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What Linear Models of an Economy Can Teach Us

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1 First Principles

Mixed strategies make sense in some economic applications such as principal-agent problems and cartels but not in models of an economy. In a 2-person game the payoff to the row player who chooses row i from the mxn matrix $A = [a_{i,j}]$ is $a_{i,j}$ if the column player chooses column j. Because both players make their choices simultaneously, they gain by concealing their strategies. This inclines them to use mixed strategies. Hence x_i is the probability the row player chooses row i and y_j the probability the column player chooses column j. It follows that the expected payoff to the row player, E[row], becomes

(1) $E[\text{row}] = x^T \text{Ay} = \sum_{j=1}^{n} \left(\sum_{i=1} x_i\, a_{i,j}\right) y_j,$

(2) $\text{x,y} \geqq 0,\ \sum_{i=1}^{m} x_i = 1,\ \sum_{j=1}^{n} y_j = 1.$

If the row and column players are in opposition, then the row player chooses x that maximizes his expected payoff given the choice of the column player. The column player chooses y to minimize the row player's expected payoff given his choice of x. The Minimax Theorem asserts there is an m-vector x_0 and an n-vector y_0 such that for all vectors x and y that satisfy the conditions in (2),

(3) $x^T \text{Ay}_0 \leqq x_0^T \text{Ay}_0 \leqq x_0^T \text{Ay}.$

The term $x_0^T \text{Ay}_0$ is the saddle value of A and the pair (x_0, y_0) is a saddle point. The saddle value is unique, but not the saddle point. A mixed strategy has at least two positive coordinates. A pure strategy has only one equal to 1. A pure strategy is deterministic but a mixed strategy is not. Given the payoff matrix A, finding the saddle value and the saddle

points is not hard for modern computers. The interpretation of the saddle points as probabilities in a game between adversaries is clear. It is altogether different for an economy.

2 An Economic Interpretation of a Payoff Matrix

To show why it makes no sense to interpret x and y as probabilities in a model of the economy, we start with an economic interpretation of the matrix A. Let the n columns of A refer to goods and the m rows to resources. Resources are inputs and goods are outputs. Let $a_{i,j}$ describe how much resource i is used per unit of good j. Figure 1 shows 2 goods and 6 resources. The coordinates of the point Ri are $(a_{i,1}, a_{i,2})$. Although it is not quite correct to say that one unit of good 1 requires $a_{i,1}$ units of resource i and that 1 unit of good 2 requires $a_{i,2}$ units of the same resource i, it is not too misleading to do so. A correct description comes later in Section 5. The axes show the quantities of goods 1 and 2. The feasible quantities of the two goods are the points in the positive quadrant below the locus of line segments connecting the points {R5, R6, R3, R2, R1, R4}. This sets the stage for Figure 2 that displays traps for an unsuspecting economist.

