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A Topological Structure on the Structure Sheaf,

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A Topological Structure on the Structure Sheaf of a Topological Ring

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The construction of the structure sheaf $\underset{\sim}{\tilde{R}}$ of a commutative ring R is well known. It is natural to ask whether further structure on R can be reflected in further structure on the sheaf $\underset{\sim}{\tilde{R}}$; in particular, for a topological ring (R, T_R) is there a topological structure on $\underset{\sim}{\tilde{R}}$ such that the sectional representation theorem produces an isomorphism which is also a homeomorphism? Under suitable hypotheses on the relationship between multiplication and the topology on R, the answer to this question is yes.

I. Topological Structures on Sheaves

Since we are asking for a topological structure on $\underset{\sim}{R}$, it would be wise to specify which of the several possible meanings of a topological structure on a sheaf we are talking about. For algebraic structures there is no difficulty-- external and internal definitions agree, so the external definition is usually used (we talk about sheaves of rings rather than ring objects in the category of sheaves, for instance). For topological structures, the internal and external definitions do not agree, nor do all internal definitions produce the same results. We will be using the internal definition obtained by stating the usual definition in terms of open subsets in the internal logic of the topos: a topological structure on a sheaf $\underline{A}$ is a subsheaf of its powerobject $\underline{PA}$ satisfying formal statements to the effect that it is closed under pairwise intersection and arbitrary internal union and contains both $\underline{A}$ and $\underline{\emptyset}$. This approach was explored to see how much of general topology carries over to the sheaf setting in [4] ; it was made possible by the observation of Lawvere and Tierney ([1] ,[2] ,[5], and [6]) that categories of sheaves have a higher order internal logic. While the internal logic makes the concepts involved in this paper more natural, we will need an external characterization of topological structures in order to work with the topology explicitly in the proof of the representation theorem. Rather than take the space here to prove that characterization, we will use it as our definition of a topological structure on a sheaf on a topological space.

Definition: Let $\underline{A}$ ba a sheaf on the topological space (X,T_X), and let $A \xrightarrow{p} X$ be its associated étale space. Then a topological structure on $\underline{A}$ is a second topology on A, T_A , which is coarser than the topology making p étale but fine enough that p is still continuous.

The value topology on the set $\Gamma(\underline{A})$ of global sections of $\underline{A}$ is the topology with basis $\{\Gamma(\underline{U}) : \underline{U} \text{ is open in } T_A\}$.

A map f: $(\underline{A}, T_A) \longrightarrow (\underline{B}, T_B)$ is continuous internally if the associated map on étale spaces is continuous with respect to the second topologies on the total spaces.

For such structures product and quotient structures may be constructed as pullbacks and quotients, respectively, in the category of topological spaces over X.

Examples:

1) If Y is a topological space, then there is a natural topological structure on the sheaf of germs of locally constant Y-valued functions on X. The associated étale space is

$$X \times (Y, \text{discrete}) \xrightarrow{pr} X$$

the second topology is the product $X \times (Y, T_Y)$.

2) For any sheaf there are two trivial topologies: the discrete topology, which is the same as the topology making p étale; and the indiscrete topology, which is the same as the topology induced by p.

In order to use these internal notions of topology on a sheaf we will need an internal construction of the structure sheaf. By making the whole construction internal we will be able to introduce the topology without having to take an associated sheaf or consider the passage from internal to external topology more than once.

II. Internal Construction of the Structure Sheaf

The classical construction of $\underset{\sim}{\tilde{R}}$ (as in, say, Macdonald [3]) proceeds by constructing the space X = Spec(R) and then defining a presheaf of rings on certain basic open sets and showing that it is a sheaf. In the exposition cited, X is the set of prime ideals, D_f is a basic open set consisting of those $x \in X$ such that no power of f is in x. The value of $\underset{\sim}{\tilde{R}}$ at D_f is $S^{-1}R$ where S is the multiplicative system of all powers of f.

