April 2, 2018

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4/2/18

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In standard core theory no participant obtains a return above his incremental contribution to the total return. According to standard economic theory applied to a production function, the wage rate equals the value of the marginal product. Hence standard economic theory equates the incremental contribution to the return of the participant. This note has a new result. It uses core theory to study a function such that the sum of the incremental contributions exceeds the total return. The feasible payments to the participants must fall short of their incremental contributions. There is a class of strictly convex, superadditive functions such that individual's payoff is in the core although below his incremental contribution to the total return.

The total return, f(t), is increasing, strictly convex and superadditive defined on the following closed convex set T.

(1) T = {s: $0 \leqq s \leqq t$ for all $t \in \Omega^n$, the non negative orthant of E^n}.

Denote the support of f at t by f*(t). In my application with differentiable f, f*(t) = $\partial_t f(t)$. It turns out that the return to the participants is below their incremental contribution to the total return. Even so, the f(t) defined in (6) has a non empty core.

By strict convexity

$$f(t - s) - f(t) > f^*(t)\ (t - t - s) = - f^*(t)\ s.$$

Therefore,

$$f(t) - f(t - s) < f^*(t)\ s \quad \text{for all } s \leqq t.$$

For s = t we have

(2) $f(t) < f^*(t)\ t + f(0)$

By superadditivity

(3) $f(s) + f(t-s) \leq f(t)$.

Take s = t and (3) implies $f(t) + f(0) \leq f(t)$ so that

(4) $f(0) \leq 0$.

From (2) and (4) it follows that

(5) $f(t) < f^*(t)\ t$.

It follows from (5) that $f(.) \geq 0$ implies f(.) is an increasing function of t.

Inequality (5) shows that the sum of the incremental returns exceeds the total return. Nor is this all. By strict convexity only vertexes of T can yield admissible coalitions. That is, if a type i is present in a coalition, then all the type i participants are also present. A coalition s is an n-vector whose coordinates are either 0 or t_i. This is important for the f(t) defined in (6). (Cf. Telser, 1987, chap. 4, sec. 4)

Let

(6) $f(t) = t' M t$,

M is nXn matrix, M = M', positive definite and nonnegative. In addition, let M ≥ 0. It turns out that M ≥ 0 is both necessary and sufficient for superadditivity as well for a non empty core.

First, we verify that this f(t) is superadditive.

Lemma. If $M \geq 0$, then $(t - s)' M (t - s) + s' M s \leq t' M t$.

Proof. $(t - s)' M (t - s) + s' M s$

$= t' M t - s' M t - t' M s + 2 s'M s - t' M t$

$= 2 (s' M s - s' M t) = 2 s' M (s - t) \leq 0$

for all $s - t \leq 0$ because $M \geq 0$ by hypothesis.

This proves $f(t) = t' M t$ is superadditive on T.

(7) $f^*(f) = 2 t' M$

Let x denote an n-vector of payoffs to the n participants. Let

(8) $x = f^*(t) - a \geq 0$,

According to (8), the payoffs are reduced by the non negative n-vector, a.

The payoffs to the participants can cover the total f(t) if

(9) $x t = f^*(t) t - a t = f(t)$.

From (7) and (9),

(10) $x t = f(t) = t' M t = 2 t' M t - a t$.

It follows from (9) and (10) that $a t = 2 t'M t - t' M t$. Hence

(11) $a = t' M$.

To verify x is in the core we must show that

(12) $x s \geq f(s)$ for all $s \in T$ and $s \leq t$.

Now $x s = [f^*(t) -a] s = t' M s$. Hence

(13) $t'M s \geq f(s) = s' M s$

if and only if

(14) $(t - s)' M s \geq 0$

for all $s \leq t$ and $s \in T$. This is true because $M \geq 0$ by hypothesis. Therefore, as claimed, the x given by (8) is in the core.

If M had some negative terms, then inequality (10) could not be assured. Negative terms in M can be interpreted as participants who are competitors, substitutes, and positive terms as those who are partners, complements.

References

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