November, 2010

II Distribution of Share Valuations

Lester G Telser, *University of Chicago*

II. Distribution of Share Valuations

L. G. Telser

Abstract. A model of the market in shares of common stock enhances understanding of the empirical results in Part I. Asset Pricing JEL G12

■ 1 Introduction

Ownership of a corporation lies in the hands of its share holders in proportion to the number of their shares. A share holder may sell his shares to whom he pleases without the consent of any other share holder. Anyone may buy shares from a current share holder without the consent of any other share holder. A transaction in shares may occur between two or more parties on terms they themselves agree on without the consent of anybody apart from the parties themselves. Partnerships are different. How they differ from corporations explains the rules for corporations. No partner may buy or sell shares in the partnership without the consent of the other partners. Among the reasons for this condition is that partners are liable for the obligations and debt of the partnership. Corporate share holders are not liable for the obligations or debts of the corporation. This is why corporate shares are fungible but shares of partnerships are not.

The conclusion may appear paradoxical. Given that the current share owners may sell or buy its shares without the consent of anybody, it follows that a corporation must win approval of at least 50 percent of its present share holders or their agents for its important decisions. Therefore, I am pleased to report a major result of my model in the following
Theorem. Let A(v) denote the cumulative distribution function (cdf) of the valuations of the shares held by the current share holders of a corporation. Hence A(v) is a nonnegative, nondecreasing function of v that approaches 1 as v increases. It is a stable distribution if and only if there is a positive number α such that A(v) = 0 for all v < α, A(v) is discontinuous at v=α where it jumps from 0 to a number not less than 1/2 and A(v) $\geq$ 1/2 for all v $\geq$ α. Therefore, A(v) is upper semi continuous at v=α.

In a stable cdf of valuations at least half the current share holders must believe that their shares are worth at least α. This means there is substantial agreement among current shareholders in a company with respect to the value of its shares because ownership in a corporation is voluntary. Each share holder decides on his own to remain an owner by virtue of his freedom to buy or sell his shares in a competitive market.

A model of the market for shares of common stock in a company begins with the assumption that current and potential share holders have in mind valuations of their shares. They compare their valuations to the current share price. If this price is below someone's valuation, then he is inclined to buy. If the price is above his valuation and he currently owns the stock, then he is inclined to sell his stock. For the sake of simplicity I shall ignore short sales and assume they are not allowed. Someone who

does not currently own stock in the company would not buy it if his valuation is below its current price. Therefore, current non owners presumably value the stock less than the current owners.

The next question is immediate. Why should a current owner retain his holdings of the stock if he values it below its current price? it would also follow that no current share owner would value his stock below the current price. This would imply that all current owners have the same valuation that always equals the current share price. This raises a difficulty since the evidence is unkind to this hypothesis. First, some current share holders sell their shares when the current share price goes up. Second, when one company bids for shares in another company, not all current owners of the target company accept the bid of the would be acquirer. The acquiring company must raise its bid enough to buy the number of shares it wants. Therefore, I reject the notion that all current share holders value their shares at the current price. Nor is this all. Apart from the evidence against this hypothesis there is another objection. To assume that share holders' valuations always equal the share price does not explain the share price.

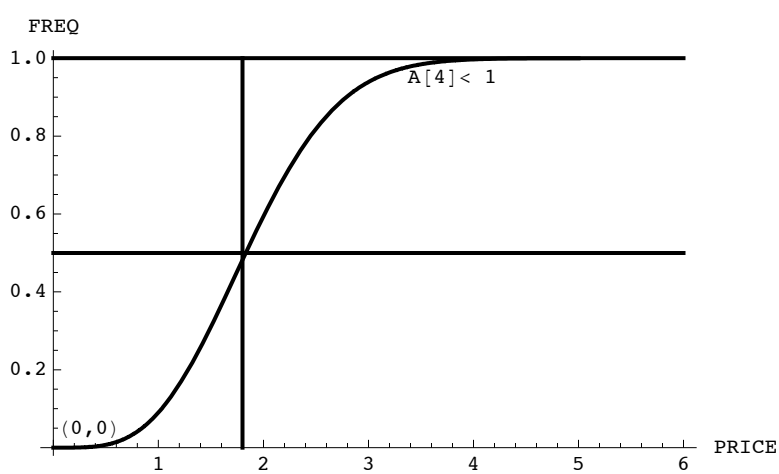
■ 2 Assumption 1

Assumption 1. There is a continuum of valuations for shares in a company described by the cumulative distribution function (cdf)

