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A B S T R A C T

This paper presents a new, stable, approximate inversion of Abel integral equation. By using the Taylor expansion of the unknown function, Abel equation is approximately transformed to a system of linear equations for the unknown function together with its derivatives. A desired solution can be determined by solving the resulting system according to Cramer’s rule. This method gives a simple and closed form of approximate Abel inversion, which can be performed by symbolic computation. The \( n \)th-order approximation is exact for a polynomial of degree up to \( n \). Abel integral equation is approximately expressed in terms of integrals of input data; so the suggested approach is stable for experimental data with random noise. An error analysis of this approach is given. Finally, several numerical examples are given to illustrate the accuracy and stability of this method.

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1. Introduction

Standard Abel integral equations refer to
\[
\int_0^s \frac{\psi(t)}{(s-t)^\nu} \, dt = f(s), \quad s > 0, \; 0 < \nu < 1,
\] (1)
and
\[
\int_s^a \frac{\psi(t)}{(t-s)^\nu} \, dt = f(s), \quad s < a, \; 0 < \nu < 1,
\] (2)
where \( f(s) \) is a known function, and \( \psi(t) \) is an unknown function to be determined. For the above two cases, the corresponding exact solutions are \cite{1}
\[
\psi(s) = \frac{\sin(\nu \pi)}{\pi} \frac{d}{ds} \int_0^s f(t) (s-t)^{-\nu-1} \, dt,
\] (3)
and
\[
\psi(s) = -\frac{\sin(\nu \pi)}{\pi} \frac{d}{ds} \int_s^a f(t) (t-s)^{-\nu-1} \, dt,
\] (4)
respectively. Nevertheless, both exact solutions fail in practical application since input function \( f(s) \) is given with a small random error and the differential operator involved is an ill-posed or unbounded operator. In other words, small noise in
the data function $f(s)$ might cause large error in the computed solution, and however an instable solution is undesired. Consequently, to avoid any instability of a desired solution, other stable methods are needed.

So far, many approaches have been proposed for determining the numerical solution to Abel equation [2]. For example, Fettis [3] proposed a numerical form of the solution to Abel equation by using the Gauss–Jacobi quadrature rule. Piessens and Verbaeten [4] and Piessens [5] developed an approximate solution to Abel equation by means of the Chebyshev polynomials of the first kind. When input signal is with noisy error, Murio et al. [6] suggested a stable numerical solution. Furthermore, Garza et al. [7] and Hall et al. [8] used the wavelet method to invert the inversion of noisy Abel equation.

In this study, we present a new, simple approach for solving the approximate solution to a class of Abel integral equations. By expanding the unknown function to be determined as a Taylor polynomial, we can convert Abel integral equation to a system of linear equations for the unknown function together with its derivatives up to order $n$. The $n$th-order approximation is given explicitly in terms of the input signal along with its integrals. An error analysis of approximate Abel integral equation is given. Finally, several examples are given to show the effectiveness of the present method.

2. Approximation of Abel inversion

For convenience, in this section we focus our attention on Eq. (1). For Eq. (2), the method can be employed in a completely similar manner. First, consider the case of $f(0) = 0$. Moreover, $f(t)$ is continually differentiable in the interval of interest. In this case, the solution to Eq. (1) can be then simplified to

$$\varphi(s) = \frac{\sin(v\pi)}{\pi} \int_0^s \frac{f'(t)}{(s-t)^{1-v}} \, dt.$$  

(5)

It is readily verified that $\varphi(s)$ has no singularity. Furthermore, we assume that the desired solution $\varphi(t)$ is $n$ times continually differentiable. Consequently, $\varphi(t)$ can be represented in terms of an $n$th-order Taylor expansion, if its $(n+1)$th-order derivative exists, i.e.,

$$\varphi(t) = \varphi(s) + \varphi'(s)(t-s) + \cdots + \varphi^{(n)}(s) \frac{(t-s)^n}{n!} + \varphi^{(n+1)}(\xi) \frac{(t-s)^{n+1}}{(n+1)!},$$

