Schrodinger's Cat Lives in an Empty Core

Lester G Telser
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L. G. Telser

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Part 1 Schrodinger’s Cat

I shall not deprive the reader of the pleasure of reading about Schrodinger’s cat. If x is binary so that x=1 with probability p and equals 0 with probability 1- p, then the mean value of x, \( \mu = 1 \cdot p + 0 \cdot (1-p) = p \) has no existence in reality because it cannot be seen. Schrodinger describes the present state, dead or alive. The outcomes of a coin toss are visible. The present state of the coin, is its two sides, heads on one side and tails on the other. Both sides are observable in the present.

A more complicated illustration of Schrodinger’s cat is a system of inconsistent inequalities. It has no solution in the same sense that Schrodinger's cat does not exist. Because such a system of inequalities occurs when the core is empty, one can explain it in simple terms with algebra and a diagram. The result proves a theorem: Schrodinger's cat lives in an empty core.

\[
\begin{align*}
C & : y[1] \beta + y[2] \gamma \\
\end{align*}
\]

The two lines AB and CD in the diagram are the boundaries of two different regions.
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All the parameters, a’s, b’s, \(\beta\) and \(\gamma\) are positive. The x’s are nonnegative variables that sum to 1. The diagram shows no points in the positive quadrant above AB and below CD. Therefore, the inequalities have no solution in this region. Now introduce two more nonnegative variables \(y[1]\) and \(y[2]\) that also sum to 1. The mean of \(\beta\) and \(\gamma\) is \(\mu\).

1. \(\mu = y[1] \beta + y[2] \gamma\).

defines a line in the positive quadrant between AB and CD. But the points on this line do not actually exist. They correspond to Schrodinger’s cat. There is no solution in this region that can satisfy both inequalities. Either there can be a point on AB or a point on CD but not both.

An empty core describes the situation that the set of inequalities which an imputation must satisfy cannot do so. We cannot actually have a point in an empty core. The reason speaks for itself.

Schrodinger’ s cat is at the surface of a deeper problem, the consistency of quantum mechanics of his time. This theory intends to understand real events. An inconsistent theory may glimpse some truth but only for a wary user.

An economic theory with an inconsistent set of inequalities cannot explain reality. It may serve another purpose, as a description of an empty core. It helps to know what prevents a core and how to repair the model without a core. Usually, while many repairs exist, it is worthwhile to see if reality appears in accord with any. One kind of repair of the empty core is a fake, a random mixture of inconsistent conditions. Even when it gives satisfactory empirical results, it offers no understanding of them.
Part 2 Two Empty cores: Voting and the Treasure of Sierra Madre

2.1 A Simple Example

A simple example illustrates an empty core. A 3-person voting game with majority rule has 4 winning coalitions, three are 2-person coalitions and the fourth is the 3-person coalition. Singletons are losers and get zero. The payoff to player i is $x_i$. The prize is 1. The three 2-person core constraints are

1. $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_2 + x_3 \geq 1$.

The most available for distribution is 1+a. It is the feasibility constraint

2. $x_1 + x_2 + x_3 = 1 + a$.

Sum the 3 inequalities in (1) and obtain $x_1 + x_2 + x_3 \geq 3/2$. Therefore,

3. $1 + a = x_1 + x_2 + x_3 \geq 3/2 \iff a \geq 1/2$.

There is an implication of an empty core if $0 \leq a < 1/2$.

The arithmetic mean payoff in any 2-person coalition is at least $1/2$. In the 3-person coalition the mean is $(1+a)/3$. This is below the mean payoff to the members of 2-person coalitions when a is below $1/2$. The amount available to the 3-person coalition is above 1 when a is positive. The total return to the coalitions increases but not enough to erase the conflict between the global optimum and the local optima. This failure is typical for an empty core. The conflict also means that each 2-person coalition can give each member more than is feasible for the 3-person coalition.

