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How Default Probability Affects Returns on Loans

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1. Introduction

Even the simplest kind of default brings out difficulties. In this case many identical borrowers are vulnerable to default similar to being struck by lightening, an independent random event that could affect any borrower with equal probability. Because an affected borrower cannot repay the loan, the lender incurs a cost. This random event is an insurable risk that can be shared equally among all the borrowers. A private enterprise in the lending business could survive if it includes a suitable insurance premium that covers the expected loss from default in the terms it offers potential borrowers. The formulas to do so are derived here. They are neither obvious nor hard.

Owing to the nature of the relations among private enterprises, default in one can affect many others. Enterprises buy and sell among themselves. A seller often extends credit to the buyer. This credit is an asset that can serve as collateral for a loan to the seller from his supplier. The network of assets and liabilities can cause default of one to spread to many others. Default is no longer an independent random event. It leads to harder problems not touched here. Still it is useful to treat correctly the easier case.

2. Basic Premise

Premise. The future is not predictable. The more distant is the future, the less predictable.
Corollary. Survival is not predictable.
This explains why a good now is more valued than a good later. An individual may not survive long enough to obtain any benefit from goods that will become available later. This also explains a positive rate of

interest. An exchange of a good now for a good later takes place only if the lender obtains compensation in return for the uncertain prospect of the good later. This explanation of positive time preference gives a plausible estimate of the real rate of interest. It is the reciprocal of life expectancy. An expected life of 80 years implies a real interest rate of 1.25 per cent. See Telser and Graves (1972, chap. 2, sec. 1, pp. 50-1),

3. Random Walk

Let S(t) denote the state of the economy at time t. It is a multivariate vector that takes a random walk. Components of S(t) could be continuous functions of t but none would have derivatives of any order. Weierstrass gives an example that Graves (1946, chap. 7, sec. 6, pp. 125-6) describes in detail.

Among the variables in S(t) are those pertinent to the means of payment. Even if one were bold enough to assume these are continuous functions of time, few would go so far as to assume they are not affected by unpredictable events. The simplest version of such events is a first-order random walk. Hence these variables would have no derivatives. No rule assuming they have derivatives is defensible.

4. Default Free Loans

At time s a lender holding a stock of means of payment x(s) in nominal terms could buy a bundle of goods y(s). By lending the nominal amount x(s) at time s to be repaid the nominal x(t) at time t > s, the lender forgoes buying this bundle for y(s) in nominal terms. In compensation the lender would require that the nominal amount x(t) received at time t could buy a bigger bundle than x(s). Let r > 0 denote the relative increase. Equation (1) states this concisely.

(1) $$\frac{y(t)}{x(t)} = \frac{y(s)}{x(s)}(1+r).$$

Assume the cost of the original bundle in nominal terms at time t satisfies

(2) $$x(t) = (1+\lambda)\, x(s).$$

Substitute expression (2) into (1) so that

(3) $$\frac{y(t)}{(1+\lambda)\,x(s)} = \frac{y(s)}{x(s)}(1+r).$$

Multiply through equation (3) by $(1+\lambda)$ x(s) and obtain

(4) $y(t) = y(s)\,(1+r)(1+\lambda)$.

Divide through equation (4) by y(s) and it becomes

(5) $y(t)/y(s) - 1 = [(1+r)(1+\lambda)] - 1$.

The nominal rate of interest is

(6) $[(1+r)(1+\lambda)] - 1$

This analysis assumes repayment of the loan is certain. This proves

Lemma 1. The nominal interest rate = $r + \lambda + r\,\lambda$ for a default free loan. The product of the terms r and λ is often omitted. The resulting error is considerable in a hyperinflation when λ is big. The so-called nominal interest rate could be negative when prices are much lower at time t than at time s. Although $\lambda < 0$ in this case, it remains always true that $1+\lambda \geq 0$. Hence the lower bound on λ is -1.

It must not escape notice that prices affect the real value only of nominal assets not encumbered by liabilities. These assets are default free. A loan by a private business to a private business results in an asset equal to a liability. Changes in prices affect both equally so no Pigou effect is present on them.

5. How Default Probability Affects Loans

Loans that can default pose subtle challenges. It is best to start from widely acceptable assumptions as follows.

Assume $r > 0$ and $1 + \lambda > 0$. Hence $\lambda > -1$. Rewrite equation (4) from the preceding section

(1) $y(t) = (1+r)(1+\lambda)y(s)$.