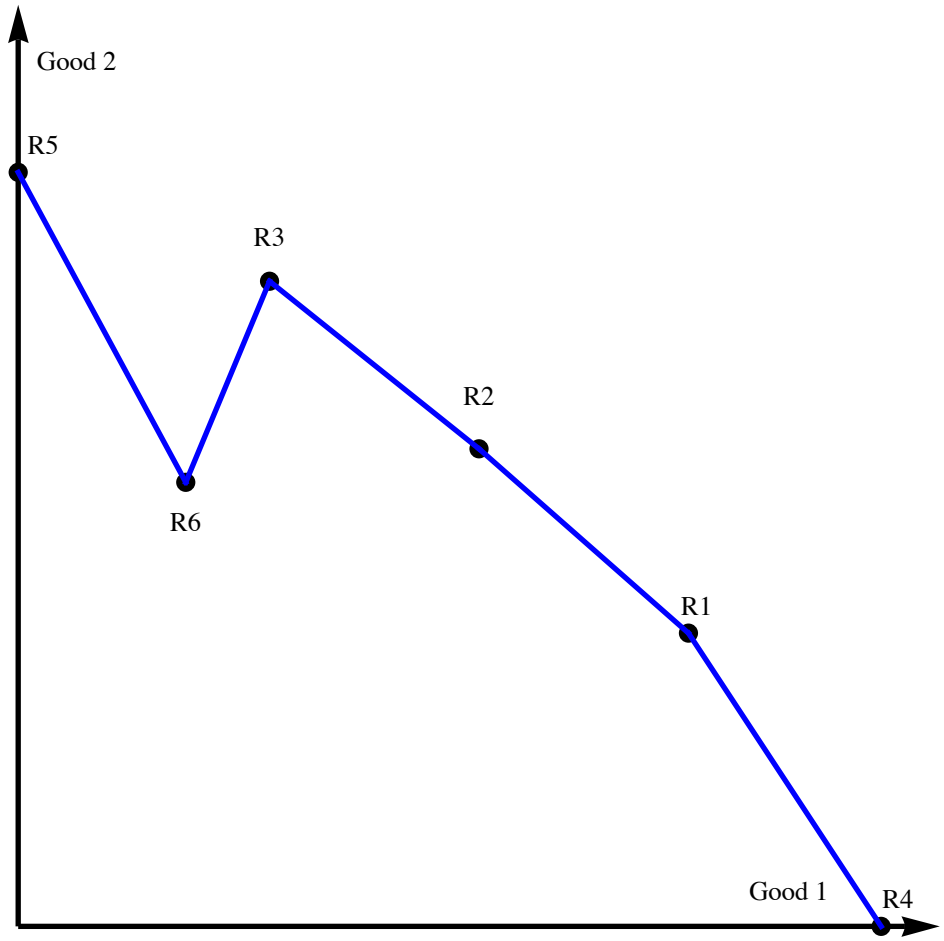

Figure 1

Figure 2 focuses on the non convex part of the feasible set. The three feasible points A, B and C do not dominate each other. The points on the line segment joining the two feasible points A and B are outside the set of feasible points apart from the end points A and B. All the points on the sub segment A1, B1 dominate the feasible point C. There is the practical question. Is it possible to realize the points on the line segment AB that are outside the feasible set?

