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In his presentation at the categories Session at Oberwolfach in 1973, Tierney defined the continuous reals for a topos with a natural numbers object (he called them Dedekind reals). Mulvey studied the algebraic properties of the object of continuous reals and proved that the construction gave the sheaf of germs of continuous functions from X to $\mathbf{R}$ in the spatial topos $Sh(X)$.

This paper presents the results of the study of the topological properties of the continuous reals with an emphasis on similarities with classical mathematics and applications to familiar concepts rephrased in topos terms.

The notations used for the constructions in the internal logic of a topos conform to that of Osius [11]. For what is needed of basic topos theory the reader is refered to the early sections of Freyd [5] and Kock and Wraith [7], or to Lawvere [8] for a quick introduction with less detail.

Useful lists of intuitionistically valid inferences may be found in Kleene [6] on pages 118, 119 and 162.

1. Definition and Characterizations of the Reals.

In a topos with a natural numbers object $\underline{N}$, we can form the object of integers $\underline{Z}$ as a ring with underlying object $\underline{N}+\underline{N}^+$, where $\underline{N}^+$ is the image of the successor map s. If we take the $\underline{N}^+$ summand as the positives, then the isomorphism of $\underline{N}$ with $\underline{N}+1$ gives rise to the validity of

$$\forall_{z\epsilon\underline{Z}}\ ((z \text{ is positive}) \vee (z=0) \vee (-z \text{ is positive})).$$

The rational numbers object $\underline{Q}$ is the ring of quotients of $\underline{Z}$ obtained by inverting the positives. The positive integers give rise to positive rationals and hence an order relation $<$. Trichotomy for the integers then implies the validity of

$$\forall_{q\epsilon\underline{Q}}\ ((q>0) \vee (q=0) \vee (q<0)).$$

Once we have the rationals we may define the reals in a number of inequivalent ways. The characterizations of the objects defined in various topoi make the continuous reals of significant interest.

DEFINITION 1.1. The *object of continuous reals*, $\underline{R}_T$, is the subobject of $P\underline{Q}\times P\underline{Q}$ consisting of pairs $(\underline{r},\overline{r})$ satisfying the following conditions:

1° $\forall_{q\in \underline{Q}}(q\in\underline{r} \iff \exists_{q'\in\underline{Q}}(q<q' \wedge q'\in\underline{r}))$.

2° $\forall_{q\in \underline{Q}}(q\in\overline{r} \iff \exists_{q'\in\underline{Q}}(q>q' \wedge q'\in\overline{r}))$.

3° $\forall_{q,q'\in\underline{Q}}((q\in\overline{r} \wedge q'\in\underline{r}) \Rightarrow q>q')$.

4° $\forall_{n\in\underline{N}}\exists_{q,q'\in\underline{Q}}(q\in\underline{r} \wedge q'\in\overline{r} \wedge q'-q<\frac{1}{n})$.

EXAMPLES.

In the topos of sheaves on a topological space X, Mulvey proved in [10] that $\underline{R}_T$ is the sheaf of germs of continuous functions from X to R. His techniques may be used to get several other characterizations.

For a measurable space (X,Σ) we may construct a topos on the site with category Σ (morphisms are inclusions) and covers countable families

$$\{B_i \to B\} \text{ with } \cup B_i = B.$$

As for sheaves on a topological space, we can show that global sections of $\underline{R}_T$ in this topos are real-valued functions. The first condition is enough to guarantee that the functions are measurable into the extended reals with measurable subsets generated by the set of intervals $(-\infty, q)$ with q rational. The second condition gives extended-real-valued functions measurable

with respect to the σ-algebra generated by the set of intervals $(q,+\infty)$ with q rational. Combined with the remaining conditions this is enough to guarantee Borel measurability.

Similarly, given a uniform space $(X,\mathfrak{U})$, we may define a site with the open subsets of the associated topological space as category (again the morphisms are inclusions) and covers generated by the uniform covers. The resulting topos has $\underline{R}_T$ the sheaf of germs of uniformly continuous functions from X with the associated locally fine uniformity to R with the additive uniformity.

For a measure space (X, Σ, μ) there are two interesting topoi. The first, studied by Scott in [12] and [13] in the guise of Boolean valued models of set theory, has category Σ with morphisms inclusions, and covers given by countable collections

$$\{B_i \to B\} \quad \text{with} \quad \mu(B - \cup B_i) = 0 .$$

He identified the continuous reals as that sheaf having as sections over a measurable subset X' the set of all random variables on X'; that is, equivalence classes of measurable functions to R under the equivalence relation with

$$f \sim g \quad \text{if} \quad f = g \text{ almost everywhere.}$$

The second construction uses the same category but takes the collection of covers generated by the set of all countable collections $\{B_i \to B\}$, such that :

$$\text{either } \cup B_i = B \quad \text{or} \quad \lim \mu(B_i) = \mu(B).$$

This gives the sheaf of random variables with equivalence classes under the relation :

$$f \approx g \quad \text{if the stationary sequence } \{f\} \text{ converges to } g \text{ in measure.}$$

There are other constructions of the reals which may be of interest in a topos. Dedekind's original definition, Cauchy's definition, and a definition traced back as far as Lorenzen by Staples giving a constructive form of the Dedekind definition are all possible.

DEFINITION 1.2. The *object of Cauchy reals*, $\underline{R}_C$, is the quotient of the subobject of $\underline{Q}^{\underline{N}}$ given by those f satisfying the statement

$$\forall_{n \in \underline{N}} \exists_{m \in \underline{N}} \forall_{h,k \in \underline{N}} ((k > m \wedge h > m) \Longrightarrow |f(k) - f(h)| < \tfrac{1}{n})$$

by the equivalence relation with f equivalent to g if

$$\forall_{n \in \underline{N}} \exists_{m \in \underline{N}} \forall_{k \in \underline{N}} (k > m \Longrightarrow |f(k) - g(k)| < \frac{1}{n}).$$

EXAMPLE. In sheaves on the unit interval, $\underline{R}_C$ is the sheaf of germs of locally constant functions. Here $\underline{N}$ and $\underline{Q}$ are the sheaves of locally constant N and Q valued functions. Since the interval is locally connected, the formation of $\underline{R}_T$ also gives locally constant functions. The two restrictions guarantee that the values are in fact real numbers. This shows that $\underline{R}_C$ is distinct from $\underline{R}_T$.

DEFINITION 1.3 (Staples [15]). The *object of Staples cuts* $\underline{R}'_S$ is the subobject of $P\underline{Q} \times P\underline{Q}$ consisting of those pairs (S, T) satisfying the following conditions :

1° $\exists_{q\in \underline{Q}}(q\in S) \wedge \exists_{q'\in \underline{Q}}(q'\in T)$.

2° $\forall_{q,q'\in \underline{Q}}((q\in S \wedge q'\in T) \Longrightarrow q < q')$.

3° $\forall_{q\in \underline{Q}} \forall_{s\in \underline{Q}}((s\in S \Longrightarrow s<q) \Longrightarrow \forall_{t\in \underline{Q}}(t>q \Longrightarrow t\in T))$.

4° $\forall_{q\in \underline{Q}} \forall_{t\in \underline{Q}}((t\in T \Longrightarrow t>q) \Longrightarrow \forall_{s\in \underline{Q}}(s<q \Longrightarrow s\in S))$.

The *object of Staples reals* $\underline{R}_S$ is the quotient of $\underline{R}'_S$ by the equivalence relation with $(S,T) \sim (S',T')$ if and only if

$$\forall_{q\in \underline{Q}}(\forall_{s\in S}(q>s) \Longleftrightarrow \forall_{s'\in S'}(q>s')).$$

EXAMPLE. In sheaves on the unit interval, $\underline{R}'_S$ is the sheaf of germs of real-valued functions with only jump discontinuities. The equivalence relation identifies functions which differ only at their points of discontinuity. Thus Staples reals are in general a larger collection than the continuous reals.

DEFINITION 1.4 (Dedekind [4], p. 317). The object of *Dedekind's cuts*, $\underline{R}'_D$, is the subobject of $P\underline{Q}\times P\underline{Q}$ consisting of those pairs (L,U) such that:

1° $\forall_{q\in \underline{Q}}(q\in L \vee q\in U)$.

