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Available at: https://works.bepress.com/limingliu/22/
A Discrete-Time Model for Common Lifetime Inventory Systems

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We consider a discrete-time $(s,S)$ inventory model in which the stored items have a random common lifetime with a discrete phase-type distribution. Demands arrive in batches following a discrete phase-type renewal process. With zero lead time and allowing backlogs, we construct a multidimensional Markov chain to model the inventory-level process. We obtain a closed-form expected cost function. Numerical results demonstrate some properties of optimal ordering policies and cost functions. When compared with the results for the constant lifetime model, the variance of the lifetime significantly affects the system behavior. Thus, the formalism that we create here adds a new dimension to the research in perishable inventory control under uncertainty in lifetime.

Key words: perishable inventory; random common lifetime; discrete-time model; optimization
MSC2000 subject classification: Primary: 90B05; secondary: 60J20
ORMS subject classification: Primary: Inventory/production, perishable items, uncertainty, stochastic; secondary: optimal control, Markov, finite state
History: Received October 1, 2002; revised October 15, 2003.

1. Introduction. There is a growing research interest in discrete-time queues (see, e.g., Gravey et al. [6], Alfa and Chakravarthy [1], Yang and Chaudhry [23], Hwang et al. [8], and Ganesh et al. [5], mainly motivated by their applications in computer and communication systems where measurements of time are often slotted (see, e.g., Hayes [7] and Liu and Neuts [13]). Though many inventory systems are conveniently characterized by fixed length intervals at which events occur and decisions are made, few articles in the literature deal with discrete-time inventory models.

In this paper, we propose a discrete-time formulation for perishable inventory systems subject to stochastic demands. We may divide perishable inventory problems, where each item on-hand has a finite lifetime, into two categories—items that behave independently and items that survive a random time in storage. It is usually assumed that all fresh items in stock share a common lifetime distribution, though they do not usually fail at the same time. We call these random lifetime models. This paper concerns the second category, where all items of the same age fail at the same time. We refer to those as common lifetime models. We contribute to the literature by proposing a model that captures the randomness in common lifetimes. It is all too common that goods fail or outdate in batch. However, the existing approach in the literature restricts the lifetime to a fixed constant. Our formalism allows us to move away from this restrictive assumption while keeping the basic common lifetime characteristic, i.e., items of the same age fail together. There are many practical inventory control applications with random common lifetimes. For example, fresh food is perishable, but its exact lifetime is not known in advance and is affected by weather, temperature control, etc. For some products, the introduction of new editions makes old editions obsolete and they are sold at a discount. Vendors do not know exactly when that will happen.

Perishable inventory problems with constant lifetime have been studied quite extensively using the periodic review policy. Periodic review models fit the constant lifetime well,
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but they usually lead to numerically difficult dynamic optimization problems. Fries [4] and Nahmias [17] use dynamic programming in a perishable inventory model with a lifetime \( m \), zero lead time, and zero ordering cost (see Nahmias [18]) for a survey of the literature on periodic review models. While the authors found properties for the optimal ordering policy, the exact form of the policy could not be identified. The alternative continuous-review approach leads to computationally efficient Markovian models for random lifetime problems once the system performance measures are obtained (see, e.g., Kalpakam and Sapna [9], Liu [14], and Liu and Yang [16]). However, the usual continuous inter-demand time assumption (for example, renewal demand processes) is not quite compatible with common lifetimes and limits the analytical tractability of the traditional continuous-time models (see, e.g., Liu and Lian [15], Lian and Liu [12], Ravichandran [20], and Weiss [22]). Perishable inventory systems are also studied as queues with impatient customers, but the decision problems are not discussed (see, e.g., Kaspi and Perry [10, 11] and Bar-Lev and Perry [2] for a discrete-time case).

We measure time in discrete units with epochs 0, 1, 2, . . . . All events can occur only at these epochs and the system state is monitored (reviewed) continuously at each and every epoch. We can now formulate a discrete-time model which is parallel to the continuous-time and continuous-review models. We assume that the demands are in batches and follow a discrete phase-type (PH) renewal process, that the common lifetime has a discrete PH distribution, and that the lead time is zero and backorders are allowed. Under these assumptions, we construct a finite discrete-time Markov chain. By using recursive formulas, the expected cycle length and the total cost can be calculated. The discrete PH distribution provides a general framework to model various discrete lifetimes and inter-demand times. For example, the geometric, binomial, and uniform distributions are discrete time PH distributions preserves the generality of the model. PH distributions have been widely adopted in other stochastic models, such as queueing and reliability models, and methodologies and algorithms have been developed systematically. As a result, while our model assumptions are general, we can borrow many existing methods for both analytical derivations and numerical computations.

This paper is organized as follows. In §2, we present the decision problem by defining the cost structure and identifying the format of the cost function. We then define the system model and construct the corresponding Markov chain in §3. In §4, we obtain the stationary probability vector of the Markov chain and derive the reorder cycle length. Based on the results from §4, we can then construct the cost function in §5 and analyze its convexity properties. In §6, we present our numerical experience, compare the results for models with different lifetime distributions, and conclude the paper in §6.