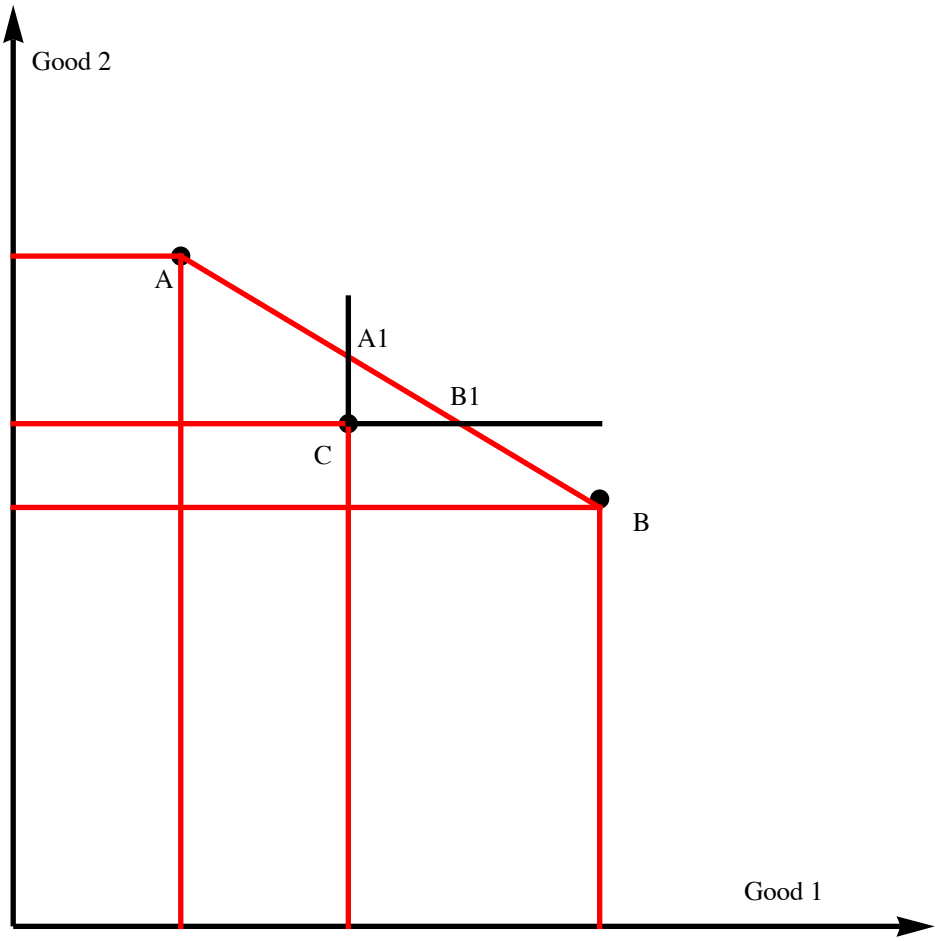

Figure 2

One interpretation goes this way. Production occurs at point A some fraction of the time and at point B the remaining fraction of the time. Therefore, points on the segment A1, B1 are weighted averages of the outputs produced at A and at B. However, this interpretation is incomplete because it is silent about what happens at one point while the other is idle. Does an idle activity incur costs?

A second Interpretation takes a more complicated road. It asserts the production processes can run at any desired level because all resources are perfectly flexible, none is fixed. This says resources are continuously divisible and can be used at any level from zero to the maximum available amount. The second interpretation implies production occurs under constant returns to scale. This needs explanation.

Let F denote the stock of resources and G the stock of goods. The flow of services from F, dF/dt, produces the flow of services from G, dG/dt.

Each flow is subject to an upper bound given by the capacity of the source. The model says flows are continuously divisible between zero and their upper bounds. However, in the actual economy not all stocks of resources can satisfy these conditions. For example, a generator of electricity is either idle or active. When it is idle it makes no electricity. When it is active it runs at capacity. Generators cannot run at any rate between 0 and their capacity. Hence the output of electricity from a generator is not continuously divisible. Although oil refineries can usually run continuously, the rate must be above a certain level. Simplicity is the enemy of accuracy.

Stocks wear out. Stocks can deteriorate as they age. Stocks require repair. Stocks incur storage costs. Therefore, it is wrong to assume given levels of stocks. Even the stock of knowledge presents complications for linear models of inputs and outputs.

A third interpretation goes more deeply into the problem and offers better results. It faces more directly the issues raised by a linear model of input and output. It also has the considerable advantage of delivering determinate solutions.

3 Special Linear Models of Inputs and Outputs

The existence of solutions to a linear programming problem is assured only by the nature of the constraints because the objective is a linear function of the control variables so it lacks finite bounds. We shall study two related linear problems. The first version is as follows. The minimal required amount of good j is g_j. The input of resource i is $x_i \geqq 0$. The cost per unit of resource i is $c_i \geqq 0$. The objective minimizes the total cost of resources used to satisfy the minimal requirements of the outputs of goods. Hence the problem seeks the minimum of $x^T c$ with respect to nonnegative x subject to the following inequality (1).

(1) $\quad x^T A \geqq g^T$

The Lagrangian for this linear programming problem is defined in equation (2). It introduces the n-vector of nonnegative Lagrangian multipliers y. These n variables are the n shadow prices of the n goods.

(2) $L_1(x, y) = x^T c + (g^T - x^T A) y.$

A solution of this LP problem, if one exists, must satisfy

(3) $c - A y \geqq 0.$

Indeed, inequality (3) is the constraint for the dual problem that seeks the maximum of $g^T y$ with respect to nonnegative y subject to (3). A good is free so that its shadow price is zero, if the solution yields more than its required amount. This is the typical case in a linear programming problem because it usually does not satisfy every constraint (1) exactly as an equality.

The second linear programming problem takes a different approach. Now the n-vector y is the output of the n goods instead of their shadow prices. The nonnegative coordinates of the given m-vector f are the available resources. Prices of the n goods are the given nonnegative coordinates of the n-vector p. The objective maximizes the value of the n goods by choosing that y which does not use more than the available amounts of the m resources. Hence the solution maximizes $p^T y$ with respect to nonnegative y subject to

(4) $A y \leqq f$.