2° $L\cap U = \emptyset$.

3° $\forall_{q,q'\in \underline{Q}}((q\in L \wedge q'<q) \Longrightarrow q'\in L)$.

4° $\forall_{q,q'\in \underline{Q}}((q\in U \wedge q'>q) \Longrightarrow q'\in U)$.

5° $\forall_{q,q'\in \underline{Q}}((q\in U \wedge q'\in L) \Longrightarrow q>q')$.

6° $\exists_{q,q'\in \underline{Q}}(q\in U \wedge q'\in L)$.

The *object of Dedekind's reals*, $\underline{R}_D$, is the quotient of $\underline{R}'_D$ by the equivalence relation identifying cuts which differ only on the boundary ; that is (L,U) is equivalent to (L',U') if and only if

$$\{q\in \underline{Q} \mid \exists_{q'\in \underline{Q}}((q'<q) \wedge q'\in U)\} = \{q\in \underline{Q} \mid \exists_{q'\in \underline{Q}}(q'<q \wedge q'\in U')\}$$

and

$$\{q\in \underline{Q} \mid \exists_{q'\in \underline{Q}}((q'>q) \wedge q'\in L)\} = \{q\in \underline{Q} \mid \exists_{q'\in \underline{Q}}(q'>q \wedge q'\in L')\}.$$

EXAMPLE. Conditions 1 and 2 specify that the cuts consist of detachable subobjects of $\underline{Q}$. In topoi of the form $Sh(X)$ this means that the statements $q\in L$ and $q\in U$ must hold on clopen sets. The rest of the conditions specify that the fibers must be real numbers, so the object of Dedekind reals is the sheaf of germs of real-valued functions constant on components.

PROPOSITION 1.5. *In any topos with a natural numbers object,*

$$\underset{\sim}{Q} \subseteq \underset{\sim}{R}_C \subseteq \underset{\sim}{R}_T \subseteq \underset{\sim}{R}_S .$$

PROOF.

The inclusion of $\underset{\sim}{Q}$ into $\underset{\sim}{R}_C$ is accomplished using stationary sequences.

The inclusion of $\underset{\sim}{R}_C$ into $\underset{\sim}{R}_T$ is obtained from two maps $\underset{\sim}{R}_C \to P\underset{\sim}{Q}$, one giving the lower cut and the other giving the upper cut. For the lower cut the map is the exponential adjoint of the characteristic morphism of the subobject of $\underset{\sim}{R}_C \times \underset{\sim}{Q}$ consisting of those pairs $([f], q)$ satisfying:

$$\exists_{q'>q}\ \exists_{f\in[f]}\ \exists_{n\in\underset{\sim}{N}}\ \forall_{k\in\underset{\sim}{N}}\,(k>n \Longrightarrow f(k)>q').$$

The upper cut is defined analogously using

$$q'<q \quad \text{and} \quad f(k)<q'.$$

It is immediate from the definition that conditions 1 through 3 in the definition of $\underset{\sim}{R}_T$ are satisfied. The Cauchy condition gives condition:

4° For a given n find m such that for k and h larger than m,

$$|f(k)-f(h)| < \frac{1}{3n} ,$$

then $f(m)+\frac{1}{3n}$ is in the upper cut and $f(m)-\frac{1}{3n}$ is in the lower cut.

The resulting morphism is monic because the pullback of the equality relation on $\underset{\sim}{R}_T$ is the equivalence relation used in defining $\underset{\sim}{R}_C$.

To show that $\underset{\sim}{R}_T \subseteq \underset{\sim}{R}_S$ observe that condition 4 in the definition of $\underset{\sim}{R}_T$ implies condition 1 for $\underset{\sim}{R}_S$. Condition 3 for $\underset{\sim}{R}_T$ is condition 2 for $\underset{\sim}{R}_S$. From conditions 1 and 4 for $\underset{\sim}{R}_T$ we may obtain condition 4 for $\underset{\sim}{R}_S$ as follows: in the hypothesis of the statement

$$\forall_{q\in\underset{\sim}{Q}}\ \forall_{t\in\underset{\sim}{Q}}\,((t\in T \Rightarrow t>q) \Longrightarrow \forall_{s\in\underset{\sim}{Q}}\,(s<q \Rightarrow s\in S)),$$

q is a lower bound of the upper cut. For any $s<q$ we can find an n such that $\frac{1}{n} < q-s$. For that n there are q' and q'' such that

$$q'\in \underline{r}\,,\quad q''\in \overline{r} \quad \text{and} \quad q''-q' < \frac{1}{n}.$$

From this we may conclude that $s<q'$ and thus, by 1, $s\in\underline{r}$. A similar argument using 2 and 4 yields 3.

It remains to show that no two continuous reals are collapsed by the equivalence relation on $\underset{\sim}{R}_S$. By condition 1 on $\underset{\sim}{R}_T$ the upper bounds may be replaced by strict upper bounds, which are in fact members of the upper

cut. The equivalence relation on $\underline{R}_S$ then says that the two continuous reals have the same upper cut. If r and s have the same upper cut, then for any $q \in \underline{r}$ we may conclude that $q \in \underline{s}$ as well: there is an n such that we have $q + \frac{1}{n} \in \underline{r}$. For that n we can find q_1 through q_4 such that

$$q_1 \in \underline{r}\,,\ q_2 \in \underline{s}\,,\ q_3 \in \bar{r} \text{ and } q_4 \in \bar{s} \text{ with } q_3 - q_1 < \frac{1}{3n} \text{ and } q_4 - q_2 < \frac{1}{3n}.$$

Now since the upper cuts are the same, $|q_3 - q_4| < \frac{1}{2n}$, since otherwise either $q_3 < q_2$ or $q_4 < q_1$ would occur. But this implies that

$$q_1 - q > \frac{1}{6n} \quad \text{and} \quad |q_1 - q_2| < \frac{1}{6n}\,,$$

so $q < q_2$ and hence $q \in \underline{s}$.

PROOF. First we construct a function from the object of Dedekind cuts to $\underline{R}_T$ and then show that its kernel pair gives the equivalence relation used to define the Dedekind reals. The desired function takes a cut (L, U) to the pair

$$(\{q \mid \exists_{q' \in \underline{Q}} (q' > q \wedge q' \in L)\}, \{q \mid \exists_{q' \in \underline{Q}} (q' < q \wedge q' \in U)\}).$$

It is clear from conditions 3, 4 and 5 in the definition of a Dedekind cut that this maps through the object of pairs satisfying conditions 1 through 3 of the definition of $\underline{R}_T$. It is also clear that the kernel pair of this map is the equivalence relation used to define the Dedekind reals. The delicate point is in showing condition 4 of the definition of $\underline{R}_T$ is satisfied. This uses two facts from the work of Coste and Sols.

LEMMA *(Sols [14]). In any topos any nonempty detachable subobject of $\underline{N}$ has a least element.*

COROLLARY *(Coste [3]). Any recursive function from $\underline{N}^n$ to $\underline{N}^m$ can be constructed in any topos.*

From 6 we know that there are q and q' with $q \in L$ and $q' \in U$. From 5 this tells us that $q' - q$ is a positive rational. The process of reducing a rational to lowest terms is recursive, so we may write this difference as $\frac{m}{n}$ with m and n relatively prime and both positive. For each $k \in \underline{N}$ there is a subobject of $\underline{N}$ consisting of those n' such that $q + \frac{n'}{4kn} \in U$. Since U is detachable in $\underline{Q}$, this set is detachable in $\underline{N}$. By Sols's lemma it must have a smallest member q^*. The desired rationals for condition 4 are

$$q^* - \frac{1}{2kn} \quad \text{and} \quad q^* + \frac{1}{4kn}\,.$$

2. Order Properties of the continuous Reals.

In defining the rationals in Section 1, we obtained a concept of positive for rationals which satisfies the usual trichotomy axiom. From this we derived an order on $\underset{\sim}{Q}$ which also satisfies trichotomy. The order on $\underset{\sim}{R}_T$ which we wish to study is the order extending this order on $\underset{\sim}{Q}$. It need not satisfy the trichotomy axiom.

