( I(1) \).

\[ \epsilon_t = m_t - \beta_0 - \beta_1 p_t - \beta_2 y_t - \beta_3 r_t \sim I(0) \]

\[ X_t' = (m_t, 1, p_t, y_t, r_t) \]

\[ \beta' = (1, -\beta_0, -\beta_1, -\beta_2, -\beta_3) \]

\[ X_t \sim CI(\ d, \ b \) \]
IMPORTANT POINTS:

1. **COINTEGRATION** refers to a linear combination of nonstationary variables.

2. COINTEGRATING VECTOR NOT UNIQUE:

   \[
   \beta' = (\beta_1, \beta_2, \ldots, \beta_n)
   \]

   \[
   \lambda \beta' = (\lambda \beta_1, \lambda \beta_2, \ldots, \lambda \beta_n)
   \]

   Normalizing: \( \lambda = \frac{1}{\beta_i} \)

3. ALL VARIABLES MUST BE INTEGRATED OF THE SAME ORDER. VARIABLES INTEGRATED OF DIFFERENT ORDERS CANNOT BE COINTEGRATED.

4. IF \( X_t \) HAS "n" COMPONENTS, THERE MAY BE AS MANY AS "n – 1" LINEARLY INDEPENDENT COINTEGRATING VECTORS. THE NUMBER OF COINTEGRATING VECTORS IS CALLED THE **COINTEGRATING RANK** OF \( X_t \).
### Phillips-Perron Unit Root Test

<table>
<thead>
<tr>
<th>Type</th>
<th>Lags</th>
<th>Rho</th>
<th>Pr &lt; Rho</th>
<th>Tau</th>
<th>Pr &lt; Tau</th>
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<tbody>
<tr>
<td>Zero Mean</td>
<td>2</td>
<td>-208.6771</td>
<td>0.0010</td>
<td>-14.2346</td>
<td>0.0010</td>
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<td>Trend</td>
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<td>0.0010</td>
<td>-14.1897</td>
<td>0.0010</td>
</tr>
</tbody>
</table>
C. COMMON STOCHASTIC TRENDS

\[ Y_t = \mu_t + \varepsilon_{Yt} \sim I(1) \]

\[ X_t = \mu_t + \varepsilon_{Xt} \sim I(1) \]

\[ \mu_t = \mu_{t-1} + \varepsilon_t \]

\[ Y_t - X_t = \mu_t + \varepsilon_{Yt} - \mu_t + \varepsilon_{Xt} = \varepsilon_{Yt} - \varepsilon_{Xt} \sim I(0) \]

\[ Y_t, X_t \sim CI(1,1) \]

\[ Z_t = \mu_{Zt} + \varepsilon_{Zt} \]

\[ X_t = \mu_{Xt} + \varepsilon_{Xt} \]

\[ Y_t = \mu_{Yt} + \varepsilon_{Yt} \]

\[ \mu_{Zt} = \mu_{Xt} + \mu_{Yt} \]

\[ X_t + Y_t - Z_t = \mu_{Xt} + \varepsilon_{Xt} + \mu_{Yt} + \varepsilon_{Yt} - \mu_{Zt} - \varepsilon_{Zt} = \varepsilon_{Xt} + \varepsilon_{Yt} - \varepsilon_{Zt} \sim I(0) \]

\( n = 3 \) Variables,

\( r = 1 \) Cointegrating Relation and Trend Linear Relationship

\( n-r = 2 \) Common Stochastic Trends
1. "n" VARIABLES REPRESENTATION:

\[ X_t = \mu_t + \varepsilon_t \]

\( X'_t \) - the vector \((X_{1t}, X_{2t}, \ldots, X_{nt})\)
\( \mu'_t \) - the vector \((\mu_{1t}, \mu_{2t}, \ldots, \mu_{nt})\)
\( \varepsilon_t \) - an "n x 1" vector of disturbances

If one trend can be expressed as a linear combination of other trends, then:

\[ \beta'\mu_t = 0 \]

\[ \beta'X_t = \beta'\mu_t + \beta'\varepsilon_t = \beta'\varepsilon_t \sim I(0) \]

If the cointegrating rank is "r," there are \( r < n \) linear relationships among the trends:

\[ \beta'\mu_t = 0 \quad \text{where} \quad \beta' \text{ is an } "r \times n" \text{ matrix} \]
V. COINTEGRATION AND ERROR CORRECTION

A. GRANGER REPRESENTATION THEOREM

1. FOR ANY SET OF $I(1)$ VARIABLES, ERROR CORRECTION AND COINTEGRATION ARE EQUIVALENT REPRESENTATIONS.

2. THE RESTRICTIONS NECESSARY TO ENSURE THAT THE VARIABLES ARE CI$(1,1)$ GUARANTEE THAT AN ERROR-CORRECTION MODEL EXISTS.