The Lagrangian for this problem is defined by equation (5). It introduces the m-vector of shadow prices, x, for the m resources.

(5) $L_2(x, y) = p^T y + x^T (f - \mathrm{Ay}).$

If this linear programming problem has a solution, then it must satisfy the inequalities in (6).

(6) $p^T - x^T A \leqq 0$

In version 2 of the linear programming problem, x is the m-vector of shadow prices of resources but in version 1, x is the m-vector of resource inputs to the n process of production. The objective of the dual problem for version 2 minimizes the sum of the shadow values of the m resources subject to the constraint (6). For version 2, a solution of this dual problem requires the n-vector of outputs of the n goods to satisfy the m inequalities in (4).

4 The Saddle Value and the Version 1 LP Problem

The expected return to the *X*-player is x^T Ay. The inequalities for the saddle value of A are given by (1)

(1) $x^T \mathrm{A}y_o \geqq x_o^T A\, y_o = \sigma \geqq x_o^T A\, y.$

The strategy for the *X*-player is a nonnegative m-vector x whose coordinates sum to 1. For the *Y*-player it is a nonnegative n-vector y whose coordinates sum to 1. Condition (2) shows how the *Y*-player prevents the *X*-player from getting more than the saddle value, σ.

(2) $\Sigma_i\, x_i\, a_{i,j} > \sigma \Longrightarrow y_j^o = 0$ and $y_j^o > 0 \Longrightarrow \Sigma_i\, x_i\, a_{i,j} = \sigma.$

Condition (3) shows how the *X*-player prevents the *Y*-player from forcing his expected return below the saddle value.

(3) $\Sigma_j\, a_{i,j}\, y_j < \sigma \Longrightarrow x_i^o = 0$ and $x_i^o > 0 \Longrightarrow \Sigma_j\, a_{i,j}\, y_j = \sigma.$

The celebrated Minimax Theorem of von Neumann supplies equation (4) that is equivalent to the inequalities (1).

(4) $\max_x \min_y x^T \mathrm{A}y = \min_y \max_x x^T \mathrm{A}y$

Let I_n be an n-vector of 1's and I_m an m-vector of 1's. The relations between the saddle value of A and the Version 1 LP problem are given in the following definitions.

(5) $\sigma\, I_n^T = x_0^T A = g_0^T > g^T$ and $\sigma\, I_m = A\, y_0 = c_0 < c.$

Hence the saddle value is an upper bound on g and a lower bound on c. Define the feasible sets for x and y in the Version 1 LP problem.

(6) $X = \{x : x^T A \geqq g\}$ and $Y = \{y : A\, y \leqq c\}.$

It is evident from inequalities (5) that $x_o \in X$ and $y_o \in Y$ give the implication that the constraint sets *X* and *Y* are both feasible so that the version 1 linear programming problems are feasible. This proves a useful sufficient condition for the existence of a solution to the primal and dual linear programming problems given by version 1. By the duality theorem of linear programming, it follows that

(7) $\min x^T c = \max g^T y$ for all $x \in X$ and $y \in Y$.

Equality (7) says the minimum cost of the resources used to make the goods y equals the maximal shadow value of these goods. The inequalities (5) also yield a pair of interesting results. Given that the coordinates of the saddle points x_0 and y_0 sum to 1, (4) implies that

(8) $\sigma = g_0^T y_0 > g^T y_0$ and $\sigma = x_0^T c_0 < x_0^T c.$

The solutions of an LP problem are on the face of a simplex that is a convex combination of a finite number of vertexes. In the case

illustrated by Figure 2, the simplex can be a line segment joining vertexes A and B. Although the line segment is a weighted average of these 2 vertexes, it also includes the two vertexes themselves. When the simplex has more than one vertex, any one of them could be a solution. Consequently, a *vertex* in the feasible set is always a solution of an LP problem of this type.