(1) $A(v) = \int_{\alpha}^{v} a(x)\, dx$

in which a(x) > 0 for all x such that $0 \leq \alpha \leq x$ and A(x) →1 for as x increases.

According to Assumption 1, no current share holder values his stock below α. Figure 1 exhibits a valuation function that starts at A[0]=0 and increases continuously as it approaches the upper bound, 1. The lower bound of x is 0. Its upper bound can be as big as you please.



3 Assumptions 2 and 3

Assumption 2. There is a continuum of valuations for shares of the company by the current non owners described by B(v) as follows:

(2) $B(v) = 1 - \int_0^v b(x)\,dx$

in which b(x) > 0 for all x such that $0 \le x \le \gamma$ and b(x) = 0 for all $x \ge \gamma$.

Assumption 3. If $0 < \alpha$, then $\gamma < \alpha$.

Assumptions 1-3 imply that valuation of any current non owner is less than the valuation of any current owner.

4 Stable CDF for Current Share Holders

Definition. A CDF of valuations for the present share holders of stock in a company is stable if it can persist in a market in which *all* valuations are always available for trade.

To claim all valuations are tradeable means the market has a continuum of valuations. Consequently, the number of traders in the market is the same as the measure of real numbers, the continuum. The principal theorem is next.

Theorem. Let A(v) denote the cumulative distribution function (cdf) of the valuations of the shares held by the current share holders of a corporation. Hence A(v) is a nonnegative, nondecreasing function of v that approaches 1 as v increases. It is a stable distribution if and only if there is a positive number α such that A(v) = 0 for all $v < \alpha$, A(v) is discontinuous at $v=\alpha$ where it jumps from 0 to a number not less than 1/2 and $1 \ge A(v) \ge 1/2$ for all $v \ge \alpha$. Hence A(v) is upper semi continuous at $v=\alpha$.

Proof. Because 1-A(v)<1/2 if $v > \alpha$ and $A(\alpha) \ge 1/2$, the market cannot clear at a price of α. A fortiori it cannot clear at a price above α. Therefore, no trade can occur at any price $\ge \alpha$ among all the incumbent share holders. Since A(v) cannot change, it is stable □

So far this model is silent about an important aspect of the current share owners' valuations. If current share owners have different valuations, then those current owners who value shares more would buy shares from those current owners who value shares less. Let us see why. Choose any p such that $p > \alpha$. Now A(p) equals the fraction of current share owners who would sell their shares at price of p or higher. Since 1-A(p) is the fraction of current share owners who value their shares above p, trades between these two groups of present share owners could occur at a price that satisfies the following equation:

(3) A(p) = 1 - A(p).

The solution is any median of the cdf A(v). Do not overlook the adjective 'any' because a median need not be unique.

Figure 2 helps follow the analysis. The cdf of current share holders, A[v], is shown by the curve starting at A[0]=0 and increasing to the upper bound at 1. The curve 1-A[v] for current share holders is symmetric around the horizontal line at 1/2. It starts at (0,1-A[0]) and decreases to its lower bound of 0. This curve for 1-A[v] shows the fraction of valuations greater than or equal to v. At the price =

2 that is the median, half the current share holders, those on the curve through A[2], value their holdings worth 2 at most while those on the curve 1-A[v] at v = 2, value their shares worth 2 at least. Hence both buyers and sellers would accept the price 2, the median. Consequently, the cdf of the valuations for the current share holders in Figure 1 cannot persist. The next step seeks and finds what cdf can persist.

