February 2, 2018

Space of Hmiltonian Circuits

Lester G Telser

The Space of Hamiltonian Circuits in an Odd m-Polygon

L. G. Telser

2/2/18

Introduction

This essay opens a bigger space for economics Hamiltonian circuits as the basic coalition in my core theory. How better to explain the virtue of Hamiltonian circuits in economics than to begin with a structure without circuits, a hierarchy. A hierarchy in a polygon with m vertexes has a 1-way arrow going from vertex 1 to vertex [i], i = 2, 3, … , m. Vertex[2] has no arrow going to vertex 1. It has 1-way arrows going to vertex [i], i = 3,4, … , m and so on to vertex m that has no outgoing arrows. Vertex [m] is at the bottom of the hierarchy. A hierarchy has no feed back from a lower level vertex i to a higher level vertex j < i. The top vertex can send messages to any vertex below but no message can go up from below to a higher level vertex. A hierarchy has no circuit.

The Hamiltonian circuit connecting m vertexes is

v[1]↔v[2]↔v[3]↔ …↔v[m]↔v[1].

The sequence of m arrows in this Hamiltonian circuit is

{v[1]↔v[2]},{v[2]↔v[3]}, … {v[m-1]↔v[m]},{v[m]↔v[1]}

A Hamiltonian circuit has feedback but a hierarchy has none. Only an unusual firm constitutes a hierarchy. The typical firm has Hamiltonian circuits of many sizes.

A space is a set of members organized by their relations. A finite dimensional linear vector space is a familiar example. Each vector in this space can be written as a linear combination of at most n linearly independent vectors. The members of the space of Hamiltonian circuits are the vertexes and selected directed arrows of an odd m-polygon. The arrows form Hamiltonian circuits. The pentagon has the simplest Hamiltonian space that illustrates most of the relations in the general case. One complication in the general case discussed later is present for composite m.

The tables of circuits by size explains the nature of cooperation and rivalry among circuits. An odd m-polygon has (m-1)/2 bands of circuits that form its

main partition. The circuits in each band neither overlap nor share vertexes. It is an especially useful vehicle for formal analysis of a joint enterprise.

This essay has an algorithm to make Hamiltonian circuits. It displays tables of circuits arranged by circuit size for the pentagon, septagon and nonagon. Study of these tables is indispensable for understanding the status of the core that uses thee circuits as its coalitions.

Pentagon

The first table shows the 10 arrows in the pentagon arranged in two columns by 5 rows. The second table shows all the 3-circuits. The 3rd table shows all the 4-circuits. In every table the entry in row i+1 equals the entry in row i increased by 1. All entries are adjusted Mod 5. However, 5 (Mod 5) is replaced by 5, not by zero, because indexes run from 1 to m, m=5 for the pentagon. The first term in the 4-circuit comes from the first term in the 3-circuit with 4 inserted between 1 and 2. Hence the basic element in this space is the 3-circuit (1,2,3). It is the only circuit in the 3-polygon, an equilateral triangle.

The 10 arrows in the two columns of the first table form two 5-circuits making the main partition of the pentagon.

10 Arrows

1, 2	1, 4
2, 3	2, 5
3, 4	3, 1
4, 5	4, 2
5, 1	5, 3

5 3-circuits

1, 2, 3
2, 3, 4
3, 4, 5
4, 5, 1
5, 1, 2

5 4-circuits

1, 4, 2, 3
2, 5, 3, 4
3, 1, 4, 5
4, 2, 5, 1
5, 3, 1, 2

The relation between pairs of arrows, 3-circuits and 4-circuits illuminates the nature of rivalry in a core theory where coalitions are Hamiltonian circuits. First, consider a pair of arrows {v[1],v[2]} and {v[2],v[3]}. An arrow goes from the source vertex, the first vertex in the pair, to the destination vertex, the second vertex in the pair. The pair can form a circuit of 3 arrows by including the arrow {v[3],v[1]} thus creating the 3-circuit (1,2,3), the first in the 3-circuit table. Second, say this 3-circuit wants to grow from 3 to 4 arrows. The first 4-circuit in the 4 circuit table is (1,4,2,3). Its arrows are