DEFINITION 2.1. The *order relation* $<$ is the subobject of $\underset{\sim}{R}_T \times \underset{\sim}{R}_T$ consisting of those pairs (r, s) such that

$$\exists_{q \in \underset{\sim}{Q}} (q \in \bar{r} \wedge q \in \underline{s}).$$

PROPOSITION 2.2. $<$ *is an order relation extending the order on* $\underset{\sim}{Q}$.

PROOF. Transitivity of $<$ on $\underset{\sim}{R}_T$ is a direct consequence of properties 1 and 2 in the definition of $\underset{\sim}{R}_T$.

If $q < q'$ in the order on $\underset{\sim}{Q}$, then $q' - q$ is positive. From this it follows that

$$q < q + \frac{q' - q}{2} < q',$$

giving the needed rational between q and q'.

If there is a rational q such that $q < q''$ and $q'' < q'$, then $q < q'$, by transitivity.

EXAMPLE. Trichotomy fails for $<$ on $\underset{\sim}{R}_T$: In sheaves on the reals consider the global sections of $\underset{\sim}{R}_T$ corresponding to the functions

$$f(x) = x \quad \text{and} \quad g(x) = 0.$$

The statement $f < g$ is true on $(-\infty, 0)$, $f = g$ is valid only at 0 and $f > g$ on $(0, +\infty)$. This means that there is no neighborhood of 0 on which one of the three alternatives hold globally. Hence the statement

$$\forall_{r, s \in \underset{\sim}{R}_T} ((r < s) \vee (r = s) \vee (r > s))$$

is not valid.

Besides the strict order, which will give us open sets, there is another order relation on $\underset{\sim}{R}_T$ useful for defining closed intervals and «intuitionistic» open intervals.

DEFINITION 2.3. The *order relation* $\leq$ is the subobject of $\underset{\sim}{R}_T \times \underset{\sim}{R}_T$ consisting of pairs

$$(r, s) \text{ with } \underline{r} \subseteq \underline{s} \text{ and } \bar{r} \supseteq \bar{s}.$$

PROPOSITION 2.4. $\leq$ *is transitive, reflexive and antisymmetric.*

PROOF. All three follow from the same properties of $\subseteq$.

REMARK. It is not the case that $r \leq s$ is the same as

$$(r < s) \vee (r = s).$$

In sheaves on R , $f(x) = |x|$ and $g(x) = 0$ satisfy

$$f \geq g \quad \text{but not} \quad (f > g) \vee (f = g).$$

PROPOSITION 2.5. $(r \leq s) \Longleftrightarrow \neg (r > s).$

PROOF. If $r \leq s$ and $r > s$, then there is a rational in $\bar{s}$ which is also in $\underline{r}$ and thus in $\underline{s}$, producing a contradiction. Thus

$$r \leq s \Longrightarrow \neg (r > s).$$

For the reverse implication we will need the following lemma:

LEMMA 2.6. *In any topos the rationals satisfy the following statement:*

$$\forall_{q \in \underline{Q}} (q > 0 \Rightarrow \exists_{n \in \underline{N}^+} (\tfrac{1}{n} < q)).$$

PROOF. Write q in lowest terms as $\frac{k}{h}$ (this can be done recursively). $2h$ is the desired n.

This fits in the proof of the proposition by guaranteeing that, if $q \in \underline{r}$, then there is an n such that $q + \frac{1}{n} \in \underline{r}$. Using that n we may choose q' and q'' such that

$$q' \in \bar{s} , \quad q'' \in \underline{s} \quad \text{and} \quad q' - q'' < \frac{1}{n} .$$

By trichotomy for the rationals,

$$(q < q'') \vee (q = q'') \vee (q > q'').$$

In either of the first two cases $q \in \underline{s}$. If $q > q''$, then

$$q' < q'' + \frac{1}{n} < q + \frac{1}{n}$$

and so $q' \in \underline{r}$. This says $s < r$, contradicting our assumption. Thus

$$\neg (s < r) \text{ implies } \underline{r} \subseteq \underline{s} .$$

An exactly analogous proof shows that $\bar{s} \subseteq \bar{r}$.

This Proposition is rather unusual in intuitionistic Mathematics in that it shows the equivalence of a negative statement with a purely positive one. We will encounter another such Proposition (an important corollary of this one) when we study the apartness relation on $\underline{R}_T$.

EXAMPLE. $(\underline{R}_T, \leq)$ need not be order complete: In sheaves on the unit interval consider the subsheaf S of $\underline{R}_T$ consisting of the germs of those continuous functions less than or equal to the characteristic function of

interval $(0, \frac{1}{2})$. This is bounded by the constant function 2, but the least upper bound (which is forced to be the characteristic function of $(0, \frac{1}{2})$) is not continuous at $\frac{1}{2}$ and thus will not be continuous in any neighborhood of $\frac{1}{2}$. Thus it is not possible to cover the interval with open sets on which there exists a continuous function which is the least upper bound of S, so the internal statement saying that there is a least upper bound is not valid.

The failure of $\underline{R}_T$ to be order complete is not critical - constructive Analysis (as in Bishop [1]) shows that its use can be avoided, although extensive use of various forms of choices distinguishes constructive Analysis from topos Analysis. The failure of the continuity theorem of intuitionistic Analysis in the topos of sets shows that topos Analysis is also distinct from intuitionistic Analysis.

Even though we cannot form the least upper bound of arbitrary bounded collections, we can form the maximum (and minimum) of a pair.

PROPOSITION 2.8. *There is a function* $max: \underline{R}_T \times \underline{R}_T \to \underline{R}_T$ *such that*

1^o $max(r, s) \geq r$ *and* $max(r, s) \geq s$,

2^o $\forall_{r, s, t \in \underline{R}_T} ((t \geq r \wedge t \geq s) \Rightarrow t \geq max(r, s))$.

PROOF. Consider the function

$$m: \underline{R}_T \times \underline{R}_T \to P\underline{Q} \times P\underline{Q} \text{ taking } (r, s) \text{ to } (\underline{r} \cup \underline{s}, \bar{r} \cap \bar{s}).$$

We will show that m factors through $\underline{R}_T$ giving rise to the function desired. It is clear that m factors through the subobject specified by the conditions 1 and 3 of the definition of $\underline{R}_T$. The only difficulties are in showing conditions 2 (upper cut) and 4 (cuts at zero distance).

Observe that we may define $max: \underline{Q} \times \underline{Q} \to \underline{Q}$ in the usual case-by-case fashion using trichotomy. The resulting function satisfies the conditions in the Proposition and the further condition that

$$\forall_{q, q' \in \underline{Q}} (max(q, q') = q \vee max(q, q') = q').$$

To show that $\overline{m(r, s)}$ is an upper cut, we observe that $q \in \overline{m(r, s)}$ implies $q \in \bar{r}$ and $q \in \bar{s}$. Thus there exist q' and q'' in $\bar{r}$ and s respectively, such that $q > q'$ and $q > q''$. Then $q > max(q', q'')$, which is in $\overline{m(r,s)}$.

Now for any $n \in \underline{N}$, choose q_1 through q_4 such that

$$q_1 \in \underline{r}, \ q_2 \in \bar{r}, \ q_3 \in \underline{s}, \ q_4 \in \bar{s}, \ q_2 - q_1 < \frac{1}{n} \text{ and } q_4 - q_3 < \frac{1}{n}.$$

Then

$$max(q_1, q_3) \epsilon \underline{r} \cup \underline{s} \quad \text{and} \quad max(q_2, q_4) \epsilon \bar{r} \cap \bar{s}.$$

Furthermore

$$max(q_4, q_2) - max(q_1, q_3) \leq max(q_2 - q_1, q_4 - q_3) < \frac{1}{n}.$$

Thus m factors through $\underline{R}_T$ giving the map $max\colon \underline{R}_T \times \underline{R}_T \to \underline{R}_T$.

Properties 1 and 2 are immediate from the definition and the universal properties of union and intersection.

