August 29, 2018

An LP Problem and the Status of the Core

Lester G Telser

8/29/18

An LP Problem and the Status of the Core

L. G. Telser

1 Introduction

This note clarifies the relation between solutions of a certain linear programing (LP) problem and the status of the core for an n-person economy with weakly superadditive valuation functions for its coalitions. It proves the
Theorem. A solution of the LP problem cannot be inefficient if the values of the coalitions are weakly superadditive.
However, the solutions of the LP problem need not be feasible. The core is empty when this is the case.

LP Problem Min $\varphi(N)\, x$ with respect to the non negative n-vector x subject to
(1) $\varphi(S)\, x \geq v(S)$, y.
The grand coalition N has n members. A non empty coalition S is a subset of N. The value of S, v(S), is measured in money. The coordinates of the n-vector $\varphi(S)$ indicate who belongs to the coalition S. It is a row vector. Its coordinate i equals 1 if individual i belongs to S and equals 0 otherwise. All the coordinates of $\varphi(N)$ equal 1. The Lagrangian for the minimum problem is $\mathcal{L}(x, y)$.
(2) $\mathcal{L}(x, y) = \varphi(N)\, x + \sum y_S [v(S) - \varphi(S)\, x]$.
v(N) denotes the value of the grand coalition.

The LP algorithm cannot apply to solve a problem unless it has been shown that all the constraints are solvable. The minimum problem does not impose inequality (3) as a constraint in the minimum problem.
(3) $\varphi(N)\, x \geq v(N)$,
because its solvability cannot be assured. The solvability of (1) is not in doubt. Nor in doubt is the solvability of (4).

A solution of the LP problem must satisfy
(4) $\varphi(N) - \Sigma\, y_S \varphi(S) \geqq 0$, $S \subset N$, x.
Let m(S) denote the number of members of S. From inequality (4),

$$\varphi(N)\, \varphi(N)^T - \Sigma\, y_S \varphi(S) \varphi(N)^T \geqq 0,$$

Therefore,

(5) $n \geq \Sigma\, y_S\, m(S)$,

so that $1 \geq \Sigma\, y_S m(S)/n$. Complementary slackness requires equation (6).

(6) $\varphi(N)x - \Sigma\, y_S \varphi(S)x = 0$.

An implication of (6) and (1) is

(7) $\varphi(N)x = \Sigma\, y_S \varphi(S)x \geq \Sigma\, y_S v(S)$.

Let x^O denote a solution of the minimum problem. Let $B=\{S_i\}$ denote the set of coalitions in the solution.

(8) $\varphi(S_i)\, x^O = v(S_i)$ for $S_i \in B$.

Dual Problem; Max $\Sigma\, y_S v(S)$ with respect to nonnegative y_S subject to (4),

The Duality Theorem of Linear Programming applies and says

(10) $\varphi(N)\, x^O = \Sigma\, y_{S_i}\, v(S_i)$.

If $x^O > 0$, then complementary slackness implies

(10) $\varphi(N) = \Sigma\, y_{S_i} \varphi(S_i) = 0$.

In this case inequality (5) becomes an equation.

(11) $1 = \Sigma\, y_{S_i} m(S_i)/n \leq \Sigma\, y_{S_i}$ Max $\{m(S_i)/n\} = \Sigma\, y_{S_i}$

because Max $\{m(S_i)/n\}=1$.

2 Status of the Core for a Solution of the LP Problem

Now we study the relation between a solution of the LP problem and the status of the core.

Definition 1. v(.) is strongly superadditive if $v(R \cup S) \geq v(R) + v(S)$ for all R and S $\subset N$ and $R \cap S = \Phi$.

Definition 2. v(.) is weakly superadditive if $v(N) \geq v(S) + v(N-S)$ for all $S \subset N$.

For my purposes weak superadditivity is pertinent so let v(.) be weakly superadditive. For other forms of superadditivity in application of cores to the determination of wages see (Telser, 1997, chap. 6, secs. 2 and 3).

Suppose S_i and $S_j = N - S_i$ were both in the set of solutions so that we have two equations

(1) $\varphi(S_i)\, x^O = v(S_i)$ and $\varphi(S_j)\, x^O = v(S_j)$.