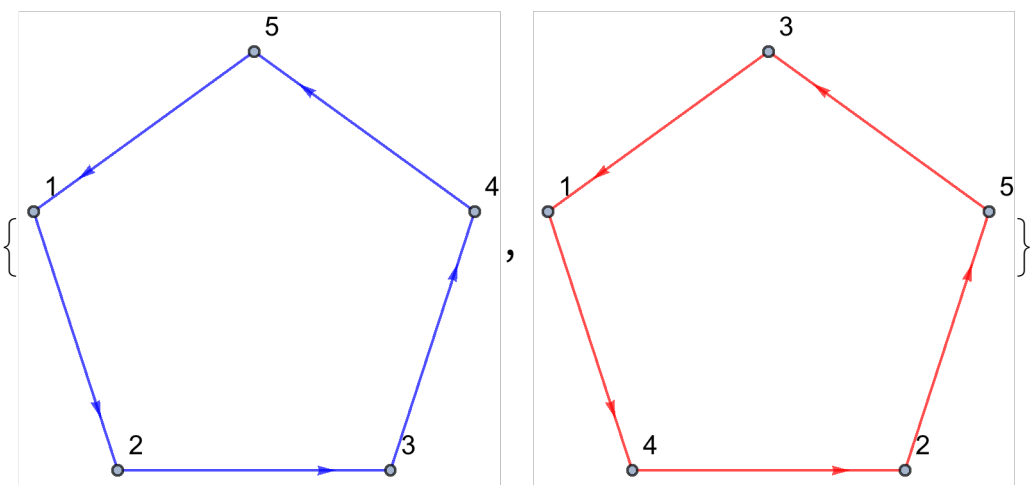
{v[1],v[4]},{v[4],v[2]},{v[2],v[3]},{v[3],v[1]}.

To obtain the 4-circuit (1,4,2,3) from the 3-circuit (1,2,3), takes two arrows {v[1,4],v[4,2]} that replace the single arrow {v[1],v[2]}. It is not true that the 4-circuit simply grows by the addition of one arrow. It needs two arrows to replace the one arrow it removes.

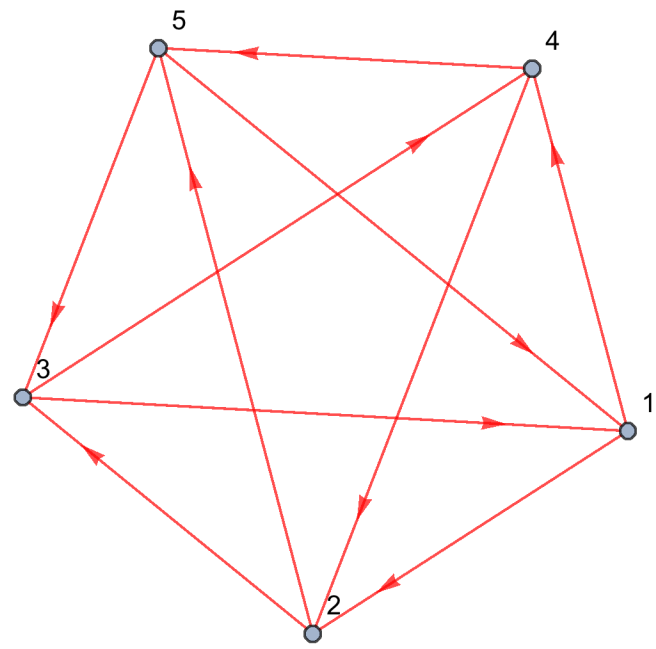
Because the arrow {v[1],v[2]} in the 3-circuit (1,2,3) does not enter the 4-circuit (1,4,2,3), let us see where it goes. TWe are led to the 3-circuit (5,1,2) that replaces its arrow {v[5],v[1]} by the pair of arrows {v[5],v[3]}, {v[3],v[1]} that obtains the 4-circuit (5,3,1,2). Moreover, the pentagon has 5 3-circuits and 5 4-circuits. The 3-circuits use 15 arrows and the 4-circuits use 20 arrows. Going from 3-circuits to 4-circuits does not cause unemployment of arrows. This is strikingly visible later when we study the nonagon and septagon.

This result is general. Going from a smaller to a bigger circuit needs more than addition of one arrow. It is a two-step process. Two related arrows replace one arrow. The algorithm for finding the structure of circuits is more concise than the preceding verbal explanation.

The main partition in the pentagon is Band[1,5]=(1,2,3,4,5) and Band[3,5]=(1,4,2,5,3) shown in the diagrams.