The topology we wish to study is the smallest topology containing the object of intervals with respect to the order $<$. There are at least two traditional inequivalent ways to define intervals which lead to objects with distinct properties and topologies on $\underline{R}_T$ with different peculiarities.

DEFINITION 2.9. The *interval* (r, s) for $r < s$ is the subobject of $\underline{R}_T$ consisting of those t such that $r < t < s$.

The *object of intervals* is the subobject Int of $P\underline{R}_T$ consisting of those S satisfying the condition

$$\exists_{r,s \epsilon \underline{R}_T} (r < s \wedge \forall_{t \epsilon \underline{R}_T} (t \epsilon S \Leftrightarrow r < t < s)).$$

Observe that with this definition an interval always has global support (indeed it always contains a rational). This keeps the object of intervals from being closed under finite intersections. In sheaves on R, the intervals $(1, 2)$ and $(x, x+1)$ intersect in a subobject of $\underline{R}_T$ which does not have global support.

DEFINITION 2.10 (Troelstra [18] or [19]). The *intuitionistic open interval* $(r, s)_I$ is the subobject of $\underline{R}_T$ consisting of those t such that

$$\neg(t \leq r \wedge t \leq s) \wedge \neg(t \geq r \wedge t \geq s).$$

The *object IInt of intuitionistic intervals* is the subobject of $P\underline{R}_T$ consisting of those S such that

$$\exists_{r,s \epsilon \underline{R}_T} \forall_{t \epsilon \underline{R}_T} (t \epsilon S \Leftrightarrow (\neg(t \leq r \wedge t \leq s) \wedge \neg(t \geq r \wedge t \geq s))).$$

Intuitionistic intervals need not have global support, and even if they do they need not contain a rational. For example, in sheaves on the reals let

$$f(x) = x \quad \text{and} \quad g(x) = 3x.$$

The interval $(f, g)_I$ is the sheaf of germs of functions from R to R such that the subobject of R on which the graph lies in the set

$$\{(x, y) | (x \geq y \wedge 3x \geq y) \vee (x \leq y \wedge 3x \leq y)\}$$

has empty interior. The function $h(x) = 2x$ is such a function since its graph falls in the forbidden zone only at 0. The statement

$$\exists_{q\in Q}(q\epsilon(f,g)_I)$$

is false since no such rational can be found in any neighborhood of 0.

PROPOSITION 2.11. $(r,s)_I = (\min(r,s), \max(r,s))_I$.

PROPOSITION 2.12. *Int is closed under pairwise intersection.*

PROOF (sketch). Let $f \geq g$ and $h \geq k$. Then $(g,f)_I \cap (k,h)_I$ is the interval $(\max(g,k), \max(\min(f,h), \max(g,k)))_I$.

DEFINITION 2.13. The *closed interval* $[r,s]$ is the subobject of $\underline{R}_T$ consisting of those t such that $r \leq t \leq s$.

EXAMPLE. It need not be the case that

$$[0,2] = [0,1] \cup [1,2] :$$

In sheaves on the interval $[0,2]$, let $f(x) = x$. Then

$$f\epsilon[0,1] \text{ on } [0,1], \quad f\epsilon[1,2] \text{ on } [1,2],$$

but there is no open cover of $[0,2]$ such that f is globally in one or the other of the two intervals on each set in the cover. In particular it cannot be done in any neighborhood of 1.

3. The Interval Topology on $\underline{R}_T$.

In [17] I showed that the usual construction of a topology on A (that is, a subobject of PA closed under pairwise intersection and arbitrary internal union and containing A and $\emptyset$) from a subbase works in the topos setting. The interval topology T is the result of applying this construction to the object of intervals. In fact it is not necessary to take closure under pairwise intersection, so Int is in fact a basis rather than just a subbase.

PROPOSITION 3.1. *Int is a basis for T.*

PROOF. Every real is in an interval, so the union of the intervals is all of $\underline{R}_T$. What needs to be shown is that the closure of Int under unions is all of T. To do this it will suffice to show that there is a subobject of T which is closed under intersections, contains Int, and is contained in the closure of Int under internal unions.

The desired object is the object of truncated intervals Int_T obtained by omitting the condition $r < s$ in the definition of Int. It is clear that Int is contained in Int_T. It remains to show that Int_T is closed under pairwise

intersection and that Int is contained in T.

The intersection of intervals (r, s) and (r', s') in Int_T is

$$(\,max(r, r'),\ min(s, s')\,).$$

This works for Int_T where it failed for Int because there is no way to guarantee that $max(r, r') < min(s, s')$.

The unrestricted interval (r, s) is the extension by zero of the partial section of Int defined on the subobjects of 1 for which $r < s$. In forming the closure under internal unions all such extensions are added as global sections (Proposition 1 in [16]).

COROLLARY 3.2. *$\underset{\sim}{Q}$ is dense in $\underset{\sim}{R}_T$.*

PROOF. Int is a basis and every element of Int has a rational member. This is one statement of density. Since every element of an open subobject has a basic neighborhood contained in that open subobject, this implies the following form of density:

$$\forall_{S \in T}\,(\,\exists_{r \in \underset{\sim}{R}_T}(\,r \in S\,) \Rightarrow \exists_{q \in \underset{\sim}{Q}}(\,q \in S\,)\,).$$

REMARK. The topology obtained using $IInt$ does not have $\underset{\sim}{Q}$ dense as the example following Definition 2.10 shows. Hence intuitionistic intervals give rise to a distinct topology.

PROPOSITION 3.3. *$(\underset{\sim}{R}_T, T)$ is second countable, i.e., it has an internally countable base.*

PROOF. The proceedure for showing that the set of pairs of rationals (p, q) with $p < q$ is countable is recursive and hence may be mimicked verbatim in a topos. Thus it will suffice to show that the object of intervals with rational endpoints is a basis for T. For this it will suffice to show that Int is contained in the closure of the object of rational endpoints intervals under unions. The interval (r, s) is the union of the internally specifiable collection of those intervals (p, q) with $r < p < q < s$.

A large number of the desireable properties of the reals may be thought of as dealing with its uniform structure, rather than its topology. Intuitionism introduces fewer complications in uniform space theory than it does in topology. The uniformity of $\underset{\sim}{R}_T$ arises from the topological group structure.

PROPOSITION 3.4. *$(\underset{\sim}{R}_T, T, +)$ is a topological group object.*

PROOF. The operation of taking additive inverse is its own inverse and

takes open intervals to open intervals, so it is a homeomorphism.

To show that addition is continuous observe that in sets this may be proved by the direct calculation

$$+^{-1}(a,b) = \bigcup_{r\in R}(r,\ r+\frac{a-b}{2})\times(a-r,\ a-r+\frac{b-a}{2}).$$

This is an internally specifiable collection of basic open sets in the product topology, so the same proof may be used in any topos.

DEFINITION 3.5 (Bourbaki [2]). A *uniform space object* in a topos is a pair (X, U) with U a subobject of $P(X\times X)$ satisfying the following conditions:

1° $\forall_{A\in P(X\times X)}\forall_{B\in P(X\times X)}((A\in U\wedge B\in U)\Rightarrow A\cap B\in U)$.

2° $\forall_{A\in P(X\times X)}\forall_{B\in P(X\times X)}((A\in U\wedge A<B)\Rightarrow B\in U)$.

3° $\forall_{A\in P(X\times X)}(A\in U\Rightarrow \Delta\leq A)$.

4° $\forall_{A\in P(X\times X)}(A\in U\Rightarrow A^{-1}\in U)$,

where A^{-1} is the image of A along the map interchanging the factors in the product.

5° $\forall_{A\in P(X\times X)}(A\in U\Rightarrow \exists_{B\in U}(B\circ B\leq A))$,

where $B\circ B$ is the image of $B\times B\cap\Delta_{2,3}$ along the projection removing the middle two factors.

As in ordinary topology we can use a uniformity to define a notion of neighborhood which can then be used to define a topology.

Topologies arising in this way are called *uniformizable.*

PROPOSITION 3.6. *Every topological group is uniformizable.*

PROOF. As it is in sets-based topology.

The most important property of the uniformity on the reals is that it is complete. Completeness involves several non-emptiness conditions in the definition of filters and convergence which are taken in the strongest sense.