(6)

where $\xi$ is between $s$ and $t$. It is readily shown that the Lagrange remainder $\varphi^{(n+1)}(\xi)(t-s)^{n+1}/(n+1)!$ is sufficiently small for an enough large $n$ provided that $\varphi^{(n+1)}(x)$ is a uniformly bounded function for any $n$ in the interval of interest. Due to this fact, in what follows we will neglect the last Lagrange remainder and approximately expand $\varphi(t)$ as

$$\varphi(t) \approx \varphi(s) + \varphi'(s)(t-s) + \cdots + \varphi^{(n)}(s) \frac{(t-s)^n}{n!}.$$  

(7)

It is worth noting that the Lagrange remainder vanishes for a polynomial of degree equal to or less than $n$, implying that the above $n$th-order Taylor expansion is exact.

Substituting (7) for $\varphi(t)$ into Eq. (1), one can get

$$\int_0^s \frac{1}{(s-t)^v} \left[ \varphi(s) + \varphi'(s)(t-s) + \cdots + \varphi^{(n)}(s) \frac{(t-s)^n}{n!} \right] \, dt = f(s),$$  

(8)

or further

$$k_{(0,0)}(s)\varphi(s) + k_{(0,1)}(s)\varphi'(s) + \cdots + k_{(0,n)}(s)\varphi^{(n)}(s) = f(s),$$

(9)

where

$$k_{(0,j)}(s) = \frac{(-1)^j \varphi^{1-j}}{(j+1-v)!}, \quad j = 0, 1, \ldots, n.$$  

(10)

Eq. (9) becomes an $n$th-order, linear, ordinary differential equation with variable coefficients for $\varphi(s)$. Instead of solving analytically the resulting ordinary differential equation, we will determine $\varphi(s), \ldots, \varphi^{(n)}(s)$ by another approach, solving a system of linear equations. To this end, other $n$ independent linear equations for $\varphi(s), \ldots, \varphi^{(n)}(s)$ are needed. This can be achieved by integrating both sides of Eq. (1) with respect to $s$ from 0 to $x$. Accordingly, we get that

$$\int_0^x \int_0^s \frac{\varphi(t)}{(s-t)^v} \, dt \, ds = \int_0^x f(s) \, ds.$$  

(11)

Changing the order of the integration of the left-hand side of Eq. (11) and considering $0 < v < 1$, we have