2. The problem definition. We consider the inventory replenishment problem for a product with a common lifetime distribution. We make the following assumptions:

(i) The inter-demand time \( Y \) follows a discrete PH distribution with \( n \) phases and representation \((\alpha, D)\), where \( D = (D_i)_{i=0}^n \) and \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \) with \( \alpha_0 = 0 \). The absorbing probability vector is \( D^0 = (D_{i0}, \ldots, D_{n0})^T = e - De \). Throughout, \( e \) denotes a column vector of 1 for all components (see Neuts [19]) for the definition of PH distribution).

(ii) Demands occur in batches with a random batch size \( B \), where \( P(B = i) = p_i, i = 1, 2, \ldots \). We write \( p_i = P(B \geq i) = \sum_{j=i}^{\infty} p_j \) for \( i \geq 0 \).

(iii) The item’s lifetime \( X \) has a discrete PH distribution with \( m \) phases and representation \((\beta, T)\), where \( T = (T_i)_{i=0}^m \), \( \beta = (\beta_0, \beta_1, \ldots, \beta_m) \), \( \beta_0 = 0 \), and \( T^0 = (T_{i0}, \ldots, T_{m0})^T = e - Te \).

(iv) The replenishment order lead time is zero. All items arrive new.

(v) All unmet demands are backordered.
To model the system properly, we adopt the following convention: at time $t$ we may have a demand arrival, the expiration of the remaining stock, or the arrival of a replenishment order—if there is a demand arrival or a expiration, the exact epoch of the event is $t^-$, just prior to $t$; if both demand and expiration occur, the expiration occurs before the demand; if there is a replenishment, it occurs just after time $t$, i.e., at $t^+$; observation of the system state is exactly at time $t$. The system state, that is, the inventory level, known to the operator/controller is denoted by $I(t) \in \{I(t^+), t = 0, 1, 2, \ldots \}$. Throughout the paper, we use $t$ to denote the time of the occurrence of any event. The exact interpretation of the time epoch depends on the type of event according to the stated convention.

Following Weiss [22], we may also show that the optimal replenishment policy for this model is of the $(s, S)$ type. This policy works as follows: at any inventory level $I(t^-) = s + 1, s + 2, \ldots, S$, if an arriving demand is such that $I(t^-)$ would be $s$ or lower, a replenishment order is placed to satisfy the latest demand, eliminate backorders, if there are any, and return the inventory level to $S$ at $t^+$.

Thus, we have $s + 1 \leq I(t) \leq S$. We note that in the optimal solution, we must have $s \leq -1$ because $s = -1$ is always better than $s \geq 0$ in that the replenishment batch size is maximized for the same order-up-to level $S$ without adding any backorder cost.

We consider the following costs parameters for the replenishment decision:

- $C_k$: the inventory carrying cost per unit per unit time;
- $C_u$: the replacement cost per unit decayed;
- $C_s$: the shortage penalty per unit short;
- $C_c$: the shortage penalty per unit per unit time;
- $C_r$: the ordering cost per order.

For convenience, we write $x = -s$ and use $x$ to indicate the backlog level. To obtain the cost function, we identify a regenerative cycle of the inventory-level process, namely, the reorder cycle. We then derive the following total expected per cycle costs: $E[HC]$, the holding cost; $E[RC]$, the replacement (disposal) cost of perished items; and $E[SC]$, the shortage cost. Let $\tau$ be the reorder cycle. The average total expected cost is

$$
C(x, S) = \frac{C_s + E[HC] + E[RC] + E[SC]}{E[\tau]}.
$$

Our problem can then be stated as follows:

$$
\min_{x, S} C(x, S), \quad S \geq 0, \quad x \geq 1.
$$

3. The system model. We construct a discrete-time Markov chain for the inventory level. Let $X(t)$ denote the age of the items at time $t$, $X(t) = 0, 1, \ldots, m$. $X(t) = 0$ corresponds to the case when no items are on hand. Let $Y(t)$ be the phase of the demand process at time $t$, where $Y(t) = 1, \ldots, n$. Then, the three-dimensional process $\{I(t), X(t), Y(t), t \geq 0\}$ is a finite Markov chain with state space $J = \{(i, j, k) | s + 1 \leq i \leq S; 0 \leq j \leq m; 1 \leq k \leq n\}$. The states are arranged in the following order:

$$
(S, 1, 1), \ldots, (S, 1, n), (S, 2, 1), \ldots, (S, 2, n), \ldots, (S, m, 1), \ldots, (S, m, n);
(S - 1, 1, 1), \ldots, (S - 1, 1, n), (S - 1, 2, 1), \ldots, (S - 1, 2, n), \ldots, (S - 1, m, 1),
\ldots, (S - 1, m, n);
\ldots, \ldots;
(1, 1, 1), \ldots, (1, 1, n), (1, 2, 1), \ldots, (1, 2, n), \ldots, (1, m, 1), \ldots, (1, m, n);
(0, 0, 1), \ldots, (0, 0, n);
$$
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{state_transition_diagram.png}
\caption{State-transition diagram.}
\end{figure}

\begin{align*}
(1, 0, 1), \ldots, (1, 0, n); \\
\ldots, \ldots, \\
(s + 1, 0, 1), \ldots, (s + 1, 0, n).
\end{align*}

We will use the Kronecker product of matrices defined below (see Neuts [19]).