DEFINITION 3.7. A *filter on* A is a subobject F of PA satisfying the following conditions:

1° $\forall_{B\in PA}(B\in F\Rightarrow \exists_{a\in A}(a\in B))$.

2° $\forall_{B,B'\in PA}((B\in F\wedge B'\in F)\Rightarrow B\cap B'\in F)$.

3° $\forall_{B,B'\in PA}((B\in F\wedge B'\supset B)\Rightarrow B'\in F)$.

4° $A\in F$.

The *object of filters*, $Filt_A$, is the subobject of $P^2 A$ specified by the conditions in the definition of a filter.

The *convergence map*, $conv: Filt_A \to PA$, is the exponential adjoint of the characteristic morphism of the subobject of $Filt_A \times A$ consisting of those pairs (F, a) such that

$$\forall_{O \epsilon T_A} (a \epsilon O \Rightarrow O \epsilon F).$$

A *Cauchy filter* is a filter satisfying

$$\forall_{E \epsilon U} \exists_{B \epsilon F} (B \times B \subseteq E).$$

A uniform space object is called *complete* if the image of the object of Cauchy filters along *conv* is contained in the object

$$\{ S \epsilon PA \mid \forall_{A' \epsilon S} \exists_{a \epsilon A} (a \epsilon A') \}.$$

THEOREM 3.8. *$(\underline{R}_T, U_+)$ is a complete uniform space object (U_+ is the additive uniformity).*

PROOF. The proof consists in the construction of a morphism *lim* from the object of Cauchy filters, *Cauchyfilt*, to $\underline{R}_T$ such that the map

$$(lim, conv): Cauchyfilt \to \underline{R}_T \times P\underline{R}_T$$

factors through ϵ.

The first step is the construction of *lim* as a map to $P\underline{Q} \times P\underline{Q}$ with components $\underline{lim}$ and $\overline{lim}$. $\underline{lim}$ is the exponential adjoint of the characteristic morphism of the subobject of $Cauchyfilt \times \underline{Q}$ consisting of those pairs (F, q) satisfying

$$\exists_{q' \epsilon \underline{Q}} (q' > q \wedge \exists_{E \epsilon P\underline{R}_T} (E \epsilon F \wedge \forall_{p \epsilon \underline{R}_T} (p \epsilon E \Rightarrow q' < p))).$$

$\overline{lim}$ is defined analogously using

$$\exists_{q' \epsilon \underline{Q}} (q' < q \wedge \exists_{E \epsilon P\underline{R}_T} (E \epsilon F \wedge \forall_{p \epsilon \underline{R}_T} (p \epsilon E \Rightarrow q' > p))).$$

It is clear, from the definition, that *lim* factors through the subobject of $P\underline{Q} \times P\underline{Q}$ satisfying the first three conditions in the definition of $\underline{R}_T$. This leaves the zero distance condition.

For this we need the Cauchy condition on filters. For any $n \epsilon \underline{N}$, the $\frac{1}{6n}$ ball around 0 gives rise to an entourage E of the additive uniformity on $\underline{R}_T$. By the Cauchy condition of F, there is an

$$A \epsilon F \quad \text{such that} \quad A \times A \leq E.$$

This implies that for r in A (which must exist by condition 1 for filters) $r+\frac{1}{3n}$ and $r-\frac{1}{3n}$ are outside A and

$$\forall_{a\in A}\left(r-\frac{1}{3n}<a<r+\frac{1}{3n}\right).$$

By the cut conditions on the reals $r-\frac{1}{3n}$ and $r+\frac{1}{3n}$, there are rationals q and q' in $\underline{r-\frac{1}{3n}}$ and $\overline{r+\frac{1}{3n}}$ respectively such that

$$r-\frac{1}{3n}-q \quad \text{and} \quad q'-r-\frac{1}{3n}$$

are both less than $\frac{1}{6n}$. Thus

$$q'-q<\frac{1}{6n}+\frac{2}{3n}+\frac{1}{6n}=\frac{1}{n}.$$

The construction guarantees that

$$q'\in\overline{lim}\,F \quad \text{and} \quad q\in\underline{lim}\,F.$$

Thus lim factors through $\underline{R}_T$.

It remains to show that $lim\,F$ is a limit point of F. For this it will suffice to show that every interval with rational endpoints containing $lim\,F$ is in F. If (a,b) contains $lim\,F$, then

$$a\in\underline{lim}\,F \quad \text{and} \quad b\in\overline{lim}\,F,$$

so there are elements of F such that a is less than every element of A and b is larger than every element of B. Then $A\cap B\leq(a,b)$, so (a,b) is in F.

PROPOSITION 3.9. *$\underline{R}_T$ is a Hausdorff uniform space, that is, the intersection of the entourages is the diagonal.*

PROOF. It will suffice to show that the intersection of all of the neighborhoods of 0 is $\{0\}$. It is clear that $\{0\}$ is contained in the intersection. Now suppose r is in every neighborhood of 0. If $q<0$, then $(q,-q)$ is a neighborhood of 0 so $q\in\underline{r}$. If $q>0$, then $(-q,q)$ is a neighborhood of 0, so $q\in\overline{r}$. Conversely suppose that $q\in\underline{r}$. Then by trichotomy

$$(q>0)\vee(q=0)\vee(q<0).$$

If $q>0$, then $q\in\overline{r}$, giving a contradiction. If $q=0$, then there is a $q'>0$ in $\underline{r}$, giving the same contradiction. Thus $q<0$. Similarly $q\in\overline{r}$ yields $q>0$ so $r=0$.

These two propositions, together with the proof of the universal property of the Hausdorff completion of a uniform space as in Bourbaki (which is intuitionistically valid), show that $\underline{R}_T$ is the completion of the additive uniformity on the rationals. This has the advantage of allowing us to define functions from $\underline{R}_T$ (and spaces derived from $\underline{R}_T$) to $\underline{R}_T$ using extension by continuity.

PROPOSITION 3.10. *Uniformly continuous functions preserve the Cauchy property for filters.*

PROPOSITION 3.11. *If C is the Hausdorff completion of A and $f: A \to B$ is uniformly continuous with B a complete Hausdorff uniform space, then there is a unique uniformly continuous function from C to B extending f.*

PROOF. The Bourbaki construction of C identifies elements of the Hausdorff completion with minimal Cauchy filters on A. Such a filter is taken to the limit point of the Cauchy filter on B whose base is given by the direct images along f of its elements. The proof that the resulting map is the desired extension is exactly as in Bourbaki.

This proposition may be used to extend the definition of multiplication from the rationals to the reals. In general it is quite difficult to define multiplication directly in terms of cuts.

4. Metric properties of $\underline{R}_T$.

DEFINITION 4.1. The *norm function* $|\ |: \underline{R}_T \to \underline{R}_T$ takes r to $max(r, -r)$.

PROPOSITION 4.2. $|\ |$ *is a norm in the usual sense, that is, it satisfies the following properties:*

1° $\forall_{r \in \underline{R}_T} (|r| \geq 0)$.

2° $\forall_{r \in \underline{R}_T} (|r| = 0 \Rightarrow r = 0)$.

3° $\forall_{r, s \in \underline{R}_T} (|r+s| \leq |r| + |s|)$.

PROOF. Using Proposition 2.5 we may replace each inequality with the negation of the strict inequality of the opposite sense. This gives the norm the intuitionistic properties used by Troelstra [18].

To show 1 assume that $|r| < 0$. Then there is a rational $q < 0$ with $q \in \overline{|r|}$. Now this means that

$$q \in \overline{r} \quad \text{and} \quad -q \in \underline{r}.$$

But $q < 0$ implies that $-q > q$, so this gives a contradiction.

To show 2, let $|r| = 0$; then

$$\underline{r} \cup \underline{-r} = \underline{Q}^- \quad \text{and} \quad \bar{r} \cap \overline{-r} = \underline{Q}^+ .$$

If $q < 0$ then $-q \epsilon \underline{Q}^+$. Then $-q \epsilon \overline{-r}$, so $q \epsilon \underline{r}$. This shows $\underline{r} = \underline{Q}^-$. If $q \epsilon \bar{r}$, then $-q \epsilon \underline{-r}$, so $-q \epsilon \underline{Q}^-$. This shows $\bar{r} = \underline{Q}^+$.