$$\frac{1}{1-v} \int_0^s \varphi(t)(s-t)^{1-v} \, dt = \int_0^s f(t) \, dt,$$  

(12)
where we have replaced variable $x$ with $s$, for convenience. Applying the Taylor expansion again and substituting (7) for $\varphi(t)$ into Eq. (12) gives
\[
k_{(1,0)}(s)\varphi(s) + k_{(1,1)}(s)\varphi'(s) + \cdots + k_{(1,n)}(s)\varphi^{(n)}(s) = \int_0^s f(t) dt, \tag{13}
\]
where
\[
k_{(1,j)}(s) = \frac{(-1)^j s^{j+2-v}}{(j+2-v)(1-v)j!}, \tag{14}
\]
Thus we have arrived at another linear equation for $\varphi^{(j)}(s)$, $(j = 0, \ldots, n)$. By repeating the above integration process for $i (1 < i \leq n)$ times, we have
\[
k_{(i,0)}(s)\varphi(s) + k_{(i,1)}(s)\varphi'(s) + \cdots + k_{(i,n)}(s)\varphi^{(n)}(s) = f_{(i)}(s), \quad i = 2, 3, \ldots, n, \tag{15}
\]
where
\[
k_{(i,j)}(s) = \frac{(-1)^j s^{1+i-j-v}}{(1-v)(1+i+j-v)j!}, \quad i = 2, 3, \ldots, n, \tag{16}
\]
\[
f_{(i)}(s) = \begin{cases} \frac{1}{(i-1)!} \int_0^s (s-t)^{i-1} f(t) dt & i \neq 0, \\
f_{(i)}(s) & i = 0. \end{cases} \tag{17}
\]
Therefore, Eqs. (9), (13) and (15) form a system of $n+1$ linear equations for $n+1$ unknowns $\varphi(s), \varphi'(s), \ldots, \varphi^{(i)}(s)$, which can be rewritten as
\[
K_{nn} \Phi_n = f_n, \tag{18}
\]
with
\[
K_{nn}(s) = \begin{bmatrix} k_{(0,0)}(s) & k_{(0,1)}(s) & \cdots & k_{(0,n)}(s) \\
k_{(1,0)}(s) & k_{(1,1)}(s) & \cdots & k_{(1,n)}(s) \\
\vdots & \vdots & \ddots & \vdots \\
k_{(n,0)}(s) & k_{(n,1)}(s) & \cdots & k_{(n,n)}(s) \end{bmatrix},
\tag{19}
\]
\[
f_n(s) = \begin{bmatrix} f(s) \\
f_{(1)}(s) \\
\vdots \\
f_{(n)}(s) \end{bmatrix}, \quad \Phi_n(s) = \begin{bmatrix} \varphi(s) \\
\varphi'(s) \\
\vdots \\
\varphi^{(n)}(s) \end{bmatrix}. \tag{20}
\]
For convenience, we have replaced $\varphi^{(i)}(s)$ with $\tilde{\varphi}^{(i)}(s)$ to stand for its approximate solution. Without confusion, we still use a simple notation $\varphi(s)$ instead of $\tilde{\varphi}(s)$. If only $\varphi(s)$ is of concern, it can be solved easily with the aid of the well-known Cramer’s rule. For this purpose, we denote
\[
F_n(s) = \begin{bmatrix} f(s) & k_{(0,1)}(s) & \cdots & k_{(0,n)}(s) \\
f_{(1)}(s) & k_{(1,1)}(s) & \cdots & k_{(1,n)}(s) \\
\vdots & \vdots & \ddots & \vdots \\
f_{(n)}(s) & k_{(n,1)}(s) & \cdots & k_{(n,n)}(s) \end{bmatrix}. \tag{21}
\]

**Lemma 1.** For an arbitrary non-negative integer $n$, the determinant $C_n(x) \neq 0$ for arbitrary positive $x$, where
\[
C_n(x) = \begin{vmatrix} \frac{1}{x} & \frac{1}{x+1} & \cdots & \frac{1}{x+n} \\
\frac{1}{x+1} & \frac{1}{x+2} & \cdots & \frac{1}{x+1+n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x+n} & \frac{1}{x+n+1} & \cdots & \frac{1}{x+2n} \end{vmatrix}. \tag{22}
\]
**Proof.** Firstly, by setting $n = 0$ and 1, we can easily get that $C_0(x) = 1/x$, and $C_1(x) = [x(2 + x)]^{-1} - (1 + x)^{-2}$, respectively; so $C_0(x) \neq 0$, and $C_1(x) \neq 0$ for $x > 0$.

Secondly, for $k = n - 1$, one assumes $C_{n-1}(x) \neq 0$. Then for $k = n$, from (22) one readily finds

$$C_n(x) = \frac{1}{(n + x) \cdots (2n + x)} \begin{vmatrix} n + x & x & \cdots & 1 \\ n + 1 & x + 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x + n & x + n + 1 & \cdots & 1 \end{vmatrix}.$$  \hspace{1cm} (23)

Furthermore, subtract the last column from the first $n$ columns, respectively, and a further simplification allows us to obtain

$$C_n(x) = \frac{1}{n!(n + x) \cdots (2n + x)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{x}{n + x} & \frac{x + 1}{1} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x + n} & \frac{1}{x + n + 1} & \cdots & 1 \end{vmatrix}.$$  \hspace{1cm} (24)