Band[1, 5] Band[3,5]



The next two tables show the frequency of the arrows in the 5 3-circuits. The arrows in the first table enter Band[1,5]= (1,2,3,4,5). The arrows in the second table enter Band[3,5]= (1,4,2,5,3).

Arrow Frequency by 3-Circuit

arrow	frequency
1, 2	2
2, 3	2
3, 4	2
4, 5	2
5, 1	2

arrow	frequency
1, 4	1
2, 5	1
3, 1	1
4, 2	1
5, 3	1

This completes a description of the space containing the Hamiltonian circuits of the pentagon.

Nonagon

36 Arrows of the Nonagon

1, 2	1, 4	1, 6	1, 8
2, 3	2, 5	2, 7	2, 9
3, 4	3, 6	3, 8	3, 1
4, 5	4, 7	4, 9	4, 2
5, 6	5, 8	5, 1	5, 3
6, 7	6, 9	6, 2	6, 4
7, 8	7, 1	7, 3	7, 5
8, 9	8, 2	8, 4	8, 6
9, 1	9, 3	9, 5	9, 7

30 3-circuits in the Nonagon

1, 2, 3	1, 2, 5	1, 2, 7	1, 4, 7
2, 3, 4	2, 3, 6	2, 3, 8	2, 5, 8
3, 4, 5	3, 4, 7	3, 4, 9	3, 6, 9
4, 5, 6	4, 5, 8	4, 5, 1	4, 7, 1
5, 6, 7	5, 6, 9	5, 6, 2	5, 8, 2
6, 7, 8	6, 7, 1	6, 7, 3	6, 9, 3
7, 8, 9	7, 8, 2	7, 8, 4	7, 1, 4
8, 9, 1	8, 9, 3	8, 9, 5	8, 2, 5
9, 1, 2	9, 1, 4	9, 1, 6	9, 3, 6

The table has 27 3-circuits for the nonagon in the first 3 columns. The total has 30 Hamiltonian 3-circuits. The 3 missing Hamiltonian 3-circuits (1,4,7), (2,5,8), (3,6,9) form Band[3,9] of the main partition for the nonagon. Column 4 these 3-circuits repeated 3 times. This explains the composite band 3.

90 4-circuits Nonagon

1, 4, 2, 3	1, 4, 2, 5	1, 4, 2, 7	1, 4, 2, 9
2, 5, 3, 4	2, 5, 3, 6	2, 5, 3, 8	2, 5, 3, 1
3, 6, 4, 5	3, 6, 4, 7	3, 6, 4, 9	3, 6, 4, 2
4, 7, 5, 6	4, 7, 5, 8	4, 7, 5, 1	4, 7, 5, 3
5, 8, 6, 7	5, 8, 6, 9	5, 8, 6, 2	5, 8, 6, 4
6, 9, 7, 8	6, 9, 7, 1	6, 9, 7, 3	6, 9, 7, 5
7, 1, 8, 9	7, 1, 8, 2	7, 1, 8, 4	7, 1, 8, 6
8, 2, 9, 1	8, 2, 9, 3	8, 2, 9, 5	8, 2, 9, 7
9, 3, 1, 2	9, 3, 1, 4	9, 3, 1, 6	9, 3, 1, 8

1, 6, 2, 3	1, 6, 2, 5	1, 6, 2, 7	1, 6, 2, 9
2, 7, 3, 4	2, 7, 3, 6	2, 7, 3, 8	2, 7, 3, 1
3, 8, 4, 5	3, 8, 4, 7	3, 8, 4, 9	3, 8, 4, 2
4, 9, 5, 6	4, 9, 5, 8	4, 9, 5, 1	4, 9, 5, 3
5, 1, 6, 7	5, 1, 6, 9	5, 1, 6, 2	5, 1, 6, 4
6, 2, 7, 8	6, 2, 7, 1	6, 2, 7, 3	6, 2, 7, 5
7, 3, 8, 9	7, 3, 8, 2	7, 3, 8, 4	7, 3, 8, 6
8, 4, 9, 1	8, 4, 9, 3	8, 4, 9, 5	8, 4, 9, 7
9, 5, 1, 2	9, 5, 1, 4	9, 5, 1, 6	9, 5, 1, 8