**Definition 3.1.** If $U = (U_{ij})$ and $M = (M_{ij})$ are rectangular matrices of dimensions $k_1 \times k_2$ and $k_1' \times k_2'$, their Kronecker product $U \otimes M$ is the matrix of dimensions $k_1k_1' \times k_2k_2'$ written in the block-partitioned form

\[
U \otimes M = \begin{pmatrix}
U_{11}M & \cdots & U_{1k_1}M \\
\vdots & \ddots & \vdots \\
U_{k_11}M & \cdots & U_{k_1k_1}M
\end{pmatrix}.
\]

If $L$, $M$, $U$, and $V$ are rectangular matrices such that the ordinary matrix products $LU$ and $MV$ are defined, then $(L \otimes M)(U \otimes V) = LU \otimes MV$ and $M(U \otimes V) = U \otimes MV$.

We introduce the notation: $T_a = T \otimes D^\alpha$, $T_a^0 = T^0 \otimes D^\alpha$, $T_a^\beta = Te \otimes D^\alpha$, $T_a^\beta_0 = T^0 \otimes D$, and $e_a = e \otimes D^\alpha$. The state-transition diagram (see Figure 1) helps to establish the transition matrix in Lemma 1.

**Lemma 3.1.** The transition probability matrix of the Markov chain \{I(t), X(t), Y(t), t \geq 0\} can be written in block-partitioned form as

\[
P = \begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix},
\]

where the matrices $P_{11}$ is of dimension $mn \times mn$, $P_{12}$ is $mn \times n(mS - m - s)$, $P_{21}$ is $n(mS - m - s) \times mn$, and $P_{22}$ is $n(mS - m - s) \times n(mS - m - s)$. They are given by

\[
P_{11} = T \otimes D + p_{s-1}^0 e \otimes D^\alpha,
\]

\[
P_{12} = (p_1 T_a, \ldots, p_{S-1} T_a, p_S e_a + T_a^0, p_{S+1} T_a^\beta + p_1 T_a^\beta_0, \ldots, p_{S-S-1} T_a^\beta + p_{S-1} T_a^\beta_0),
\]

\[
P_{21} = \begin{pmatrix}
p_{S-1}^0 e \otimes D^\alpha \\
(p_{S-1}^0 T e \otimes D^\alpha) \otimes D^\alpha \\
(p_{S-1}^0 T e \otimes D^\alpha) \otimes D^\alpha \\
p_{S-1}^0 T e \otimes D^\alpha \\
p_{S-1}^0 T e \otimes D^\alpha
\end{pmatrix}.
\]
and $P_{22}$ is itself partitioned as $(P_{23}, P_{24})$, where

$$
P_{23} = \begin{pmatrix}
T \otimes D & p_1 T_a & p_2 T_a & \cdots & p_{s-2} T_a \\
T \otimes D & p_1 T_a & p_2 T_a & \cdots & p_{s-3} T_a \\
T \otimes D & \vdots & \vdots & \ddots & \vdots \\
T \otimes D & \vdots & \vdots & \ddots & p_{s-4} T_a
\end{pmatrix}
$$

and

$$
P_{24} = \begin{pmatrix}
p_{s-1} T_a + T_D & p_2 T_a + p_1 T_D & \cdots & p_{s-2} T_a + p_{s-1} T_D \\
p_{s-2} T_D & p_{s-1} T_a + p_1 T_D & \cdots & p_{s-3} T_a + p_{s-1} T_D \\
\vdots & \vdots & \ddots & \vdots \\
p_1 T_a + T_D & p_2 T_a + p_1 T_D & \cdots & p_{s-2} T_a + p_{s-1} T_D \\
D & p_1 D^a & \cdots & p_{s-1} D^a \\
D & \cdots & p_{s-2} D^a \\
& \ddots & \vdots \\
& \cdots & D
\end{pmatrix}
$$

**Proof.** The construction requires familiarity with the formalism of PH distribution. Here, we explain how one arrives at (2) and (3), which govern the transitions from level $S$ to level $S-1, \ldots, 1, 0, -1, \ldots, s+1$. 

$P\{I(t+1) = S, X(t+1) = j_2, Y(t+1) = k_2 | I(t) = S, X(t) = j_1, Y(t) = k_1\}$ 

$\quad = P\{no\ demand\ arrives, X\ changes\ from\ j_1\ to\ j_2\ and\ Y\ from\ k_1\ to\ k_2;\ or\ a\ demand\ of\ size\ \geq S - s\ arrives\ triggering\ the\ arrival\ of\ a\ new\ order,\ the\ initial\ phase\ of\ the\ new\ lifetime\ is\ j_2\ and\ that\ of\ the\ next\ demand\ is\ k_2\}$ 

$\quad = T_{j_1 j_2} D_{k_1 k_2} + p_{S-s}^0 \beta_{j_2} D_{k_1} \alpha_{k_2}$ for $j_1, j_2 = 1, 2, \ldots, m; k_1, k_2 = 1, 2, \ldots, n$. 