A similar manipulation for rows can lead to

$$C_n(x) = \frac{1}{n!(n + x)^2 \cdots (2n - 1 + x)^2(2n + x)} \begin{vmatrix} n + x & n + x + 1 & \cdots & 1 \\ \frac{x}{n + x} & \frac{x + 1}{n + 1 + x} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}. \hspace{1cm} (25)$$

Now, subtracting the last column from other columns leads to

$$C_n(x) = \frac{1}{n!(n + x)^2 \cdots (2n - 1 + x)^2(2n + x)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{x}{1} & \frac{x + 1}{1} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = \frac{1}{n!(n + x)^2 \cdots (2n - 1 + x)^2(2n + x)} C_{n-1}(x). \hspace{1cm} (26)$$

Consequently, $C_n(x) \neq 0$ for $x > 0$ immediately follows from the assumption of $C_{n-1}(x) \neq 0$ for $x > 0$. Therefore, based on induction we have $C_n(x) \neq 0$ for an arbitrary non-negative integer $n$. □

From the above Lemma 1, taking into account

$$k_{i,j}(s) = \frac{(-1)^j s^{1+i+j-v}}{(1 - v) \cdots (i - v)(1 + i + j - v)!}, \hspace{1cm} (27)$$

and using direct computation, one gets

$$\det K_{nn} (s) = \frac{(-1)^{(n+1)n/2} s^{(n+1-v)(n+1)}}{(1 - v)^n (2 - v)^{n-1} \cdots (n - v)!}! \cdot n! \cdot C_n(1 - v), \hspace{1cm} (28)$$

and

$$F_n (s) = \frac{(-1)^{(n+1)n/2} s^{(n+2-v)n}}{(1 - v)^n (2 - v)^{n-1} \cdots (n - v)!}! \cdot n!.$$
Assume that Lemma 1 together with the well-known Cramer’s rule we can obtain the following:

**Theorem 1.** Assume that \( \psi(s) \) is \((n + 1)\) times continually differentiable, and \( f(0) = 0 \). Then the solution to Abel equation (1) can be approximated by

\[
\varphi_n(s) = \frac{1}{s^{1-v}C_n (1-v)} \begin{vmatrix}
1 - v & \int_0^s f(t) \, dt \\
\frac{1}{s} & 1 \\
& \vdots \\
\Gamma(n + 1 - v) & 1 \\
\Gamma(n) \Gamma(1-v) & \frac{1}{s^n} \int_0^s (s-t)^{n-1} f(t) \, dt
\end{vmatrix},
\]

(29)

where \( C_n = \frac{\Gamma(n+1)}{\Gamma(n)} \). This approximation is given by explicit expressions, so that this approximate solution can be implemented by symbolic computation using a personal computer.

The above relation into Eq. (1), we find that \( \phi(s) \) should satisfy Abel integral equation

\[
\int_0^s \frac{\phi(t)}{(s-t)^v} \, dt = h(s),
\]

with

\[
h(s) = f(s) - f(0).
\]

**Remark 1.** The above nth-order approximation is given by explicit expressions, so that this approximate solution can be implemented by symbolic computation using a personal computer.

**Remark 2.** For the case of \( f(0) \neq 0 \) for Eq. (1), in view of

\[
\int_0^s \frac{1}{t^{1-v}(s-t)^v} \, dt = \frac{\pi}{\sin(v\pi)},
\]

we can assume

\[
\psi(s) = \frac{\sin(v\pi) f(0)}{s^{1-v}} + \phi(s).
\]

Inserting the above relation into Eq. (1), we find that \( \phi(s) \) should satisfy Abel integral equation

\[
\int_0^s \frac{\phi(t)}{(s-t)^v} \, dt = h(s),
\]

with

\[
h(s) = f(s) - f(0).
\]