To show 3, suppose that $|a| + |b| < |a+b|$. Then there is a rational q such that

$$q = q' + q'' \quad \text{with} \quad q' \epsilon \overline{|a|},\; q'' \epsilon \overline{|b|}\; ,$$

and $q \epsilon \underline{|a+b|}$. The conditions on q' and q'' say that

$$q' \epsilon \bar{a},\; q'' \epsilon \bar{b},\; -q' \epsilon \underline{a} \text{ and } -q'' \epsilon \underline{b} .$$

This means

$$q = q' + q'' \epsilon \overline{a+b} \quad \text{and} \quad -q = -q' - q'' \epsilon \underline{a+b},$$

Thus $q \epsilon \overline{|a+b|}$, giving the needed contradiction.

COROLLARY 4.3. *$(\underline{R}_T, T)$ is a metric space with metric d taking (a, b) to $|a-b|$.*

PROOF. The fact that d satisfies the usual axioms for a metric follows directly from the properties of the norm. The topology associated with d has a basis given by the balls of positive real radius around each point. Each interval (a, b) is the $\frac{b-a}{2}$ ball of $\frac{b+a}{2}$ and each ball

$$\{ x \mid d(x, a) < r \}$$

is an interval $(a-r, a+r)$.

Thus we have shown that the interval topology on $\underline{R}_T$ is metrizable. The density of $\underline{Q}$ shows that the metric space is separable; it remains to show that it is complete.

DEFINITION 4.4. A *Cauchy sequence in a metric space object (A, δ)* is a function $f : \underline{N} \to A$ such that

$$\forall_{n \epsilon \underline{N}} \exists_{m \epsilon \underline{N}} \forall_{k, k' \epsilon \underline{N}} ((k > m \wedge k' > m) \Rightarrow \delta(f(k), f(k')) < \tfrac{1}{n}).$$

The *object of Cauchy sequences*, *Cauchyseq*, is the subobject of $A^{\underline{N}}$ specified by this predicate.

The *convergence map* form *Cauchyseq* to PA is the exponential adjoint of the characteristic morphism of the subobject of $Cauchyseq \times A$ consisting of those pairs (f, a) satisfying

$$\forall_{n \epsilon \underline{N}} \exists_{m \epsilon \underline{N}} \forall_{k \epsilon \underline{N}} (k > m \Rightarrow \delta(f(k), a) < \tfrac{1}{n}).$$

A metric space is called *complete* if *conv* factors through

$$\{ S \subset PA \mid \exists_{a \in A} (a \in S) \}.$$

COROLLARY 4.5 *(to Theorem 3.8). $(\underline{R}_T, d)$ is complete.*

PROOF. We define a map

$$l: \mathit{Cauch\,seq} \to \mathit{Cauchy\,filt}$$

such that

$$conv_{(sequences)} = conv_{(filters)} l .$$

Metric completeness will then follow from uniform completeness.

We start by defining l as a map from $\underline{R}_T^{\underline{N}}$ to $P^2 \underline{R}_T$ as the exponential adjoint of the characteristic morphism of the subobject of $\underline{R}_T^{\underline{N}} \times P \underline{R}_T$ consisting of those pairs (f, A) satisfying

$$\exists_{n \in \underline{N}} \forall_{m \in \underline{N}} (m > n \Rightarrow f(m) \in A).$$

We need to show that this takes Cauchy sequences to Cauchy filters. It is clear that l factors through the object of filters. To get Cauchy filters it will suffice to show that for any fundamental entourage E of the uniformity and any Cauchy sequence f there is a subobject B of $\underline{R}_T$ such that: $B \times B \subseteq E$ and (f, B) satisfies the predicate used to define l.

A fundamental system of entourages is given by the N-indexed family of pairs

$$\{ (r, s) \mid |r - s| < \frac{1}{n} \} .$$

For any n the Cauchy criterion on f guarantees the existence of an m such that, for k and h larger than m,

$$d(f(k), f(h)) < \frac{1}{n} .$$

This says that the object B of values of f at natural numbers larger than m is E-small for the entourage E associated with n. By construction the pair (f, B) satisfies the predicate used to define l.

To show the convergence condition it will suffice to show that the pullback along $l \times \underline{R}_T$ of the object of pairs (F, r) such that

$$\forall_{n \in \underline{N}} ((r - \frac{1}{n}, r + \frac{1}{n}) \in F)$$

is the object of pairs (f, r) such that

$$\forall_{n \in \underline{N}} \exists_{m \in \underline{N}} \forall_{k \in \underline{N}} (k > m \Rightarrow d(f(k), r) < \frac{1}{n}).$$

Now $(r-\frac{1}{n}, r+\frac{1}{n})$ is in the image of f under l if and only if there is an m beyond which all the values of f are in $(r-\frac{1}{n}, r+\frac{1}{n})$, which is precisely what is required.

Applying this result to the topoi mentioned in Section 1 gives some new results. In his book on stochastic convergence[9], Lukacs gives proofs that for non trivial probability spaces convergence in measure and convergence almost everywhere are incompatible with a norm in the classical sense. In appropriate topoi these kinds of convergence of random variables become convergence of real numbers with respect to the internal norm. Furthermore, the resulting spaces are internal Banach spaces. As a concrete application it may be shown that a regular stochastic matrix with random variables as entries instead of real numbers (corresponding, for instance, to a Markov chain with uncertain transition probabilities) converges almost surely (or in measure) to a steady state matrix of the same type.

The same application may be made with continuous functions and uniform convergence merely by changing to a spatial topos (even though the computational technique normally used to find the steady state matrix is not continuous).

5. Separation properties.

Separation properties in topos topology are a bit delicate. In general the conditions used in ordinary topology using inequality and disjointness need to be replaced with conditions using various forms of apartness.

EXAMPLE. $\underline{R}_T$ need not satisfy the Hausdorff axiom

$$\forall_{r,s\in \underline{R}_T}(r\neq s \Rightarrow \exists_{U,V\in T}(r\in U \wedge s\in V \wedge U\cap V=\emptyset)):$$

In sheaves on the reals the identity function f and the constant function 0 satisfy $f\neq 0$, but they cannot be separated in any open neighborhood of 0.

The problem here is that $\neq$ is not a strong enough form of inequality since it allows sections to agree so long as the set on which they agree has no interior. We need a statement which says «nowhere equal» or «everywhere apart».

DEFINITION 5.1. The *apartness relation* $\between$ is the subobject of $\underline{R}_T\times\underline{R}_T$ consisting of those pairs (r,s) such that $(r>s)\vee(r<s)$.

PROPOSITION 5.2. $r \between s \iff |r-s| > 0$.

PROOF. Direct.

PROPOSITION 5.3. $\between$ *is an apartness relation in the sense of Troelstra* ([19] p. 15); *that is, it satisfies the following conditions:*

1^0 $\forall_{r,s \in \underline{R}_T}(\neg(r \between s) \iff r = s)$.

2^0 $\forall_{r,s \in \underline{R}_T}(r \between s \iff s \between r)$.

3^0 $\forall_{r,s,t \in \underline{R}_T}(r \between s \Rightarrow (r \between t \vee s \between t))$.

PROOF. Condition 2 and the reverse implication of condition 1 are trivial.

By Proposition 5.2,

$$\neg(r \between s) \text{ says } \neg(|r-s| > 0),$$

so by Proposition 2.5, $|r-s| \leq 0$. Thus $|r-s| = 0$, so $r = s$. This proves condition 1.

To show 3 it will suffice to show that

$$r > s \Rightarrow (t < r \vee t > s).$$

Now $r > s$ says that there is a $q \in \underline{Q}$ such that $q \in \bar{s}$ and $q \in \underline{r}$. By the cut properties we can sharpen this to saying that there is an $n \in \underline{N}$ such that

$$q - \frac{1}{n} \in \bar{s} \text{ and } q + \frac{1}{n} \in \underline{r}.$$

Now find q' and q'' such that

$$q' \in \bar{t}, \quad q'' \in \underline{t} \text{ and } q' - q'' < \frac{1}{n}.$$

If $q < q'$, then

$$q'' = q' - (q' - q'') > q' - \frac{1}{n} > q - \frac{1}{n}.$$

So $q'' \in \bar{s}$ and $t > s$. If $q = q'$, then $t < r$. If $q > q'$, then certainly $q \in \bar{t}$. This tells us that $r > t$. This exhausts the three choices allowed by trichotomy for rationals.