This leads to the following block matrix and then (2):

$$
P_{11} = \begin{pmatrix}
T_{111} + p_{S-1} \beta_1 D_{10} & \cdots & T_{1m} + p_{S-1} \beta_m D_{00} \\
\vdots & \ddots & \vdots \\
T_{m1} + p_{S-1} \beta_1 D_{10} & \cdots & T_{mm} + p_{S-1} \beta_m D_{00}
\end{pmatrix}
$$

For the matrix in (3),

$P\{I(t+1) = i, X(t+1) = j_2, Y(t+1) = k_2 | I(t) = S, X(t) = j_1, Y(t) = k_1\}$ 

$\quad = P\{a\ demand\ of\ size\ S - i\ arrives, X\ changes\ from\ j_1\ to\ j_2, and\ the\ initial\ phase\ for\ the\ next\ demand\ is\ k_2\}$ 

$\quad = p_{S-i} T_{j_1 j_2} D_{k_1} \alpha_{k_2}$ for $i = 1, \ldots, S-1; j_1, j_2 = 1, \ldots, m; k_1, k_2 = 1, \ldots, n$. 

Similarly,

\[ P\{I(t+1) = 0, X(t+1) = 0, Y(t+1) = k_2 \mid I(t) = S, X(t) = j_1, Y(t) = k_1 \} \]

\[ = P\{a \text{ demand of size } S \text{ arrives and the initial phase of the next demand is } k_2; \]

or no demand arrives, \( Y \) changes from \( k_1 \) to \( k_2 \), the remaining items expire at time \( t+1 \)
\[ = p_S D_{k_1,0} \alpha_{k_2} + T_{j_0} D_{k_2,0}, \quad \text{for } j_1 = 1, \ldots, m; \ k_1, k_2 = 1, \ldots, n. \]

\[ P\{I(t+1) = i, X(t+1) = 0, Y(t+1) = k_2 \mid I(t) = S, X(t) = j, Y(t) = k_1 \} \]

\[ = P\{a \text{ demand of size } S - i \text{ arrives, the initial phase of the next demand is } k_3; \]

or the remaining items expire at \( t+1 \) and a demand of size \(-i\) arrives
\[ = p_{S-i} \left( \sum_{l=1}^{m} T_{jl} \right) D_{k_1,0} \alpha_{k_2} + p_{-i} T_{j_0} D_{k_2,0} \alpha_{k_2} \]

for \( i = -1, \ldots, s + 1; \ j = 1, \ldots, m; \ k_1, k_2 = 1, \ldots, n. \)

Again, these three formulas lead to expression (3). \( \Box \)

4. Performance analysis. In this section, we derive the performance measures that are needed for the construction of the cost function. We will first derive the stationary probability vector of the Markov process defined in §3 and then provide an expression for the expected reorder cycle.

4.1. The stationary probability vector. The following lemma can be easily proved.

Lemma 4.1. The matrices \( I - P_{11} \) and \( I - P_{22} \) are nonsingular.

Proof. We know that \( P \) is a transition probability matrix, so \( P_{11} e + P_{12} e = e \). Then,
\[ (I - P_{11}) e + P_{12} e = 0. \]

Denote the set of states in level \( S \) by \( J_S \), and the elements of matrix \( (I - P_{11}, P_{12}) \) by \( w_{ij} \), \( i \in J_S, j \in J \). Then,
\[ |w_{ij}| = \left| \sum_{j \in J} w_{ij} \right| > \left| \sum_{j \in J_1} w_{ij} \right|. \tag{6} \]

Therefore, the matrix \( I - P_{11} \) is diagonally dominant, and hence nonsingular (see Fiedler \([3]\)). By a similar argument, so is \( I - P_{22} \). \( \Box \)

Let the partitioned vector \( \pi = (\pi_S, \pi_{S-1}, \ldots, \pi_{s+1}) \) be the stationary probability vector, where \( \pi_i \in R^m \) and \( \pi_j \in R^n \) contain the probabilities corresponding to level \( i \) and level \( j \), respectively, for \( i = 1, 2, \ldots, S \) and \( j = s + 1, s + 2, \ldots, -1, 0 \). We write \( \hat{\pi} = (\pi_{S-1}, \ldots, \pi_{s+1}) \).

Theorem 4.1. The stationary probability vector \( \pi = (\pi_S, \hat{\pi}) \) satisfies
\[ \pi_S = v(\beta \otimes \alpha)(I - P_{11})^{-1}, \tag{7} \]
\[ \hat{\pi} = v(\beta \otimes \alpha)(I - P_{11})^{-1} P_{12} (I - P_{22})^{-1}, \tag{8} \]
where the constant \( v \) is given by
\[ v = \frac{1}{(\beta \otimes \alpha)(I - P_{11})^{-1} e + (\beta \otimes \alpha)(I - P_{11})^{-1} P_{12} (I - P_{22})^{-1} e}. \]
It turns out that the reorder cycle is the key to the construction of the cost function. The cycle length is defined as the time between two consecutive replenishment points. Because replenishments can only be triggered by a demand arrival that consumes all remaining items, if there are any, each replenishment corresponds to a replenishment point. Because replenishments can only be triggered by a demand arrival, the reorder cycle is the key to the construction of the cost function. It turns out that the reorder cycle is the key to the construction of the cost function.