Evidently, \( h(0) = 0 \); so one can solve Abel equation (34) according to Theorem 1.
Remark 3. For the case of \( f (a) \neq 0 \) for Eq. (2), in view of
\[
\int_{s}^{a} \frac{1}{(a-s)^{1-\nu} (s-t)^{\nu}} \, dt = \frac{\pi}{\sin (\nu \pi)},
\]
we can assume
\[
\varphi (s) = \frac{\sin (\nu \pi)}{\pi} \frac{f (a)}{(a - s)^{1-\nu}} + \phi (s).
\]
Substituting the above relation into Eq. (2), we find that \( \phi (s) \) satisfies Abel integral equation
\[
\int_{s}^{a} \frac{\phi (t)}{(t - s)^{\nu}} \, dt = h (s),
\]
with
\[
h (s) = f (s) - f (a).
\]
On account of \( h (a) = 0 \), Abel equation (38) can be solved according to Theorem 2.

Finally, it should be noted that such methods can be further extended to look for generalized Abel inversion [9], and it also gives a very fast convergence ratio, even in the presence of noise, as compared to existing techniques for inverting Abel inversion.

3. Error analysis

In what follows we only give an error analysis of Eq. (1) subject to the boundary condition \( f (0) = 0 \). For convenience, suppose that the exact solution to be determined is infinitely differentiable in the interval \( I = (0, H] \), \( H \) being a positive constant. In other words, we can expand \( \varphi (t) \) as a uniformly convergent Taylor series in \( I \):
\[
\varphi (t) = \varphi (s) + \sum_{j=1}^{\infty} \varphi^{(j)} (s) \frac{(t - s)^{j}}{j!}.
\]

Then using the above suggested process, Abel integral equation (1) can be transformed to the following equivalent infinitely systems of linear equations for unknown \( \varphi^{(j)} (s) \), \( j = 0, 1, \ldots \),
\[
K \Phi = f,
\]
with
\[
K = \lim_{n \to \infty} K_{nn}, \quad \Phi = \lim_{n \to \infty} \Phi_{n}, \quad f = \lim_{n \to \infty} f_{n},
\]
where \( K_{nn}, \Phi_{n}, \) and \( f_{n} \) are defined as follows
\[
K_{nn} = [k_{ij} (s)]_{(n+1) \times (n+1)}, \quad \Phi_{n} = [\varphi^{(j)} (s)]_{(n+1) \times 1}, \quad f_{n} = [f^{(i)} (s)]_{(n+1) \times 1},
\]
respectively. Thus the unique solution can be expressed as
\[
\Phi = Lf,
\]
where \( L = K^{-1} \). Alternatively, the above relation can be rewritten as
\[
\begin{bmatrix} \Phi_{n} \\ \Phi_{\infty} \end{bmatrix} = \begin{bmatrix} L_{nn} & L_{n\infty} \\ L_{\infty n} & L_{\infty\infty} \end{bmatrix} \begin{bmatrix} f_{n} \\ f_{\infty} \end{bmatrix}.
\]

Consequently, one finds that the vector \( \Phi_{n} \) composed of the first \( n + 1 \) elements of the exact solution vector \( \Phi \) must satisfy the following relation
\[
\Phi_{n} = L_{nn} f_{n} + L_{n\infty} f_{\infty}.
\]