5 A General Linear Programming Model of Inputs and Outputs

The input-output matrix of this model offers a logical framework for cost-benefit analysis of a complicated linear economy. The model starts with a given stock of resources capable of yielding a continuous flow of services bounded above by the capacity of the stock of resources. It assumes lower bounds on the required quantities of goods.

Let the n-vector b denote the unit benefits derived from the n goods. Let the m-vector c denote the unit costs of the m services from the available resources. The coordinates of the m-vector f are the upper bounds on the resources. The coordinates of the n-vector g are the lower bounds on the quantities of the required goods. Let all coordinates of f and g be nonnegative so that f≥0, g≥0 and neither f nor g = 0. Hence at least one coordinate of f and of g must be positive. The nonnegative m-vector x shows the inputs from the stock of resources used to produce the n-vector y of outputs of the goods. It takes $a_{i,j}$ units of resource i services to make one unit of good j. The m X n matrix $A = [a_{i,j}]$ describes the technology. Assume A≥0. As we shall see, it is also necessary to assume each column of A has at least one positive term in order to sidestep certain anomalies. The objective is the maximum net benefit from the m resources used to produce the n goods subject to the constraints given by the linear inequalities in (1).

(1) $x^T A - g^T \geqq 0$ and $A y \leqq f$.

At the outset the inequalities (1) raise the question of whether the available resources can satisfy the minimal demand requirements.

Lemma. The economy described by the mXn input output matrix A, the stock of available resources given by the m-vector f ≥ 0 and the

required output of goods given by the n-vector g ≥ 0 is feasible if and only if the inequalities (1.1) can be satisfied.

(1.1) $f^T A \geqq g^T$ and $A g \leqq f$.

Proof.

Sufficiency is obvious. To prove necessity suppose that

(1.2) $f^T A < g^T$ or $A g > f$.

In either case it would not be possible to satisfy all the constraints of the economy given by (1.1) unless both inequalities in (1.2) were false. Therefore, the economy is feasible only if

(1.3) $f^T A \nless g^T$ and $A g \ngtr f$.

The two inequalities in (1.3) are equivalent to (1.1) as was to be shown.

These conditions that ensure feasibility of the economy are reasonable. The constraints cannot be met unless it is possible to satisfy all the minimal demands, namely g, by using all the available services of the available resources, namely f.

Let us walk around the Lemma to learn more about it. The inequality

(2) $\sum_{i=1}^{m} x_i a_{i,j} \geq g_j$

implies that no resource is essential to produce good j unless column j has only one positive term. Moreover, suppose resource 1 were an input to the production of every good so that $a_{1,j} > 0$ for all j=1,2, ... , n. It would follow that

(3) $0 < \sum_{j=1}^{n} a_{i,j} g_j \leq f_1$

if at least one g_j were positive. Therefore, the Lemma implies that if any resource is an input to every process of production, then the minimum amount of that resource must be positive. It is also useful to see what the Lemma says about a situation in which some resource is also a good, say $f_1 = g_1$. It would follow that

(4) $(a_{11} - 1) f_1 + \sum_{j=2}^{n} a_{1,j} g_j \leq 0$

and since $\sum_{j=2}^{n} a_{1,j} g_j \geq 0$, given that $f_1 > 0$, then

(5) $a_{11} \leq 1$.

As we shall see, inequality (2) becomes an equation if the shadow price of good j is positive. Assume this is the case so that

(6) $\sum_{i=1}^{m} x_i a_{i,j} = g_i$.

It follows from equation (6) that if at least 3 coefficients of the resource inputs are positive for the production of good j, then the relative amounts of these inputs can change. It follows that these inputs can be substitutes for each other. To illustrate, assume the first 3 coefficients are positive and let Δx_i, i=1,2,3 denote changes in the inputs so that $\Delta g_j = 0$.