Equation (1) shows the return on a default free loan at time t on a loan made at time $s < t$. Let α denote y(t)/y(s).

(2) $\alpha = y(t)/y(s) = (1+r)(1+\lambda)$.

The assumptions imply that α must be positive.

Lemma 2. $r > 0$ and $1+ \lambda > 0$ imply $\alpha > 0$.

Let θ denote the probability the loan is repaid so that $1-\theta$ is the probability of default. A necessary condition to offer a loan is that

default is not certain. Consequently, the following formula is valid if and only if there is a positive probability of repayment so that $\theta > 0$. The lender subtracts the expected loss of default from the loan. This resembles an insurance premium. Hence the loan to the borrower at time s is

(3) $y(s) - (1-\theta)y(t)$.

The borrower who does not default repays y(t). The lender's return is

(4) $$\begin{cases} y(t)+(1-\theta)\,y(t) & \text{with probability } \theta \\ 0+(1-\theta)\,y(t) & \text{with probability } (1-\theta) \end{cases}$$

Write E[L] for the expected return on the loan.

(5) $E[L] = \theta\,[y(t)+(1-\theta)y(t)]+(1-\theta)[0+(1-\theta)y(t)]$.

The algebra in (5) reduces to (6).

(6) $E[L]= y[t]$.

Therefore, the expected rate of return on a loan subject to default is given by the following ratio.

(7) $y(t)/[y(s)-(1-\theta)y(t)]$.

To obtain a more informative expression, divide through formula (7) by y(s) and obtain

(8) $[y(t)/y(s)]\,[1/[1 - (1-\theta)y(t)/y(s)]$.

Let β denote the return on a loan that is not certain to default. Using α from (2), formula (8) becomes

(9) $$\beta = \alpha\,\frac{1}{1-(1-\theta)\,\alpha}$$

Formula (9) shows that $\beta=\alpha$ if $\theta = 1$.

Lemma 3. $\beta > 0$ if and only if $1-(1-\theta)\alpha > 0$.

Proof. By Lemma 2 and (9) it is evident that β is positive if and only if $1-(1-\theta)\alpha$ is positive.

Corollary. $1-(1-\theta)\alpha > 0$ if and only if

(10) $(1-\theta) < 1/\alpha$.

The upper bound on the default probability in (10) is meaningful provided $\alpha \geq 1$. Moreover, there would be no positive loan unless (10) were satisfied. Suppose $\alpha < 1$. Hence $(1+r)(1+\lambda) < 1$. Because $(1+\lambda)>0$, it follows that λ is subject to the following upper bound.

(11) $(1+r)(1+\lambda) < 1 \Longleftrightarrow \lambda < - r/(1+r)$.

Because $-1 < \lambda$ and $-1 < -r/(1+r)$, inequality (11) seems no problem in itself but since it allows α below 1, it leads to a major difficulty in

inequality (10). For $\alpha < 1$ removes a meaningful upper bound on the default probability. It seemingly says a loan sure to default would be offered if the price level at the time of repayment would be low enough, that is, would satisfy inequality (11). This completes the proof of the

Theorem. If $r > 0$, $\lambda > -r/(1+r)$, then the nominal interest rate on a default free loan is positive, the probability of default is bounded above by $1/\alpha$, and $\beta > \alpha$.
Proof. The hypotheses imply that the denominator in (9), $1 - (1-\theta)\alpha < 1$ and the numerator $\alpha > 1$.

It is instructive to deduce the relation between θ, the probability of repayment, and α, the nominal return on a default free loan, keeping (2) in mind. Inequality (10) implies that
(12) $\theta > 1 - 1/\alpha$.
A glance at the figures yields a useful perspective.

α	5 / 4	3 / 2	2	5 / 2	3	4
lower bound θ	1 / 5	1 / 3	1 / 2	3 / 5	2 / 3	3 / 4
upper bound $1 - \theta$	4 / 5	2 / 3	1 / 2	2 / 5	1 / 3	1 / 4

As $\alpha \to \infty$, inf $\theta \to 1$ and sup $(1- \theta) \to 0$. The variable that most likely affects the nominal return is the price level. The pattern in the table says that repayment becomes more likely, the higher the price level. Some may prefer to state this in terms of inflation.

References

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