In addition, based on the analysis of the foregoing section, the unique solution of Eq. (18) is denoted as
\[
\Phi_{n} = K_{nn}^{-1} f_{n}.
\]
Subtracting (47) from (46) yields
\[
\Phi_{n} - \Phi_{n} = A_{nn} f_{n} + L_{n\infty} f_{\infty},
\]
where \( A_{nn} = L_{nn} - K_{nn}^{-1} \).
Expanding the right-hand side of (48), the first element of the vector at the left-hand side of (48) is expressed by

\[ \varphi (s) - \bar{\varphi} (s) = \sum_{j=0}^{n} a_{ij} f_{ij} (s) + \sum_{j=n+1}^{\infty} l_{ij} f_{ij} (s). \]

where \( a_{ij} \) and \( l_{ij} \) are the elements of \( A_{nn} \) and \( L_{\infty, n} \), respectively. Thus

\[ |\varphi (s) - \bar{\varphi} (s)| \leq \left( \sum_{j=0}^{n} |a_{ij}|^2 \right)^{1/2} \left( \sum_{j=0}^{n} |f_{ij} (s)|^2 \right)^{1/2} + \left( \sum_{j=n+1}^{\infty} |l_{ij}|^2 \right)^{1/2} \left( \sum_{j=n+1}^{\infty} |f_{ij} (s)|^2 \right)^{1/2} \]

follows from the well-known Cauchy–Schwarz inequality. This is what we desire to obtain, which gives an error between the exact and approximate solution. Since \( \lim_{n \to \infty} A_{nn} = 0, \lim_{n \to \infty} L_{\infty, n} = 0 \), so we can obtain \( \lim_{n \to \infty} |\varphi (s) - \bar{\varphi} (s)| = 0. \)

Next, let us consider the case of an input function with noise. In this case, we take

\[ f^* = f + \varepsilon, \]

with

\[ \varepsilon = \left[ \begin{array}{c} \varepsilon (s) \\ \int_0^s \varepsilon (t) \, dt \\ \vdots \\ \frac{1}{(n-1)!} \int_0^s (s-t)^{n-1} \varepsilon (t) \, dt \\ \vdots \end{array} \right], \]

where \( \varepsilon (y) \) denotes random noise. From the above, other elements except the first one are clearly expressed by the integrals of \( \varepsilon (y) \). Obviously, when random noise is present, there is no crucial change in the above derivation. Since only integrals related to input data function \( f(s) \) are involved in the above solution, it can be predicted that the algorithm is stable when input signal \( f(s) \) is with random noise.

Note that although the function \( \varphi (s) \) considered in this section is assumed to be infinitely differentiable in the interval \( I = (0, H) \), practical application to be given in the following section indicates that such a constraint can be further relaxed.

4. Numerical examples

In this section, several test examples are presented to illustrate the accuracy and stability of the method described in this paper. To examine the accuracy of the results, absolute errors are employed to assess the efficiency of this method. As for the stability of this method, we take the right-hand side prescribed function \( f(s) \) with small errors. To model this character, in the following examples we take the right-hand side prescribed function as \( f^* (s) = f (s) + \theta (s) \), where \( f (s) \) denotes a true function corresponding to the exact solution, \( \varepsilon \) is a constant and \( \theta (s) \) denotes a uniform random variable with values in \( [-1, 1] \) such that the maximum relative error \( \max |f^* (s) - f (s)| / |f (s)| \leq \varepsilon. \)

**Example 1.** We consider Abel equation

\[ \int_0^s \frac{\varphi (t)}{(s-t)^{1/3}} \, dt = f (s), \quad 0 < s \leq 1, \]

with \( f (s) = s^{5/3} \), and its exact solution is \( \varphi (s) = 10s/9. \)

Based on Theorem 1, we can obtain explicit forms of several lower-order approximate solutions. By direct computation, one can find that \( \varphi_n (s) \) reduces to the desired exact solution only if \( n \geq 1 \), i.e., \( \varphi_n (s) = 10s/9, \) as \( n \geq 1. \) This can be explained as a result of the fact that the desired solution is just a polynomial of degree one. Therefore, if a desired solution is a polynomial of degree \( n \), the \( n \)-th order approximation collapses to its exact solution.