4.2. The reorder cycle. A reorder cycle is defined as the time between two consecutive replenishment points. Because replenishments can only be triggered by a demand arrival, the reorder cycle is the key to the construction of the cost function. It turns out that the reorder cycle is the key to the construction of the cost function.

Theorem 4.2. The cycle length $\tau$ has a discrete PH distribution with representation $((\beta \otimes \alpha, O_{n(m-s)}), \tilde{P})$. Its expectation is

$$E \tau = (\beta \otimes \alpha, O_{n(m-s)})(I - \tilde{P})^{-1}e,$$

where $O_{n(m-s)}$ is a zero vector of dimension $n(mS - m - s)$, and the partitioned matrix

$$\tilde{P} = \begin{pmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{pmatrix},$$

where blocks are given by

$$\tilde{P}_{11} = \begin{pmatrix} T \otimes D & p_1T_a & \cdots & p_{S-1}T_a \\ T \otimes D & \cdots & p_{S-2}T_a \\ \vdots & \ddots & \vdots \\ T \otimes D & \cdots & \cdots & \cdots \end{pmatrix},$$

$$\tilde{P}_{12} = \begin{pmatrix} p_ST_a + T_0 \otimes D & p_{S+1}T_a + p_1T_a & \cdots & p_{S-1}T_a + p_{S-1}T_a \\ p_{S-1}T_a + T_0 \otimes D & p_ST_a + p_1T_a & \cdots & p_{S-2}T_a + p_{S-1}T_a \\ \vdots & \ddots & \ddots & \vdots \\ p_1T_a + T_0 \otimes D & p_2T_a + p_1T_a & \cdots & p_ST_a + p_{S-1}T_a \end{pmatrix},$$

$$\tilde{P}_{22} = \begin{pmatrix} D & p_1D^0\alpha & \cdots & p_{S-1}D^0\alpha \\ D & \cdots & p_{S-2}D^0\alpha \\ \vdots & \ddots & \ddots \\ D & \cdots & \cdots & D \end{pmatrix}. $$
PROOF. Corresponding to the Markov chain \{I(t), X(t), Y(t)\}, we construct a Markov chain with state space \( J \cup \{s\} \), where \( s \) is an absorbing state. When the inventory level reaches \( s \), the Markov chain is absorbed. The transition matrix of the Markov chain \( \{\tilde{I}(t), \tilde{X}(t), \tilde{Y}(t)\} \) can thus be given by

\[
\tilde{Q} = \left( \begin{array}{cc}
\tilde{P} & \tilde{P}^0 \\
O & 1
\end{array} \right),
\]

where \( \tilde{P}^0 = e - \tilde{P}e \), and \( O \) is a zero row vector.

The initial probability vector of the Markov chain \( \{I(t), X(t), Y(t)\} \), i.e., \( (\beta \otimes \alpha, O_{n(mS-m-3)}) \), is also the initial probability vector of the Markov chain \( \{\tilde{I}(t), \tilde{X}(t), \tilde{Y}(t)\} \). Because the cycle length \( \tau \) is the first passage time to state \( s \), the density and mean of the discrete PH distribution with representation \((\beta \otimes \alpha, O_{n(mS-m-3)}), \tilde{P}\) are given by

\[
P(\tau = k) = (\beta \otimes \alpha, O_{n(mS-m-3)}) \tilde{P}^{k-1} \tilde{P}^0, \quad k \geq 1,
\]

\[
E\tau = \sum_{k=1}^{\infty} k P(\tau = k) = (\beta \otimes \alpha, O_{n(mS-m-3)})(I - \tilde{P})^{-1}.
\]

\( E\tau \) can be calculated recursively. Let

\[
y_S, \ldots, y_{s+1} = (\alpha \otimes \beta, O_{n(mS-m-3)})(I - \tilde{P})^{-1},
\]

where, for \( i = 1, \ldots, S \), \( y_i \) is an \( mn \)-dimensional vector; for \( i = s + 1, \ldots, -1, 0 \), \( y_i \) is an \( m \)-dimensional vector. Then, \( y_i \) is the expected sojourn time of the system in level \( i \), \( i = s - 1, \ldots, S \). From (12), when \( S > 0 \), we have that

\[
y_S = (\beta \otimes \alpha)(I - T \otimes D)^{-1},
\]

\[
y_S-i = \sum_{l=1}^{i} p_l y_{S-i+l}(T \otimes D^0 \alpha)(I - T \otimes D)^{-1}, \quad 1 \leq i \leq S - 1,
\]

\[
y_0 = \sum_{l=1}^{S} [p_l y_l (T \otimes D^0 \alpha) + y_j (T_0 \otimes D)](I - D)^{-1},
\]

\[
y_j = \left[\sum_{l=1}^{S} y_l (p_{l-1} T e + p_{l-1} T^0) \otimes D^0 \alpha + \sum_{l=j+1}^{0} y_l p_{l-j} D^0 \alpha \right](I - D)^{-1}, \quad s + 1 \leq j \leq -1.
\]