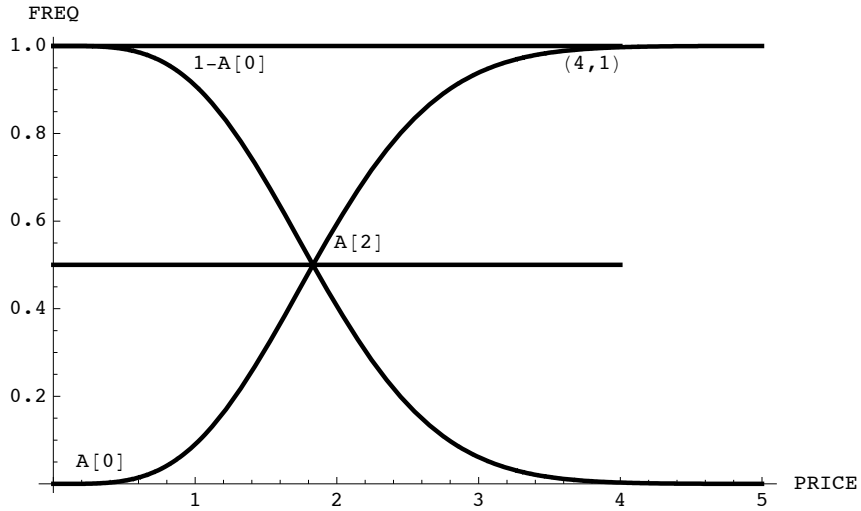


Figure 2

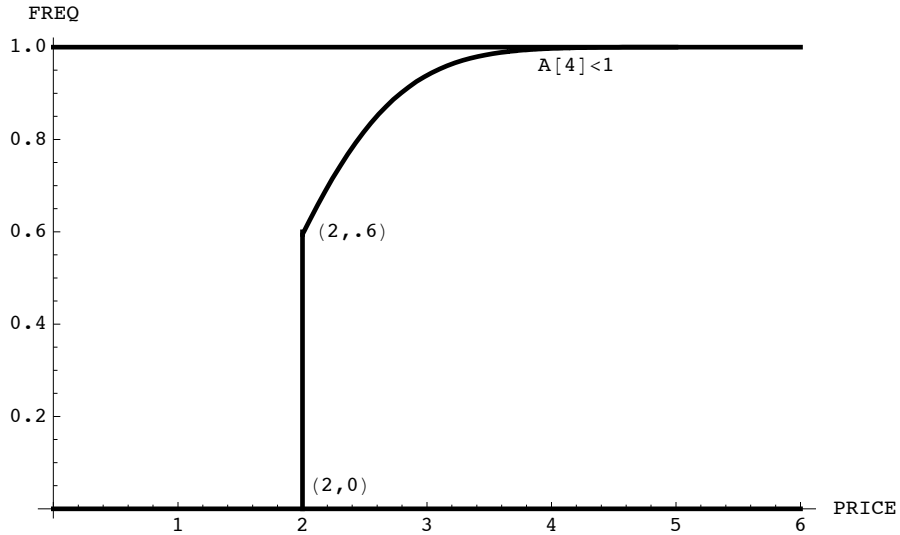


Figure 3

With the help of Figure 3 we can deduce the properties of a stable distribution of valuations among existing share holders. Figure 3 shows that the stable cdf for current share holders has 3 parts. The first part is the horizontal line from the origin to (2,0). There are no current share holders with valuations below 2. The second part of the cdf is the vertical line from (2, 0) to (2, 0.6). The third part starts at (2, 0.6) and moves on the curve to the upper bound 1 as shown by A[4]<1. Up to 3/5 of the current valuations equal 2, the rest are on the curved part of the cdf. No trade is possible between those whose valuations are 2 and those whose valuations exceed 2 because there would

require more offers than bids at 2. In this fashion the Figures illustrate the theorem on sustainable CDF's.