(7) $\Delta x_1\, a_{1,j} + \Delta x_2\, a_{2,j} + \Delta x_3\, a_{3,j} = 0.$

Let $\Delta x_1 = \lambda\, \Delta x_2$ such that λ is an arbitrary positive scalar. Replacing Δx_1 in (7) by this expression we obtain

(8) $\Delta x_2 \left(\lambda a_{1,j} + a_{2,j}\right) + \Delta x_3\, a_{3,j} = 0.$

Equation (8) implies

(9) $\Delta x_3 / \Delta x_1 = -\left(\lambda\, a_{1,j} + a_{2,j}\right) / a_{3,j}.$

However, because λ is arbitrary, no fixed relation between the inputs of factor 2 and 3 is required to produce a given quantity of good j. Therefore, none of the 3 resources is essential for the production of good j. It follows that substitution among resources dominates for any good that utilizes the services of at least 3 resources.

The Lagrangian for this problem is equation (10). The n-vector of shadow prices for the n goods is η. The m-vector of shadow prices for the m resources is ξ.

(10) $L(x, y, \xi, \eta) = b^T y - x^T c + \left(x^T A - g^T\right)\eta + \xi^T (f - A y).$

If there is a solution, then it must satisfy

(11) $b^T - \xi^T A \leqq 0$ and $-c + A\eta \leqq 0.$

The first inequality in (11) says that the shadow cost of producing good j is not less than its unit benefit. This is inequality (12).

(12) $\sum_{i=1}^{m} \xi_i\, a_{i,j} \geqq b_j.$

If $\sum_{i=1}^{m} \xi_i\, a_{i,j} > b_j$ then $\eta_j = 0$. If $\eta_j > 0$, then $\sum_{i=1}^{m} \xi_i\, a_{i,j} = b_j$. The shadow price of good j is zero if its shadow cost exceeds its unit benefit. If the shadow price is positive, then its shadow cost equals its unit benefit. For the second inequality in (11), the shadow revenue for resource i does not exceed its unit cost, (13),

(13) $\sum_{j=1}^{n} a_{i,j}\, \eta_j \leqq c_i.$

If $\sum_{j=1}^{n} a_{i,j}\, \eta_j < c_i$, then $\xi_i = 0$. If $\xi_i > 0$, then $\sum_{j=1}^{n} a_{i,j}\, \eta_j = c_i$. These

conditions say the shadow value of a resource i is zero if the shadow revenue it can generate does not cover its unit cost. The shadow value of resource i is positive only if its shadow revenue can cover its unit cost.

The objective of the dual problem minimizes $\xi^T f - g^T \eta$ with respect to nonnegative ξ and η. The duality theorem says that

(14) $\max\left(b^T y - x^T c\right) = \min\left(\xi^T f - g^T \eta\right)$.

Theorem. Let (y_o, x_o) and (ξ_o, η_o) be solutions of the primal and dual problems. Therefore,

(15) $b^T y_o - x_o^T c = \xi_o^T f - g^T \eta_o = 0$.

Proof. The actual input of resource i X shadow value of resource i equals the shadow cost of resource i X the given quantity of resource i.

(16) $\xi_i^o \Sigma_j a_{i,j} y_j^o = \xi_i^o f_i$.

The actual output of good j X shadow price of good j equals the required output of good j X shadow price of good j.

(17) $\left(\Sigma_i x_i^o a_{i,j}\right) \eta_j^o = g_j \eta_j^o$.

Therefore, (16) says that $\xi_i^o x_i^o = \xi_i^o f_i$ and (17) says that $y_j^o \eta_j^o = g_j \eta_j^o$.

Summarizing, we have

(18) $b^T y_0 - x_0^T c = \eta_0^T y_0 - \xi_0^T x_0 = \xi_o^T f - \eta_0^T g$.

Since $\xi_0^T x_0 = \xi_0^T f$ and $\eta_0^T y_0 = \eta_0^T g$, the latter two equalities in (18) assert in effect that Z=-Z. This is possible if and only if Z = 0. Since

(19) $Z = b^T y_0 - x_0^T c$,

this proves the desired result. □

Because the maximum net benefit is zero, it may seem that any departure from the best inputs and outputs would lead to a negative net benefit. This inference would be correct if and only if the pair (x_o, y_o) given by the solution of the primal were unique. However, linear programming problems need not have unique solutions. Suppose the primal had another solution (x_1, y_1). All convex combinations of these two solutions would also yield a maximum net benefit equal to zero. Only feasible departures from these solutions would reduce the net benefit to a loss.