**Example 2.** Consider the same Abel equation in **Example 1**, but with \( f (s) = s^{7/6} \). In this case, its exact solution can be found to be

\[ \varphi (s) = \frac{7 \Gamma (1/6)}{18 \sqrt{\pi} \Gamma (2/3) \sqrt{s}}. \]
Table 1
Approximate solutions and maximum absolute errors between \( \varphi (s) \) and \( \varphi_n (s) \) of Example 2

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \varphi_n (s) )</th>
<th>( \max { | \varphi_n (s) - \varphi (s) | } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1107 \sqrt{s} )</td>
<td>0.0383</td>
</tr>
<tr>
<td>2</td>
<td>( 5920 \sqrt{s} )</td>
<td>0.0142</td>
</tr>
<tr>
<td>3</td>
<td>( 54560 \sqrt{s} )</td>
<td>( 7.11 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

Table 2
Maximum absolute errors between \( \varphi (s) \) and \( \varphi_n (s) \) of Example 3

<table>
<thead>
<tr>
<th>( x )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.21529</td>
<td>0.22085</td>
<td>0.21321</td>
</tr>
<tr>
<td>0.2</td>
<td>0.32588</td>
<td>0.33262</td>
<td>0.32298</td>
</tr>
<tr>
<td>0.3</td>
<td>0.42757</td>
<td>0.43413</td>
<td>0.42403</td>
</tr>
<tr>
<td>0.4</td>
<td>0.52933</td>
<td>0.53457</td>
<td>0.52524</td>
</tr>
<tr>
<td>0.5</td>
<td>0.63503</td>
<td>0.63775</td>
<td>0.63037</td>
</tr>
<tr>
<td>0.6</td>
<td>0.74704</td>
<td>0.74597</td>
<td>0.74176</td>
</tr>
<tr>
<td>0.7</td>
<td>0.86719</td>
<td>0.86088</td>
<td>0.86118</td>
</tr>
<tr>
<td>0.7</td>
<td>0.99709</td>
<td>0.98393</td>
<td>0.99019</td>
</tr>
<tr>
<td>0.9</td>
<td>1.13830</td>
<td>1.11642</td>
<td>1.13029</td>
</tr>
<tr>
<td>1</td>
<td>1.29239</td>
<td>1.25968</td>
<td>1.28299</td>
</tr>
<tr>
<td>( \max | \varphi_n (s) - \varphi (s) | )</td>
<td>0.03271</td>
<td>9.398 \times 10^{-3}</td>
<td>2.647 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Table 3
Approximate solutions and maximum absolute errors between \( \varphi (s) \) and \( \varphi_n (s) \) of Example 4

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \varphi_n (s) )</th>
<th>( \max { | \varphi_n (s) - \varphi (s) | } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sin(0.8s) \frac{1}{\pi^{1/5}} + \frac{147}{125} s^{4/5} )</td>
<td>( 1.33 \times 10^{-3} )</td>
</tr>
<tr>
<td>2</td>
<td>( \sin(0.8s) \frac{1}{\pi^{1/5}} + \frac{1252}{75} s^{4/5} )</td>
<td>( 2.85 \times 10^{-4} )</td>
</tr>
<tr>
<td>3</td>
<td>( \sin(0.8s) \frac{1}{\pi^{1/5}} + \frac{228872}{7565625} s^{4/5} )</td>
<td>( 9.90 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

Using Theorem 1, we can obtain explicit expressions of several lower-order approximate solutions, and the results are presented in Table 1. From Table 1, it can be found that the obtained approximations are very satisfactory, in particular for large \( n \).

Example 3. Consider Abel equation

\[
\int_0^s \frac{\varphi (t)}{(s-t)^{1/2}} dt = e^s - 1,
\]

with an exact solution \( \varphi (s) = e^s \text{ erf} (\sqrt{s}) / \sqrt{s} \).

According to Theorem 1, we can obtain explicit forms of several lower-order approximate solutions, which are omitted for saving space. A comparison between the exact and approximate solutions at ten equidistant mesh points in \([0, 1]\) is made for \( n = 1, 2, 3 \) in Table 2. From Table 2, it can be found that the obtained approximations are very accurate.