When \( S = 0 \), \( y_i \) can be easily given by

\[
y_0 = \alpha (I - D)^{-1},
\]

\[
y_j = \sum_{l=j+1}^{0} y_l p_{l-j} D^0 \alpha (I - D)^{-1}, \quad s + 1 \leq j \leq -1.
\]

We can then compute \( E\tau \) by \( E\tau = \sum_{i=s+1}^{S} y_i \) after obtaining \( y_i \) by these recurrence relations.

5. Optimization. Here, we construct the cost function and discuss its properties useful for optimization.

5.1. Cost function. The inventory holding cost incurs only when the inventory level is positive. As the expected sojourn time at level \( i \) is \( y_i e \), the total holding cost in a reorder cycle is given by

\[
E(HC) = C_h \sum_{i=1}^{S} i y_i e.
\]
To obtain the expected number of expired items in a cycle, we consider \(N_0(t)\), the number of demands in the interval \((0, t]\). Let
\[
\omega_j(k, t) = P[N_0(t) = j, Y(t) = k], \quad j = 1, \ldots, n; \quad k, t \geq 0,
\]
(19)
\[
\omega(k, t) = (\omega_1(k, t), \ldots, \omega_n(k, t)).
\]
(20)
Let \(R(t)\) be the expected number of expired items given that the lifetime of the items is \(t\). Then,
\[
R(t) = E \max \{S - N_0(t), 0\} = \sum_{k=0}^{S-1} (S - k) \omega(k, t) e, \quad t \geq 0.
\]
Because the lifetime of all items \(X\) has a discrete PH distribution with representation \((\beta, T)\), the expected number of expired items in a cycle is
\[
ER(X) = \sum_{t=1}^{\infty} R(t) \beta T^{t-1} T^0 = \sum_{t=1}^{\infty} \sum_{k=0}^{S-1} (S - k) \omega(k, t) \beta T^{t-1} T^0.
\]
The expected replacement cost in a reorder cycle is \(E(RC) = C_r E(RX)\). To compute \(\omega(k, t)\), we can use the following recursive equations:
\[
\omega(0, 0) = \alpha; \quad \omega(k, 0) = 0, \quad k \geq 1,
\]
\[
\omega(k, t) = \omega(k, t-1) D + \sum_{i=1}^{k} p_i \omega(k-i, t-1) D^0 \alpha, \quad k \geq 0, \quad t \geq 1.
\]
When the inventory level is \(i\) at time \(t\) on the condition that the inventory level is \(s\) at time \(t+1\), \(s+1 \leq i \leq -1\), the number of backorders in the cycle is \(-i\). Thus, in the steady state, the total backorders in a cycle can be obtained by
\[
\lim_{t \to \infty} \frac{\sum_{i=1}^{x-1} i P[I(t+1) = -i | I(t) = s]}{\sum_{i=1}^{x-1} P[I(t+1) = -i]} = \frac{\sum_{i=1}^{x-1} i p_i 0 \gamma_{-1} e}{\sum_{i=1}^{x-1} p_i 0 \gamma_{-1} e}.
\]
The total shortage penalty in a cycle is therefore
\[
E(SC) = C_s \sum_{i=1}^{x-1} i y_{-1} e + C_s \left(\sum_{i=1}^{x-1} i p_{i-1} 0 \gamma_{-1} e\right) / \left(\sum_{i=1}^{x-1} p_{i-1} 0 \gamma_{-1} e\right).
\]
(21)
Summing up these three costs and the ordering cost \(C_o\), and dividing the sum by the mean cycle length \(E(\tau)\), we obtain the expected total cost per unit of time as given by (1).

5.2. Analysis. To minimize the total cost rate, we first consider \(C(x, S)\) as a function of \(x\) for a fixed \(S\).

Consider the singleton demands. \(p_1 = 1\) and from (16) and (18), we have
\[
y_j e = y_{j+1} D^0 \alpha (I - D)^{-1} e = y_{-1} e, \quad -x + 1 \leq j < -1 \quad \text{if} \quad x > 1.
\]
Therefore,
\[
\sum_{j=x}^{x-1} y_j e = (x - 1) y_{-1} e. \quad \text{Denote} \quad AC = C_o + E(HC) + E(RC) \quad \text{and} \quad m_s = \sum_j y_j e. \quad \text{Then}, \quad E(\tau) = m_s + (x - 1) y_{-1} e. \quad \text{We consider two different cases of backorder penalty costs:} \quad C_s = 0 \quad \text{and} \quad C_s > 0.
\]
(a) \(C_s = 0\).
\[
C(x, S) = \frac{AC + (x - 1) C_u}{m_s + (x - 1) y_{-1} e}.
\]
Letting $\Delta C(x, S) = C(x + 1, S) - C(x, S)$, we obtain
\[
\Delta C(x, S) = \frac{C_u m_S - AC y_{-1} e}{\mu [m_S + (x - 1) y_{-1} e] [m_S + xy_{-1} e]}.
\] (22)

We note that both $m_S$ and $AC$ are independent to the backorder level $x$. Hence, $\Delta C(x, S) \geq 0$ for all $x > 0$ or $\Delta C(x, S) \leq 0$ for all $x > 0$. $C(x, S)$ is a monotone function of $x$ (see Figure 2).