In reality an economy has many resources and many goods. Because the constraints take the form

(20) $x^T A \geq g^T \geq 0$ and $0 \leq A\,y \leq f$ with $f \geq 0$,
it seems possible that some coordinates of g and f could be zero. The situation on the resource side is more complicated. At the outset note that $\sum_{j=1}^{n} a_{i,j}\, y_j^o < f_i$ and $f_i = 0$ is impossible because $a_{i,j}\, y_j \geq 0$. Hence the case when $f_i = 0$ could happen only when
(21) $\sum_{j=1}^{n} a_{i,j}\, y_j^o = f_i$ and $\xi_i^o > 0$.
Even so, since at least some goods are produced so that some y_j^o ' s are positive, it would be possible to have $f_i = 0$ only if for these y's, the $a_{i,j}$' s were zero and for the positive a's, the corresponding y_j' s would be zero. However, according to the Lemma,
(22) $0 \leq \sum_{i=1}^{m} a_{i,j}\, g_j \leq f_i$
and $\max_j \{a_{i,j}\, g_j\} > 0$. It follows from (22) that $f_i > 0$ for all i. Therefore, by virtue of the Lemma, the lower bounds are positive for all resources. Hence if $\xi_i^o > 0$, then since $f_i > 0$, $\max_j \{a_{i,j}\, y_j^o\} > 0$ and $\xi_i^o\, f_i > 0$.

Next, consider the demand constraint to see whether some g_j could be zero.
(23) If $\sum_{i=1}^{m} x_i^o\, a_{i,j} > g_j$, then $\eta_j^o = 0$. If $\eta_j^o > 0$, then $\sum_{i=1}^{m} x_i^o\, a_{i,j} = g_j$.
By hypothesis, $x_i^o\, a_{i,j} \geq 0$ for all i and j. It is an implication of (21) that $\eta_j^o\, g_j > 0$ if $\eta_j^o > 0$. Suppose that $\eta_j^o > 0$ and $g_j = 0$ so there is the question of whether good j could have a positive shadow price if the minimum quantity wanted were zero. That is, could we have
(24) $\eta_j^o > 0$, and $\sum_{i=1}^{m} x_i^o\, a_{i,j} = 0$?
The Lemma applies to (24) and says that for good j
(25) $\sum_{i=1}^{m} f_i\, a_{i,j} = 0$.
However, the Lemma also implies that the minimally available amounts of all m resources are positive, that is, $f_i > 0$ for all i. It follows that to satisfy (25) would require $f_i\, a_{i,j} = 0$ for all i, given good j. This is possible only if $a_{i,j} = 0$ for all i and for this given j. Hence good j could not be produced at all and yet its shadow price could be positive. Several conditions on the input-output matrix can rule out this anomaly. Perhaps the simplest says that each column of A has at least one positive term. The next result does the job.

Corollary. If A, f and g satisfy the hypotheses of the Lemma, then each column of A has at least one positive term.

6 Conclusions

Let the economy satisfy the hypothesis of the Lemma so that the constraints are feasible. There exist optimal inputs and outputs that maximize the net benefit. The maximum net benefit equals zero. The shadow prices of the optimal outputs cover the total costs of the inputs evaluated at the shadow unit costs. The model determines the relative, not the absolute, shadow prices. This follows because multiplying all nominal variables including unit costs of resources and unit benefits of goods by a positive scalar does not change the solution. Because the model assumes constant returns to scale, it also follows that multiplying the upper bounds on resources and the lower bounds on the required outputs of goods by the same positive scalar would multiply all the optimal inputs and outputs by that positive scalar without changing the relative quantities.

1. If good i is produced, then its shadow cost equals its benefit.
2. If resource i is utilized, then it yields a shadow revenue that suffices to cover its unit cost.
3. If good j is not produced, then its shadow cost of production exceeds its benefit.
4. If resource i is idle, then the shadow revenue it would generate would not cover its unit cost.