PROPOSITION 5.4. $(\underline{R}_T, T)$ *satisfies the Hausdorff axiom*

$$\forall_{r,s \in \underline{R}_T}(r \between s \Rightarrow \exists_{U,V \in T}(r \in U \wedge s \in V \wedge U \cap V = \emptyset)).$$

PROOF. $r \between s$ says that there is a rational between r and s; call it q. Then U and V are the intervals $(-\infty, q)$ and $(q, +\infty)$, the choice depending on whether $r > s$ or $r < s$.

COROLLARY 5.5. $(\underline{R}_T, T)$ *satisfies the Hausdorff axiom*

$$\forall_{r,s \in \underline{R}_T}(r \neq s \Rightarrow \neg\neg\exists_{U,V \in T}(r \in U \wedge s \in V \wedge U \cap V = \emptyset)).$$

PROOF. This is a direct consequence of Propositions 5.4 and 5.3.

It would be desireable to have a more topological positive form of the Hausdorff property. Forms using convergence and closure are convenient.

PROPOSITION 5.6. *Filter convergence for* $(\underline{R}_T, T)$ *factors through* $\tilde{\underline{R}}_T$.

PROOF. This is the same as saying that two limt points of the same filter must be equal. Suppose that r and s are limit points of F and $r \lessgtr s$. Since r and s have disjoint neighborhoods by Proposition 5.4, it is not possible for F to converge to both since that would require F to have two disjoint members. Thus $\neg(r \lessgtr s)$, but this says $r = s$ by Proposition 5.3.

DEFINITION 5.7. The *closure operator* $cl : PA \to PA$ associated with a topology T_A on A is the exponential adjoint of the characteristic morphism of the subobject of $PA \times A$ consisting of those pairs (S, a) satisfying

$$\forall_{O \in T_A} (a \in O \Longrightarrow \exists_{b \in A} (b \in O \cap S)).$$

A subobject is called *closed* if it is fixed by the closure operator.

This closure does not have all of the usual properties desired for a closure operator; in particular, the union of two closed subobjects need not be closed (Stout [16], last Section). It does have the advantage of being the right concept in terms of filter convergence and in agreeing with the standard intuitionistic concept (Troelstra [18], p, 26).

Using this concept of closure and the Hausdorff property of the uniform space structure of $\underline{R}_T$, we can prove the following propositions precisely as in set-based topology.

PROPOSITION 5.8. *The diagonal is closed in* $\underline{R}_T \times \underline{R}_T$.

PROPOSITION 5.9. *Each* $r \in \underline{R}_T$ *has a fundamental system of closed neighborhoods.*

In ordinary topology the normality of a metric space (and hence of R) follows directly from the existence of a continuous function giving the distance from a point to a closed set. In topos topology such a function need not exist (or at least need not have a real value), as the example used to show that $(\underline{R}_T, \leq)$ is not complete shows. Following the lead of the intuitionists we obtain the following definition.

DEFINITION 5.10. For a metric space object (A,δ) and a subobject S of A, the *map* $d(-,S)$ *from* A *to* $P\underline{Q}\times P\underline{Q}$ is defined as the map with components given by the exponential adjoint of the characteristic morphism of the subobject of $A\times\underline{Q}$ consisting of those pairs (a,q) satisfying

$$\exists_{q'\in\underline{Q}}(q<q'\wedge\forall_{s\in S}(\delta(a,s)>q'))$$

for the first component, and a similar construction using

$$\exists_{q'\in\underline{Q}}(q>q'\wedge\forall_{s\in S}(\delta(a,s)<q'))$$

for the second. If this defines a continuous morphism to $\underline{R}_T$, S is called *located.*

It is possible that the hypothesis of continuity in the definition of located is superfluous. In intuitionistic Mathematics the continuity follows from the continuity theorem, which fails for general topoi. In the classical case the proof depends heavily on trichotomy in a way that apparently cannot be dodged using trichotomy for the rationals. In any case located closed subobjects are still not good enough for normality.

EXAMPLE. There exist disjoint located closed subobjects of $\underline{R}_T$ which cannot be separated using open subobjects: In sheaves on the reals let A be the sheaf of germs of continuous real-valued functions whose value is always greater than or equal to that of the absolute value function and let B be the sheaf of germs of continuous non-positive real-valued functions. Then A and B are disjoint, but they cannot be separated in any neighborhood of 0.

This example shows that disjointness is not sufficient; we need apartness. One such condition may be given for located closed subobjects and another for closed subobjects in general.

PROPOSITION 5.11. *The internal statement which says*

if C *and* C' *are located closed subobjects of* $\underline{R}_T$ *such that*

$$d(x,C)+d(x,C')>0,$$

then there is a continuous function f *from* $\underline{R}_T$ *to* $[0,1]$ *such that*

$$f^{-1}(1)=C \quad \text{and} \quad f^{-1}(0)=C'$$

is valid.

PROOF. The desired function is the one which takes x to

$$\frac{d(x,C')}{d(x,C')+d(x,C)}.$$

This is continuous because the functions it is composed of are and the denominator is a unit. To show that the function has the desired property it will suffice to show that for any x and located subobject A,

$$d(x, A) = 0 \Longrightarrow x \in cl\, A.$$

If $d(x, A) = 0$, then for any $q \in \underline{Q}^+$ there is a member a of A such that $d(x, a) < q$. But the intervals $(x-q, x+q)$ form a fundamental system of neighborhoods of x, which is all that needs to be considered in forming closures. Thus $x \in cl\, A$.

DEFINITION 5.12. Two subobjects A and B of $\underline{R}_T$ are called *apart* (written $A \# B$) if they satisfy

$$\forall_{a \in A} \exists_{n \in \underline{N}} \forall_{b \in B} \left(d(a, b) > \frac{1}{n} \right) \wedge \forall_{b \in B} \exists_{n \in \underline{N}} \forall_{a \in A} \left(d(b, a) > \frac{1}{n} \right).$$

For singletons this is the same as the apartness relation $a \# b$.

PROPOSITION 5.13. *If A and B are closed subobjects of $\underline{R}_T$ with $A \# B$, then there are open subobjects O and O' such that*

$$A \subseteq O, \quad B \subseteq O' \text{ and } O \cap O' = \emptyset.$$

PROOF. O is the subobject of $\underline{R}_T$ consisting of those x satisfying

$$\forall_{a \in A} \exists_{n \in \underline{N}} \forall_{b \in B} \left(d(x, b) - d(x, a) > \frac{1}{n} \right).$$

O' is defined similarly with the roles of A and B reversed. It is clear from the definitions that O and O' are disjoint. The condition $A \# B$ guarantees that $A \subseteq O$ and $B \subseteq O'$. The only remaining point is to show that O and O' are open.

Let $x \in O$. Then for each $a \in A$ there is an $n \in \underline{N}$ such that, for $b \in B$,

$$d(x, b) - d(x, a) > \frac{1}{n}.$$

Take the $\frac{1}{3n}$ ball of x. Then if y is in this ball around x, we have

$$d(x, b) - d(x, y) - d(x, y) - d(x, a) > \frac{1}{3n}.$$

Now from the properties of the metric,

$$d(x, y) + d(x, a) \geq d(y, a) \quad \text{and} \quad d(x, b) - d(x, y) \leq d(y, b).$$

Combining these two results gives

$$d(y, b) - d(y, a) \geq d(x, b) - d(x, y) - d(x, y) - d(x, a) > \frac{1}{3n}.$$

Thus y is also in O (with $3n$ giving the required excess distance).

We conclude this Section with a complete regularity condition which does not follow from any of the forms of normality we have considered.