From (1), for all $S \geq 0$, we have $C(1, S) = AC/m_S$ and
\[
C(\infty, S) \doteq \lim_{x \to \infty} C(x, S) = C_u/y_{-1} e.
\] (23)

From (22) and (23), we identify the following three scenarios of the cost functions, as shown in Figure 2:

(i) When $C(1, S) < C(\infty, S)$, $\Delta C(x, S) > 0$ and $C(x, S)$ is a strictly increasing function of $x$. Obviously, no backorder should be permitted and the optimal $x = 1$.

(ii) When $C(1, S) = C(\infty, S)$, the cost function becomes independent of $x$.

(iii) When $C(1, S) > C(\infty, S)$, $C(x, S)$ is a strictly decreasing function of $x$ and all demands should be backordered.

Technically, there is no optimal solution for the third scenario ($x \to +\infty$). One simple solution to this problem is to identify a maximum waiting time that the management of a company can tolerate as part of the company’s customer service level requirement. This maximum waiting time will then limit $x$ to a maximum level $x_0$.

(b) $C_s > 0$. Letting $\Delta^2 C(x, S) = [C(x - 1, S) + C(x + 1, S)]/2 - C(x, S)$, we have
\[
\Delta^2 C(x, S) = \frac{C_u m_S (m_S - y_{-1} e)/2 - \Phi}{\mu [m_S + (x - 2) y_{-1} e] [m_S + (x - 1) y_{-1} e] [m_S + xy_{-1} e]},
\] (24)

where $\Phi = C_u m_S - AC y_{-1} e$. We also have
\[
\Delta C(x, S) = C(x + 1, S) - C(x, S) = \frac{[C_s (y_{-1} e)^2/2] x^2 + [C_s m_S y_{-1} e - C_s (y_{-1} e)^2/2]}{[m_S + (x - 1) y_{-1} e] [m_S + xy_{-1} e]} + \Phi.
\] (25)

Note here that
\[
\Delta C(1, S) = \frac{C_u m_S y_{-1} e + \Phi}{m_S (m_S + y_{-1} e)}.
\]
and

\[ \Delta C(\infty, S) = \lim_{x \to \infty} \Delta C(x, S) = C_i/2. \]  

(26)

With (24) and (26), we can identify the following two scenarios for the cost functions, as shown in Figure 3.

(i) When \( \Delta C(1, S) \geq \Delta C(\infty, S) \), \( \Delta^2 C(x, S) \leq 0 \) and \( C(x, S) \) is a concave function of \( x \). Because \( \lim_{x \to \infty} C(x, S) = +\infty \), \( C(x, S) \) is also a strictly increasing function of \( x \). Thus, the optimal \( x \) value is 1 and no backorder is permitted.

(ii) When \( \Delta C(1, S) < \Delta C(\infty, S) \), \( \Delta^2 C(x, S) > 0 \) and \( C(x, S) \) is a convex function of \( x \). By solving the equation \( [x-e]C_i/2 + [m_i C_i (y_{i-1})^2 - C_i (y_{i-1})^2]/2 + \Phi = 0 \), we can obtain the solution for \( x \). Let \( x_1 = \lfloor x \rfloor \) and \( x_2 = \lceil x \rceil + 1 \). If \( C(x_1, S) < C(x_2, S) \), the cost function is minimized at \( x_1 \), otherwise it is minimized at \( x_2 \). Here, \( \lfloor x \rfloor \) is the largest integer that is smaller than or equal to \( x \).

For batch demands, extensive numerical results show that the properties identified above do not hold. However, as we can observe from Figures 4 and 5, the overall patterns of change remain the same as for singleton demand cases, except the curves are no longer smooth. Similar to the continuous-time model (see Lian and Liu [12]), we can develop an effective algorithm to compute the optimal solution.
We have derived the optimal $x$ based on the different cases $C_s = 0$ and $C_s > 0$. We denote the optimal $x = x(S)$ as a function of $S$, and $C(S) = \min_x C(x, S)$. Based on the overall pattern of $C(S)$ (see Figure 6), we can compute $S^* = \arg\min_S C(S)$. Finally, the optimal policy is given by $(x(S^*), S^*)$.

6. Numerical implementation. In this section, we study the properties of $x^*, S^*$, and $C(x^*, S^*)$ for different lifetime distributions and different demand batch sizes.

We slot a day into 10 time units, so that $t = 10(k-1) + 1, k \geq 1$, represents the first time epoch of the $k$th day. This time scale reflects situations typical in practical applications.