1, 8, 2, 3	1, 8, 2, 5
2, 9, 3, 4	2, 9, 3, 6
3, 1, 4, 5	3, 1, 4, 7
4, 2, 5, 6	4, 2, 5, 8
5, 3, 6, 7	5, 3, 6, 9
6, 4, 7, 8	6, 4, 7, 1
7, 5, 8, 9	7, 5, 8, 2
8, 6, 9, 1	8, 6, 9, 3
9, 7, 1, 2	9, 7, 1, 4

The 4 Bands in the main partition of the nonagon are from the arrow table.
Band[1,9]=(1,2,3,4,5,6,7,8,9); Band[3,9]=((1,4,7),(2,5,8),(3,6,9));
Band[5,9]=(1,6,2,7,3,8,4,9,5); Band[7,9]=(1,8,6,4,2,9,7,5,3)
Circuits in composite bands have no junctions, e.g., Band[3,9].

Nonagon Frequency Table For Arrows in 3-Circuits

Band	1, 9	3, 9	5, 9	7, 9
Frequency	4	3	2	1

Septagon

The first table shows the 21 arrows in the septagon. Each column is a 7-circuit of the septagon. They are the 3 bands in the main partition of the septagon. The second table includes the 14 Hamiltonian 3-circuits. The entry in row i+1 equals the entry in row i plus 1 adjusted Mod 7. The septagon has no composite bands because 7 is a prime. The septagon has 28 4-circuits. The second table applies a similar rule to its 4-circuits. It has all 28 4-circuits. The

3rd table has all 28 5-circuits. The frequency table for 3-circuits shows the distribution of the arrows in the 3 bands of the main partition of the septagon.

21 Arrows,

1, 2	1, 4	1, 6
2, 3	2, 5	2, 7
3, 4	3, 6	3, 1
4, 5	4, 7	4, 2
5, 6	5, 1	5, 3
6, 7	6, 2	6, 4
7, 1	7, 3	7, 5

14 3-Circuits

1, 2, 3	1, 2, 5
2, 3, 4	2, 3, 6
3, 4, 5	3, 4, 7
4, 5, 6	4, 5, 1
5, 6, 7	5, 6, 2
6, 7, 1	6, 7, 3
7, 1, 2	7, 1, 4

28 4-Circuits Septagon

1, 2, 5, 3	1, 4, 2, 5	1, 2, 7, 3	1, 2, 7, 5
2, 3, 6, 4	2, 5, 3, 6	2, 3, 1, 4	2, 3, 1, 6
3, 4, 7, 5	3, 6, 4, 7	3, 4, 2, 5	3, 4, 2, 7
4, 5, 1, 6	4, 7, 5, 1	4, 5, 3, 6	4, 5, 3, 1
5, 6, 2, 7	5, 1, 6, 2	5, 6, 4, 7	5, 6, 4, 2
6, 7, 3, 1	6, 2, 7, 3	6, 7, 5, 1	6, 7, 5, 3
7, 1, 4, 2	7, 3, 1, 4	7, 1, 6, 2	7, 1, 6, 4

28 5-Circuits Septagon

1, 4, 2, 5, 3	1, 4, 2, 7, 5	1, 4, 5, 6, 7	1, 6, 4, 5, 3
2, 5, 3, 6, 4	2, 5, 3, 1, 6	2, 5, 6, 7, 1	2, 7, 5, 6, 4
3, 6, 4, 7, 5	3, 6, 4, 2, 7	3, 6, 7, 1, 2	3, 1, 6, 7, 5
4, 7, 5, 1, 6	4, 7, 5, 3, 1	4, 7, 1, 2, 3	4, 2, 7, 1, 6
5, 1, 6, 2, 7	5, 1, 6, 4, 2	5, 1, 2, 3, 4	5, 3, 1, 2, 7
6, 2, 7, 3, 1	6, 2, 7, 5, 3	6, 2, 3, 4, 5	6, 4, 2, 3, 1
7, 3, 1, 4, 2	7, 3, 1, 6, 4	7, 3, 4, 5, 6	7, 5, 3, 4, 2