Example 4. Let us consider an example of \( f(0) \neq 0 \),

\[
\int_0^s \frac{\varphi (t)}{(s-t)^{4/5}} dt = s + 1,
\]

with an exact solution \( \varphi (s) = (1 + 1.25s) \sin(0.8\pi s) / \pi s^{1/5} \).

Now, we first solve the resulting Abel equation

\[
\int_0^s \frac{\phi (t)}{(s-t)^{4/5}} dt = s,
\]

then making use of Remark 2, the solution to Eq. (57) can be expressed as

\[
\varphi (s) = \frac{\sin(0.8\pi s)}{\pi s^{1/5}} + \phi (s).
\]

For Abel equation (57), several lower-order approximate solutions of \( \varphi (s) \) can be reconstructed from Theorem 1. Evaluated results for \( \phi (s) \) are obtained, and the corresponding \( \varphi (s) \) are listed in Table 3.
Example 5. Consider Abel equation
\[ \int_0^s \frac{\psi(t)}{(s-t)^{1/2}} \, dt = f(s) \] (59)
where
\[ f(s) = \begin{cases} \frac{4}{3} s^{3/2}, & 0 \leq s < \frac{1}{2}, \\ \frac{4}{3} s^{3/2} - \frac{8}{3} \left( s - \frac{1}{2} \right)^{3/2}, & \frac{1}{2} \leq s \leq 1. \end{cases} \] (60)

Its exact solution is
\[ \varphi(s) = \begin{cases} s, & 0 \leq s < \frac{1}{2}, \\ 1 - s, & \frac{1}{2} \leq s \leq 1. \end{cases} \] (61)

It is noted that the desired exact solution actually is continuous, but not differential at \( s = 0.5 \), which violates the assumption of Theorem 1. However, according to the method described above, lower-order approximations except for \( n = 1 \) still give quite satisfactory accuracy. This can be observed from Fig. 1, in which the exact and corresponding approximate solutions for \( n = 1, 2, 3, 4 \) are plotted. It is seen from Fig. 1 that when \( n = 3, 4 \), the corresponding approximations are very close to the exact solution, implying that the suggested methods are also applicable to a class of Abel equations having non-differential solutions.

Example 6. Finally, we consider an example of the stability of approximations of Abel inversion, i.e., consider
\[ \int_0^s \frac{\psi(t)}{(s-t)^{1/2}} \, dt = \left( 4 \frac{s^{3/2}}{3} - \frac{32}{35} \right) \left( 1 + \varepsilon \theta(s) \right), \] (62)

where \( \varepsilon \) is a small parameter and \( \theta(s) \) denotes a uniform random variable with values in \([-1, 1]\).

It is easily found that when \( \varepsilon = 0 \) the exact solution is
\[ \varphi(s) = s - s^3. \] (63)

We adopt Theorem 1 and get the second- and third-order approximations as
\[ \varphi_2(s) = s - \frac{226}{231} s^3, \] (64)
\[ \varphi_3(s) = s - s^3, \] (65)
respectively. In the following, we set \( \varepsilon = 0.01 \), and reconstruct \( \varphi(s) \) in the case of random errors. For comparison, Figs. 2 and 3 show the second- and third-order reconstructions from the right-hand side non-homogeneous term with random noise.
5. Conclusions

A new and simple approach is suggested to approximately determine the inversion of a class of Abel integral equations. The suggested method is related to the input function as well as its integrals, but not related to its derivatives; so this method is suitable for measurement data with random noise. Moreover, it can be readily implemented by symbolic computation using a personal computer. Numerical examples demonstrate that the accuracy of lower-order approximations is very satisfactory, and the nth-order approximate solution also reduces to its exact solution when the exact solution is a polynomial of degree less than or equal to n.

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References