The second equality says $\varphi(N-S_i)\, x^O = v(N-S_i)$. It would follow that

(2) $\varphi(N)\, x^O = \varphi(S_i) x^O + \varphi(N-S_i)\, x^O = v(S_i) + v(N-S_i) \leq v(N)$.

The latter inequality is a consequence of weak superadditivity. This proves Lemma. If S_i and $S_j = N - S_i$ were both in the set of solutions, then x^O would be in the core.

Let us ask whether a solution of the LP problem is feasible.

If $R \subset N$, $N-R \subset N$ and $R \neq \Phi$, then $\varphi(R)\, x^O \geq v(R)$ and $\varphi(N-R) x^O \geq v(N-R)$ imply

(3) $\varphi(N)\, x^O = \varphi(R)\, x^O + \varphi(N\text{-}R)\, x^O \geq v(R) + v(N\text{-}R)$.
Either
(4.1) $\varphi(N)\, x^O \leq v(N)$
or
(4.2) $\varphi(N)\, x^O > v(N)$
are consistent with (3). The solution of the LP Problem is not feasible In (4.2). Indeed in this case the core would be empty. However, inequality (4.1) apparently permits an x* such that
(5) $\varphi(N)\, x^* = v(N)$ and $v(N) = \varphi(N)\, x^* > \varphi(N)\, x^O \geq v(R) + v(N\text{-}R)$.
According to (5), x* dominates x^O via the grand coalition N but not by both components of any pair of coalitions, R and N-R. Thus x* could not dominate x^O by means of any subcoalition of N such as R or N-R, unless it could satisfy at most one with equality. But (5) shows this is impossible.

Therefore, although $x^* > x^O$, it cannot dominate any solution of the LP problem. The Lemma suggests that an inefficient solution may not be possible. Indeed, suppose it were possible so that $\varphi(x^O) < v(N)$. Take the most valuable coalition in $\{S_i\}$, label it R_i.
(6) $\varphi(R_i)\, x^O = v(R_i)$ and $\varphi(N\text{-}R_i)\, x^O > v(N\text{-}R_i)$.
We obtain
(7) $\varphi(N)\, x^O = \varphi(R_i)\, x^O + \varphi(N\text{-}R_i)\, x^O \geq v(R_i) + v(N\text{-}R_i)$.
By weak superadditivity $v(N) \geq v(R_i) + v(N\text{-}R_i)$. Therefore,
(8) $v(N) - \varphi(N)\, x^O = v(R_i) - \varphi(R_i)\, x^O + v(N\text{-}R^i) - \varphi(N\text{-}R_i)\, x^O$.
By hypothesis since $v(R_i) - \varphi(R_i)\, x^O = 0$, (8) implies
(9) $v(N) - \varphi(N)\, x^O = v(N\text{-}R^i) - \varphi(N\text{-}R_i)\, x^O > 0$.
But the latter inequality contradicts the result that x^O is a solution of the LP problem. This proves the
Theorem. A solution of the LP problem satisfies
(10) $\varphi(N) x^O \geq v(N)$.
The Theorem says a solution of the LP problem cannot be inefficient if there is weak superadditivity of the valuation functions of the coalitions.

3 Summary

The core is empty when
(1) $\varphi(N)\, x^O = \Sigma\, y_{S_i}\, v(S_i) > v(N)$.
Since
(2) $\varphi(S_i) x^O = v(S_i)$ for all $S_i \in B$

condition (1) is equivalent to

(3) $\varphi(\mathrm{N})\ x^{o} = \Sigma\ y_{S_i}\ \varphi(S_i)\ x^{o} > \mathrm{v}(\mathrm{N})$.

An empty core means the economy described by the model is unable to satisfy all the requirements expressed by a solution of the LP problem.

Therefore, the structure of coalitions in the model does not determine the value of the grand coalition N. However, in a circuit core model, the grand coalition is a partition of the participants joined in legitimate circuits. A partition is an Eulerian circuit. The core constraints for the circuits in the partition are always solvable.

References

Telser, Lester G. 1997. Joint Ventures of Labor & Capital. Ann Arbor, MI : Univ. of Michigan Press.