2.2 Feasibility

The lower bound on the least acceptable payoffs to members of a coalition g must be feasible. When the value of a coalition equals the upper bound, the constraint on the payoffs to the members of that coalition must be an equality because claims for payoffs above the upper bound are not feasible. A small change of the example at the beginning of sec.1 shows when a system of equations has no nonnegative solution. 

$x_1 + x_2 = 1$, $x_1 + x_3 = 1$, $x_2 + x_3 = 1$, $x_1 + x_2 + x_3 = 1$-
a, a > 0.  
Sum the 4 equations and obtain \( x_1 + x_2 + x_3 = 4/3 - a = 1 \)

There is a non negative solution if and only if \( 0 < a \leq 1/3 \). Hence \( a > 1/3 \) implies there is no solution. An empty core results when adding a player to a coalition does not raise the value of the coalition enough to prevent the incumbent members from being made worse off.

2.3. Treasure of Sierra Madre

Another interpretation of the 3-person game uses the Treasure of Sierra Madre. It has three individuals. The Treasure is under a rock too big for one person to move but not too big for two. If the feasible coalitions are either two or three individuals, then the 3-person coalition gets no more than any 2-person coalition. The inequalities are the same as in the voting game so there is no core.

The core for this 3-person game is empty because it has no undominated imputation. The value of any 3-person coalition is 1 and is the same for any 2-person coalition. The following diagram illustrates dominance.

![Diagram](image.png)
Figure 3

One imputation divides the treasure equally among the three. The graph of \( v(N) \) shows this imputation. Another imputation give nothing to the first person and divides the treasure equally among 2 and 3. This imputation is shown by the graph of \( v(S) \). It dominates the first imputation via this 2-person coalition. It would easy to show with the coalition for players 1 and 3 that \( v(S) \) is also dominated.

Part 3 A Phony Resolution: The Algebra Behind Schrodinger’s Cat

The core conditions in 2.1 are repeated in matrix form. With \( a = 1/4 \) the core is empty.

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\begin{pmatrix}
\end{pmatrix}
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\[
\begin{vmatrix}
A \\
X \\
v
\end{vmatrix}
\]

\[
\begin{vmatrix}
0
\end{vmatrix}
\]

YA with nonnegative y’s that sum to 1 and with \( Y.v \) yields two solutions obtained by eliminating either \( X \) or \( Y \) but, of course, not both. These solutions are shown below but they are phony. They are not actually attainable.

\[
A := \{\{1, 1, 0\}, \{1, 0, 1\}, \{0, 1, 1\}, \{1, 1, 1\}\}
\]

\[
X := \text{Array}[x, 3]
\]

\[
v := \{1, 1, 1, 1 + 1/4\}
\]

\[
A.X
\]


\[
\text{MatrixForm}[A.X - v]
\]
\[
\begin{pmatrix}
\end{pmatrix} \leq \begin{pmatrix} \vdots \end{pmatrix} \leq 0
\]

\[
Y := \text{Array}[y, 4]
\]

\[
Y
\]

\[
\{y[1], y[2], y[3], y[4]\}
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\[
Y.v
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\[
\text{Reduce}[\{Y.A.X = Y.v, \text{Sum}[y[j], \{j, 1, 4\}] = 1\}, X]
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\text{||}
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\text{||}
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\[
2 + y[4] \neq 0 \&\& x[1] = \frac{4 + y[4]}{4 (2 + y[4])} \}
\]

\[
\text{||}
\]

\[
\text{Sum}[y[j], \{j, 1, 4\}] = 1
\]

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\]
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\[
\text{Reduce}\left[\{Y.A.X = Y.v, \text{Sum}[y[j], \{j, 1, 4\}] = 1\}, Y\right]
\]
\[
\left(-1 + 4 x[1] \neq 0 \&\& \right.
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\[
\left(x[1] = \frac{1}{4} \&\& -1 + 4 x[2] \neq 0 \&\& \right.
\]
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\]
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\]
\[
\left(x[2] = \frac{1}{4} \&\& x[1] = \frac{1}{4} \&\& -1 + 4 x[3] \neq 0 \&\& \right.
\]
\[