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Mixed Strategies and the Core Status

Lester G Telser

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L. G. Telser

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Abstract.

An economy is not a zero sum game. Only a non zero sum game can have a core but this is not enough. An economy must also produce a big enough pie to make no one worse off. A circuit core model explains how this can happen. JEL C71 Cooperative Games

Introduction

A fair coin has heads on one side and tails on the other. When tossed it lands on a side. If it is equally likely to land on either side, then one can say it is a hypothetical coin 1/2 heads and 1/2 tails, the same as an actual coin. Schrödinger's cat is a hypothetical cat unlike any actual cat. Not only is there a a theory based on this creature but it also predicts actual events very well. Some call this an 'as if' theory. As if there really is a Schrödinger's cat.

Economics also has its fake cat. A mixed strategy assigns probabilities to the alternative choices. Suppose they are a trip to New York or a trip to Chicago with probabilities p and q and p+q =1. Cincinnati lies midway between them. Suppose going to Cincinnati is the minimax of the strategic problem. A player never actually goes to Cincinnati but will go either to New York or to Chicago depending on the probabilities of the strategies behind the minimax. Suppose the payoff for Chicago is 10 and for New York is 5. If p and q are equal, then the expected payoff under the minimax would be 1/2 of 15. Actually, a player would get 5 or 10, never 7.5. Why is the minimax considered a security value?

One reason a mixed strategy seems to offer security is its secrecy. Neither a user nor his opponent knows in advance what will happen. Thus neither knows in advance the outcome of the random event that determines which strategy is actually used. Let p and q denote the probabilities of the two alternatives strategies. Even if p > q, a player would not deliberately choose the strategy with the bigger probability because this would destroy secrecy. In some versions of core theory the value of a coalition is the minimax of its payoff. It sets a lower bound on the amount acceptable to the coalition although as an expected value, like Schrodinger's cat, it is a fiction. It is as if the minimax offers security. The expected value is not visible in the real world nor is Schrödinger's cat visible in the real world. John v. Neumann criticized quantum mechanics because it includes the Cat. Yet its counterpart enters his own game theory.

Two Person Zero Sum Games

Consider how game theory treats a two-person zero-sum game. The payoff to the first person is described by the function, r = f(x,y), who controls x and the second person, his opponent, controls y. The payoff to the second person is described by the function s = g(x, y). In a zero sum game the payoffs sum to zero, r+s=0. The minimax as the security value for each person seems reasonable because there can be no gain from cooperation apart from the pleasure of playing the game. However, the saddlevalue for a zero sum game need not be zero. The saddlevalue of a skew symmetric matrix is zero but this is a special case. When the saddlevalue is not zero, the minimax for one player, σ, is positive and is $-\sigma$ for the other player. The zero sum game is unfair when its saddlevalue is not zero. Who would be willing to be the player in a zero sum game whose minimax is negative.

Economic situations are decidedly different. An economy is not a zero-sum game. Theories of the economy assume this is the case.

Status of the Core

Let N denote the grand coalition of the n players. The grand coalition can attain a return given by v(N). A coalition of players is a subset S of N. Let v(S) denote the most this coalition cn attain under the least favorable conditions. Thus v(S) is the security value of the coalition. It is not foolish to assume the most formiadable coalition facing S is N-S, the coalition of everybody outside S. Assume $R \subset (N-S)$ is the most formidable opponent to S. Whatever R can do, N-S can do since N-S includes R.

The algebra simplifies with the help of the indicator function $\varphi(.)$. It is an n-vectors whose coordinates are one or zero. Coordinate i of $\varphi(S)$ is one if player i belongs to S and is zero if player i does not belong to S. All coordinate of $\varphi(N)$ are 1 because all the players belong to the grand coalition. Assume all payoffs are nonnegative.

The n-vector x is in the core of this game if $\varphi(N)x = v(N)$ so that x is an imputation and there is no coalition S that can offer its members a feasible amount y, $\varphi(S)y = v(S)$, that is more than each would get under x, $\varphi(S)y>\varphi(S)x$ with y also an imputation, $\varphi(N)y = v(N)$. In this situation we have

(1) $\varphi(S)x < \varphi(S)y = v(S)$.

(2) $\varphi(S)x + \varphi(N\text{-}S)x = \varphi(N)x = v(N) \geq v(S) + v(N\text{-}S)$.

Because the security values of the coalitions are saddlevalues, a well-known theorem of v. Neumann shows these security values are superadditive for non overlapping coalitions. Leading cases are S and N-S. It follows from (1) and (2) that

(3) $\varphi(N\text{-}S)x > v(N\text{-}S)$.

A similar argument applies to y.

(4) $\varphi(S)y + \varphi(N\text{-}S)y = \varphi(N)y = v(N) \geq v(S) + v(N\text{-}S)$.

Because $\varphi(S)y=v(S)$, it follows from (4) that

(5) $\varphi(N\text{-}S)y > v(N\text{-}S)$.

Consequently, x does not dominate y via N-S nor does y dominate x via N-S.

We continue study of the status of the core using two linear programming problems. In these problems y now refers to nonnegative Lagrangian multipliers.

Primal. min $\varphi(N)x$ with respect to $x \geqq 0$ subject to $\varphi(S)\, x \geq v(S)$ for all $S \subset N$.

The Lagrangian for the Primal is

$$\mathcal{L}(x,y) = \varphi(N)x + \Sigma y_S[v(S)\text{-}\varphi(S)x]$$

A solution must satisfy

(6) $\partial_x \mathcal{L} = \varphi(N) - \Sigma\, y_S \varphi(S) \geqq 0$, x

Dual. Max $\Sigma y_S v(S)$ with respect to $y_S \geq 0$ subject to (6).

The Lagrangian for the Dual is

$$\mathcal{K}(x,y)= \Sigma y_S v(S) + [\varphi(N) - \Sigma\, y_S \varphi(S)]x$$

A solution must satisfy

(7) $\partial_y \mathcal{K} = v(S) - \varphi(S)\, x \leq 0, \quad y_S$.

By the Duality Theorem of linear programming,

(8) min $\varphi(N)x$ = Max $\Sigma y_S v(S)$.

Theorem. If $v(N) \geq$ min $\varphi(N)x$ = Max $\Sigma y_S v(S)$, then the core is not empty.

Corollary. The core is empty if $v(N) <$ min $\varphi(N)x$ = Max $\Sigma y_S v(S)$.

How do we handle an empty core?

One way is case by case that prohibits any coalition that causes an empty core. A different approach employs the geometry of regular polygons. The simplest example is a triangle. It has 3 one-way arrows: $\{1 \longrightarrow 2, 2 \longrightarrow 3, 3 \longrightarrow 1\}$ that make a circuit of its three vertexes. I define a coalition for this triangle to be this circuit.

More generally, a regular polygon with m vertexes has $\binom{m}{2}$=m(m-1)/2 one-way arrows that can form simple circuits with from 3 to m arrows. These arrows connect certain pairs of vertexes. They are the elements of a core model whose coalitions are simple circuits. This approach leads to a non empty core under appealing conditions that relies on the gains from cooperation.