B. ERROR CORRECTION MODEL

1. SHORT-TERM DYNAMICS ARE INFLUENCED BY THE DEVIATION FROM EQUILIBRIUM:

$$\Delta r_{st} = \delta_s (r_{lt-1} - \beta r_{st-1}) + \epsilon_{st}$$
$$\Delta r_{lt} = -\delta'_l (r_{lt-1} - \beta r_{st-1}) + \epsilon_{lt}$$

$\delta_s, \delta'_l > 0$ - Speed of Adjustment Parameters
$r_{st}$ - Short-Term Interest Rate
$r_{lt}$ - Long-Term Interest Rate
C. TESTING FOR COINTEGRATION

1. ENGLE-GRANGER METHODOLOGY

ESTIMATE:

\[ Y_t = \hat{\beta}_0 + \hat{\beta}_1 X_t + \hat{\epsilon}_t \]

\[ \Delta \hat{\epsilon}_t = \gamma \hat{\epsilon}_{t-1} + \tau_t \]

If we accept \( H_0 : \gamma = 0 \), \( Y_t \) and \( X_t \) are not cointegrated.

2. JOHANSEN, STOCK AND WATSON METHODOLOGY

\[ \Delta X_t = \sum_{i=1}^{p-1} \pi_i \Delta X_{t-i} + \pi X_{t-p} + \varepsilon_t \]

If the \( \text{Rank}(\pi) = 0 \), no cointegration and model is VAR in first differences.

If the \( \text{Rank}(\pi) = n \), vector process is stationary and model is VAR in levels.

If the \( 0 < \text{Rank}(\pi) < n \), multiple cointegrating vectors.
A) TESTING FOR COINTEGRATING RELATIONSHIPS

\[ \lambda_1 > \lambda_2 > \ldots > \lambda_n \] - "n" Characteristic Roots of \( \pi \)

\[ H_0 : \text{# of Cointegrating Vectors} \leq r \]
\[ H_0 : \text{# of Cointegrating Vectors} > r \]

\[ \lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{n} \ln(1 - \hat{\lambda}_i) \]

\[ H_0 : \text{# of Cointegrating Vectors} = r \]
\[ H_0 : \text{# of Cointegrating Vectors} = r + 1 \]

\[ \lambda_{\text{max}}(r,r+1) = -T \ln(1 - \hat{\lambda}_{r+1}) \]
B) EXAMPLE

\[ X_t' = (m_{2t}, y_t, i_{td}, i_{tb}) \quad t = 1974:1, \ldots, 1987:3 \]

\( m2 \) - Log of Real Money Supply (M2)
\( y \) - Log of Real Income
\( i_{td} \) - Deposit Rate on Money
\( i_{tb} \) - Bond Rate

\begin{array}{ccc}
\lambda_i & \lambda_{\max} & \lambda_{\text{trace}} \\
.4332 & 30.09 & 49.14 \\
.1776 & 10.36 & 19.05 \\
.1128 & 6.34 & 8.69 \\
.0434 & 2.35 & 2.35 \\
\end{array}

\( H_0 : r = 0 \)
\( H_A : r = 1, 2, 3, \) or 4

\( \lambda_{\text{trace}} = 49.14 \)
\( \text{Prob}(\lambda_{\text{trace}} > 49.14) > .10 \)
Accept \( H_0 \), No Cointegration

\( H_0 : r = 0 \)
\( H_A : r = 1 \)

\( \lambda_{\max} = 30.09 \)
\( \text{Prob}(\lambda_{\max} > 30.09) < .05 \)
Reject \( H_0 \), One Cointegrating Vector