28 6-Circuits Septagon

1, 6, 4, 2, 5, 3	1, 6, 4, 2, 7, 5	1, 6, 4, 7, 5, 3	1, 6, 4, 2, 7, 3
2, 7, 5, 3, 6, 4	2, 7, 5, 3, 1, 6	2, 7, 5, 1, 6, 4	2, 7, 5, 3, 1, 4
3, 1, 6, 4, 7, 5	3, 1, 6, 4, 2, 7	3, 1, 6, 2, 7, 5	3, 1, 6, 4, 2, 5
4, 2, 7, 5, 1, 6	4, 2, 7, 5, 3, 1	4, 2, 7, 3, 1, 6	4, 2, 7, 5, 3, 6
5, 3, 1, 6, 2, 7	5, 3, 1, 6, 4, 2	5, 3, 1, 4, 2, 7	5, 3, 1, 6, 4, 7
6, 4, 2, 7, 3, 1	6, 4, 2, 7, 5, 3	6, 4, 2, 5, 3, 1	6, 4, 2, 7, 5, 1
7, 5, 3, 1, 4, 2	7, 5, 3, 1, 6, 4	7, 5, 3, 6, 4, 2	7, 5, 3, 1, 6, 2

Crucial for success is the choice for the first row. It must be a row that does not intersect the diagonal of the Adjacency Matrix. Either the first or the last row will do. The septagon has 28 4-, 5- & 6- circuits. The 7-circuits are the hardest. The last table shows them. The septagon has 17 7-circuits. They are in the first two columns of the table. I obtain the first row in the table from certain exchanges of the vertexes in the perimeter 7-circuit. This missing 3 7-circuits are the bands in the main partition of the septagon: (1,2,3,4,5,6,7);

(1,4,7,3,6,2,5); (1,6,4,2,7,5,3).

14 7-Circuits Septagon

1, 4, 2, 5, 6, 7, 3	1, 4, 2, 3, 6, 7, 5
2, 5, 3, 6, 7, 1, 4	2, 5, 3, 4, 7, 1, 6
3, 6, 4, 7, 1, 2, 5	3, 6, 4, 5, 1, 2, 7
4, 7, 5, 1, 2, 3, 6	4, 7, 5, 6, 2, 3, 1
5, 1, 6, 2, 3, 4, 7	5, 1, 6, 7, 3, 4, 2
6, 2, 7, 3, 4, 5, 1	6, 2, 7, 1, 4, 5, 3
7, 3, 1, 4, 5, 6, 2	7, 3, 1, 2, 5, 6, 4

Three Bands in the main partition of the Septagon
Band[1,7]=(1,2,3,4,5,6,7);
Band[3,7]=(1,4,7,3,6,2,5); Band[5,7]=(1,6,4,2,7,5,3)

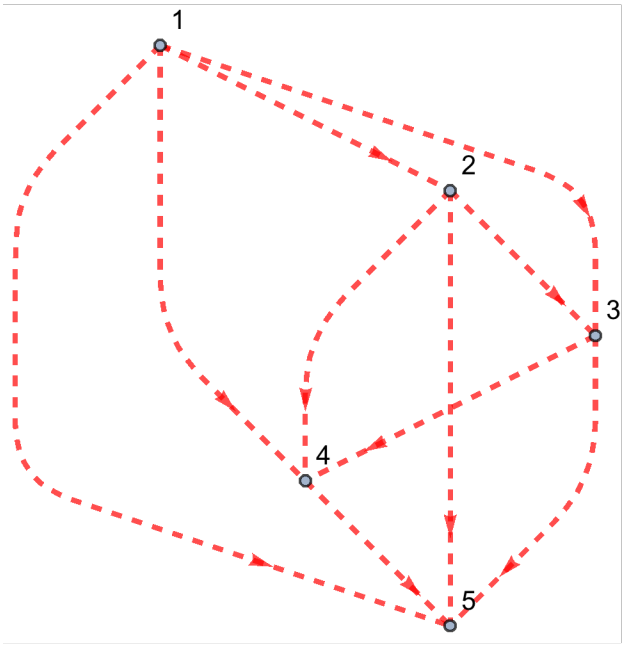
Septagon Frequency Table For Arrows in 3-Circuits

Band	1, 7	3, 7	5, 7
Frequency	3	2	1

Hierarchy Pentagon

```
h5 := {{1 ↦ 2, 1 ↦ 3, 1 ↦ 4, 1 ↦ 5, 2 ↦ 3, 2 ↦ 4,
    2 ↦ 5, 3 ↦ 4, 3 ↦ 5, 4 ↦ 5}}
```

```
fig[Flatten[h5], 3]
```



References

Telser, L. G. 2017. A Guide to Circuit Core Models. Berkeley Electronic Press

_____. 2017. Circuits as Coalitions in the Core. Berkeley Electronic Press