PROPOSITION 5.14. *If* $a \in O \in T$, *there is a continuous function* $f: \underline{R}_T \to \underline{R}_T$ *such that*

$$f^{-1}(1) = \{a\} \text{ and } f^{-1}(\underline{R}_T^{+}) \subseteq O.$$

PROOF. First choose intervals with rational endpoints such that

$$a \in (p', q') \subset (p, q) \subseteq O.$$

Using trichotomy for the rationals we may use piecewise definitions to define a function $g: \underline{Q} \to \underline{Q}$ which is uniformly continuous as follows:

if $x < p$, then $g(x) = 0$,

if $p < x < p'$, then $g(x) = \dfrac{x}{p'-p} - \dfrac{p}{p'-p}$,

if $p' < x < q'$, then $g(x) = 1$,

if $q' < x < q$, then $g(x) = \dfrac{q}{q-q'} - \dfrac{x}{q-q'}$,

if $q < x$, then $g(x) = 0$.

Extension by continuity gives the function g from $\underline{R}_T$ to $\underline{R}_T$. The desired function f is defined by

$$f(x) = g(x)\,\frac{1}{1 + d(x, a)}\,.$$

6. Connectedness.

In set-based algebraic topology R is used as a yardstick with which to measure the connectivity of other spaces. Positive fòrms of connectedness play an important role in Analysis, at least in the classical setting. In more general topoi $\underline{R}_T$ satisfies the negative continuity conditions used in topology but not the positive forms used in Analysis.

THEOREM 6.1. $(\underline{R}_T, T)$ *is connected in the following sense:*

$$\neg \exists_{U,V \in T}((\underline{R}_T = U \cup V) \wedge (\exists_{r \in \underline{R}_T} r \in U) \wedge (\exists_{r \in \underline{R}_T} r \in V) \wedge (U \cap V = \emptyset)).$$

PROOF. Suppose that, on the contrary, there is such a disconnection of $\underline{R}_T$. Then there is also a disconnection of the rationals obtained by

$$U' = U \cap \underline{Q} \quad \text{and} \quad V' = V \cap \underline{Q}\,.$$

Since the rationals are dense, there are rationals p and q such that $p \in U'$ and $q \in V'$. Let us assume $p < q$. Then we obtain a Dedekind cut (L, W) as follows:

$$L = \{\, x \mid \exists_{q' \in \underline{Q}}(q' < q \wedge q' \in U' \wedge x < q')\},$$

$$W = \{\, x \mid \forall_{q' \in \underline{Q}}(x < q' \Rightarrow (q' \in V' \vee q' > q))\}.$$

It is clear that the pair of subobjects so constructed satisfies conditions 3 through 6 in Definition 1.4. The difficulty is in showing that L and W are complementary subobjects of $\underline{Q}$.

If the logic were classical this would be no problem since the statements defining L and U have the forms

$$\exists_{q' \epsilon \underline{Q}} P(x, q') \quad \text{and} \quad \forall_{q' \epsilon \underline{Q}} (\neg P(x, q')).$$

In this case P is a subobject of $\underline{Q} \times \underline{Q}$ which has a complement (since U is the complement of V and trichotomy holds for the rationals). This reduces the problem to showing that

$$\exists_{q' \epsilon \underline{Q}} P(x, q') \Longleftrightarrow \neg \neg \exists_{q' \epsilon \underline{Q}} P(x, q'),$$

when P is fixed by double negation. Existantiation along a projection preserves terminals and conjunctions, so

$$\exists_{q' \epsilon \underline{Q}} P(x, q') \cup \exists_{q' \epsilon \underline{Q}} \neg P(x, q') = \underline{Q}.$$

Now $\underline{Q}$ has global support, so

$$\forall_{q' \epsilon \underline{Q}} \neg P(x, q') \Longrightarrow \exists_{q' \epsilon \underline{Q}} \neg P(x, q').$$

Thus we may conclude that

$$\exists_{q' \epsilon \underline{Q}} P(x, q') \cup \forall_{q' \epsilon \underline{Q}} \neg P(x, q') = \underline{Q}.$$

Thus $L \cup W = \underline{Q}$. It is always the case that

$$\forall_{q' \epsilon \underline{Q}} \neg P(x, q') \Longrightarrow \neg \exists_{q' \epsilon \underline{Q}} P(x, q'),$$

which implies that $L \cap W = \emptyset$.

By Proposition 1.6 this induces a continuous real r. If $r \epsilon U$, then there is a rational greater than r in U', which is not possible. If $r \epsilon V$, then there is a rational less than r in V', which is also not possible. Thus it is not possible that $U \cup V = \underline{R}_T$.

EXAMPLE. $(\underline{R}_T, T)$ need not satisfy the following form of connectedness :

$$\forall_{A \epsilon P\underline{R}_T} ((A \epsilon T \wedge A = cl\, A) \Longrightarrow (A = \underline{R}_T \vee A = \emptyset)):$$

In sheaves on the reals truncate the open subobject $\underline{R}_T$ to the subobject of one corresponding to the open interval $(0, 1)$. This is an open subobject as a result of the union axiom (it is the union of the subobject

$$U_{(0,1)} \times \ulcorner \underline{R}_T \urcorner \quad \text{of} \quad P\underline{R}_T).$$

It is not difficult to show that it is also closed, but there is no neighborhood of 1 on which either $A = \emptyset$ or $A = \underline{R}_T$ holds globally.

EXAMPLE. The intermediate value theorem may fail for $(\underline{R}_T, T)$: In sheaves on the unit interval we may represent a continuous function from $\underline{R}_T$ to $\underline{R}_T$ as a continuous function $g: I \times R \to R$ (whether or not all continuous functions can be so represented is a side issue - we will only need one). On an open subset $U \subseteq I$, $\underline{R}_T$ has global sections corresponding to continuous functions $f: U \to R$. The function defined by g takes f to the function from U to R taking x to $g(x, f(x))$. Since f and g are continuous, so is this composition. Thus we have defined a function from $\underline{R}_T$ to $\underline{R}_T$. An interval in the codomain is specifiable as the collection of germs of functions which have graph in an open strip between the graphs of two functions in $I \times R$. Imbed the graph of g into $I \times R \times R$ as a surface. The intersection of the surface with the subobject which is the strip in the first and third factors and all of R in the second (the inverse image of the strip under the projection removing the second factor) is open. Since the graph of g is homeomorphic to $I \times R$ by the projection removing the third factor, the image of the open section of the surface under that projection is an open region in $I \times R$. Since open rectangles form a basis for the product topology and each open rectangle may be thought of as an interval truncated to its I-factor, the inverse image of an interval in $\underline{R}_T$ along the map we have constructed is open in $\underline{R}_T$. Hence we have described a proceedure for obtaining examples of continuous functions from $\underline{R}_T$ to $\underline{R}_T$.

Now restrict both R-factors to intervals $[-1, 1]$ and consider the continuous function giving rise to the surface with level curves as illustrated in Figure 1. The level curve giving the value 0 is not a function in any neighborhood of $1/2$, so there is no cover on which every element has a global section taken to 0 by g. However, g is 1 at the constant function

1 and *- 1* at the constant function *-1* . Hence this is an example where the intermediate value theorem fails.

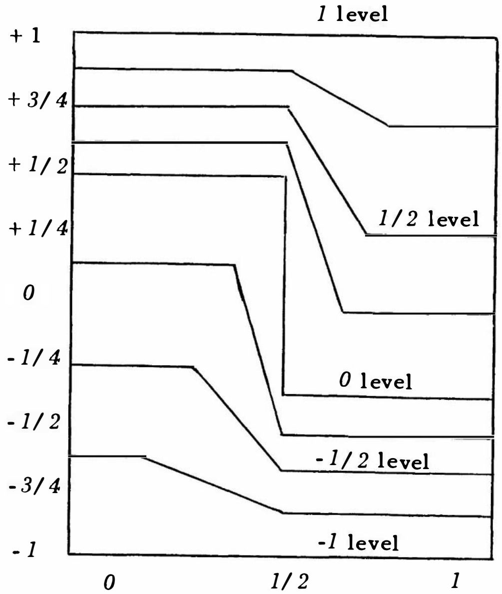

FIGURE 1

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