We assume that $C_o = \$5\text{ per order}, C_h = \$0.005\text{ per item per day}, C_r = \$8\text{ per item expired}, C_u = \$0.05\text{ per item per day},$ and $C_u = \$1\text{ per item}$. Moreover, the inter-demand time is geometrically distributed with parameter $q$. The demand batch size is $P\{B = i\} = p_i, i = 1, 2, \ldots$.

We examine the properties of $x^*, S^*$, and $C(x^*, S^*)$ under three different demand processes:

Case 1. $p_1 = 1; EY = 1, \text{Var}(Y) = 0$.
Case 2. $p_1 = 0.5, p_2 = 0.2, p_3 = 0.2, p_4 = 0.1; EY = 1.9, \text{Var}(Y) = 1.09$.
Case 3. $p_1 = 0.1, p_2 = 0.2, p_3 = 0.2, p_4 = 0.5; EY = 3.1, \text{Var}(Y) = 1.9$. 

![Figure 5. Cost function $C(x, S)$ with batch demands ($C_s = 0.05$).](image)

![Figure 6. Cost function $C(S)$ for different demand structures.](image)
where day to seven days. The results in Table 1 suggest the following conclusions: (i) For fixed 

$$\varphi = \frac{\lambda}{\sigma^2},$$

sufficiently large $R_{\lambda} \lambda T$, the cost $C(x^*, S^*)$ increases with the batch size; the reason is that the variance of demand increases when the batch size increases.

We note that for the same demand process, $x^*$ and $C(x^*, S^*)$ decrease when the expected lifetime $EX$ increases, but $S^*$ increases in $EX$. To compare $x^*$, $S^*$, and $C(x^*, S^*)$ under different lifetimes, we consider the following lifetime distribution:

$$p[X = t] = \frac{\beta_t}{1 - (1 - \rho)^m}, \quad t = 1, 2, \ldots, m \text{ days},$$

where $\beta_t = \binom{m}{t}(1 - \rho)^{m-t}$ is a binomial mass function with parameters $m$ and $\rho$. That is, $P[X = t]$ is a binomial distribution under the condition that $X \neq 0$. We call it the quasi-bin distribution. We note that the binomial distribution is in fact the summation of $n$ independent and identically distributed 0-1 random variables; thus by probability theory, the quasi-bin behaves like an asymmetric normal distribution, albeit taking discrete values $1, \ldots, m$. We may also offer the following interpretation: the lifetime is determined by $m$ factors. If $i$ out of $m$ factors is positive (taking value 1), the lifetime is $i$ periods. To reach the maximum lifetime $m$, all $n$ factors have to be positive simultaneously, and thus has the smallest probability. It is easy to see that this distribution is reasonable for practical applications.

Table 1. Optimal $(s, S)$ policies for geometrically distributed lifetimes ($C_a = 5$, $C_a = 0.005$, $C_i = 8$, $C_i = 1$, $C_i = 0.05$).

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is larger than the variance of the quasi-bin lifetime. That is, with the same expected lifetimes, the variance of geometric lifetime respectively. We know that the variances of the geometric lifetimes with the above expected lifetimes are then 1.9, 3.2, 4.8, and 6.4 days, and batch demand, fixing the expected lifetimes, the cost increases with the variance of the lifetime, but this property does not hold for the batch demand case.

The numerical results are given in Tables 2 and 3. Table 2 is for the singleton demand, and Table 3 is for batch demands with \( p_1 = 0.1, p_2 = 0.2, p_s = 0.2, \) and \( p_s = 0.5 \). We choose \( n = 8, \rho = 0.2, 0.4, 0.6, \) and 0.8. The expected lifetimes are then 1.9, 3.2, 4.8, and 6.4 days, respectively. We know that the variances of the geometric lifetimes with the above expectations are 3.5, 10.2, 22.6, and 40.3, and the variances of quasi-bin lifetimes are 1.5, 2.0, 1.9, and 1.3. That is, with the same expected lifetimes, the variance of geometric lifetime is larger than the variance of the quasi-bin lifetime.

From Tables 2 and 3, we draw the following conclusions: (i) For both singleton demand and batch demand, fixing the expected lifetimes, the cost \( C(x^*, S^*) \) increases with the variance of the lifetime; (ii) For singleton demand, fixing the expected lifetime, \( x^* \) increases with the variance of the lifetime, but this property does not hold for the batch demand case.
To conclude this paper, we point out that the formalism we create is quite powerful. It has been shown that any discrete distribution with a finite support of nonnegative integers is a discrete PH distribution (see, e.g., Shi et al. [21]). Hence, our model is applicable for any discrete lifetime and inter-demand time distributions, provided that the supports of the distributions are finite, such as that of the quasi-bin distribution.

Acknowledgments. We thank the referees and the associate editor for constructive comments and suggestions on an earlier version of this paper that were helpful to our revision. The first author thanks the University of Macau for partial support through Grant RG002/01-02W(S)/LZT/FBA. The second author thanks Research Grant Council of Hong Kong for partial support through Grant N_HKUST023/00.

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