5 The Market in Shares

The preceding model describes the stable distribution of valuations such that all current share holders value their shares more than the non share holders. Therefore, there would be no trade between current share holders and current non share holders. To complete the model assume that traders in the market are a finite random sample from the stable cdf of valuations of current share holders. This sample must come from the cdf of current share holders because no valuation drawn at random from the cdf of non share holders can exceed any valuation drawn at random from the cdf of current share holders. Yet this seems counter to the claim that the cdf of current share holders is stable. That is, if all the valuations in the market come from the stable cdf of valuations and trade occurs, then it would seem to follow that the cdf of valuations for current owners would change and so could not persist. However, trade among a finite sample of current share holders does not alter the cdf of the continuum of valuations.. A finite sample, even a countably infinite sample, has measure zero. It is of negligible size relative to the continuum of valuations. A finite sample from a continuum of valuations that implies a finite number of shares changes hands and has no discernible effect on the underlying cdf of the continuum of valuations.

The empirical study in Part I has data on the daily volume of trade and the number of shares outstanding. It shows that the volume is usually a very small fraction of the number of shares outstanding.

We observe not only that changes in share prices of a company evoke no response from most current share holders but also that the higher the current price, the more shares are offered for sale by current owners. Similarly, the lower the current price, the more shares are bought by current owners. A numerical example can explain this. The pertinent data for 4 traders drawn at random from the population of current share holders are in the following table. The middle column shows their valuations. Traders 1 and 2 would be willing to buy a total of 2 shares at a price not to exceed 130. Traders 3 and 4 would be willing to sell up to 2 shares at a price not less than 100. This is room for mutually advantageous trades at a price between 100 and 130.

Next consider a somewhat more complicated case. Because transactions are costly and because capital gains are taxable to the extent they are not offset by capital losses, what happens in the market is consistent with the hypothesis that each trader sets upper and lower bounds on his valuations.
Assumption 4. $p < v_i - b_i$ implies buy; $p > v_i + c_i$ implies sell; and $v_i - b_i \leq p \leq v_i + c_i$ implies stand pat.
The columns Lower Bound and Upper Bound show $v_i - b_i$ and $v_i + c_i$.

Trader	Lower Bound	Valuation	Upper Bound
1	109	131	150
2	121	130	140
3	88	100	110
4	95	97.5	100

We can use the numbers in the table to construct supply and demand schedules as step functions from which we can deduce the market clearing price and quantity. Figure 4 shows the supply schedule as an increasing function of offers and the demand schedule as a decreasing function of bids. While Trader 3 values his shares at 100, he would be willing to sell a share at any price above 110 and buy one share at any price below 88. Trader 4 values his shares shares to be worth 97.5. He would be willing to sell one share at any price above100 and buy one share at any price below 95. Figure 4 shows the market clears with one share sold by Trader 3 to Trader 2 at a price between 100 and 121 .

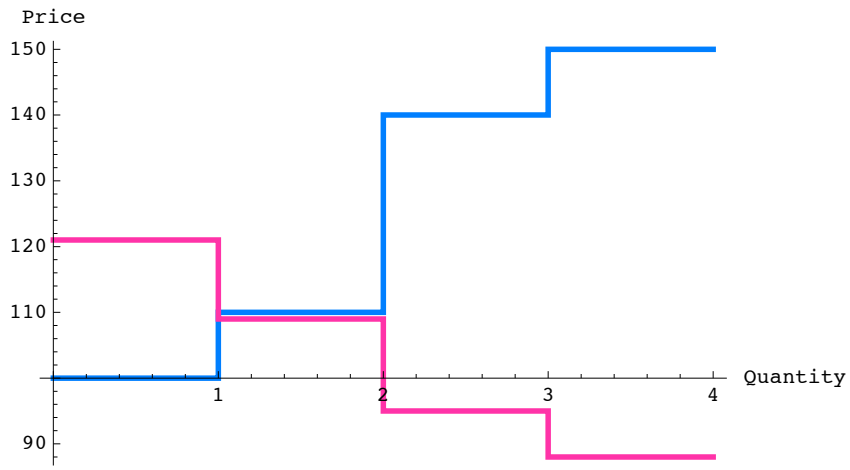


Figure 4

Because the cdf of share holder's valuation, the source of the random samples, is stable, it is plausible to conjecture that the asymptotic distribution of market clearing prices is not normal, is bounded below by the median and that the mode is the lower bound. Simulations may be the easiest way to learn more about the sampling distribution.