This construction could easily be carried out at the level of presheaves on X as a formation of a ring of quotients inverting a presheaf of multiplicative systems in the constant presheaf of rings with value R. Since this process is carried out by forming a quotient of $S \times R$ by an equivalence relation defined in terms of multiplication and equality, we may obtain the associated sheaf of the ring of quotients by forming the ring of quotients of the associated sheaf of rings at the associated sheaf of multiplicative systems in the topos of sheaves on X. Since the presheaf $\underline{\tilde{R}}$ is in fact a sheaf, this construction will yield the structure sheaf.

The associated sheaf of the constant presheaf R is the sheaf $\underline{R}$ of locally constant R-valued functions on X. The associated sheaf of multiplicative systems is easily specifiable as the subsheaf of $\underline{R}$ with stalk at x the set of all $r \in R$ not in x. Since each $f \in R$ is in S precisely on D_f, this is a sheaf; since each x is prime, it is a sheaf of multiplicative systems. If we now shift our foundations from sets to the category of sheaves on X, we have a ring equipped with a multiplicative system which we want to invert. This is the formation of a quotient of $\underline{R} \times \underline{S}$ by the equivalence relation internalizing $(r,s) \sim (r',s')$ iff $rs' = r's$. This equivalence may be obtained as the equalizer of two maps from $\underline{R} \times \underline{S} \times \underline{R} \times \underline{S}$ to $\underline{R}$.

All of this may be made explicit at the level of étale spaces: $\underline{R}$ is the étale space $X \times (R, \text{discrete}) \xrightarrow{pr} X$; $\underline{S}$ is the subspace of $X \times R$ consisting of those pairs (x,r) with $r \notin x$; and the equivalence relation is

$$(r,s,x) \sim (r',s',x) \quad \text{iff} \quad rs' = r's.$$

III. Introduction of the Topology

The étale space for $\underline{R}$ looks suspiciously familiar. It was the total space for our first example of topological structures. It has a natural structure given by the product topology $X \times (R, T_R)$.

We have three obvious options for the topology on $\underline{S}$: the subspace topology induced by the inclusion into $X \times R$, the discrete topology, and the indiscrete topology. In the proof of the representation theorem our main difficulty will be in showing that there are enough open subsheaves in $\underline{S} \times \underline{R}$, so we will use the topology on $\underline{S}$ which gives the most open subsheaves, the discrete topology.

We will then give $\underline{R} \times \underline{S}$ the product topological structure and endow $\underline{R}$ with the quotient structure. Our theorem will then give a sufficient condition for the topology of R to be recovered as the value topology on $\Gamma(\underline{\tilde{R}})$.

In any topological ring multiplication by an element of R is a continuous map; in general it need not be open (in particular, except for discrete rings, multiplication by a nilpotent is never open). For our proof we will need the following hypothesis, which is akin to the condition that multiplication by non-nilpotents be open, but which is at least potentially stronger:

(H): If $a \in R$ is not nilpotent, then for any $r \in R$ and any open set U containing r, there is an open neighborhood of ar, N, such that $((\ )a)^{-1}N \subseteq U$.

Theorem: For a topological ring satisfying the condition (H), the isomorphism exhibited in the classical sheaf representation theorem between $\Gamma(\underline{\tilde{R}})$ and R is also a homeomorphism, where $\Gamma(\underline{\tilde{R}})$ is given the value topology for the quotient structure defined above.

Proof: The classical representation theorem shows that a global section of $\underline{\tilde{R}}$ may be thought of as a constant section of $\underline{R} \times \underline{S}$ lying entirely in the slice $\underline{R} \times \underline{\{1\}}$. This is done by noting that a global section of $\underline{\tilde{R}}$ results from a cover of X by basic open sets

D_{f_i} and a constant section of $\underline{R \times S}$ on each D_{f_i} such that on the overlap between two basic open sets D_{f_i} and D_{f_j} the two pairs (r, f_i^n) and (r', f_j^m) are in the equivalence relation. It is then shown algebraically that any such collection of constant partial sections gives rise to a unique $r \in R$ such that the constant section with value $(r, 1)$ is in the saturation of the subobject of $\underline{R} \times \underline{S}$ determined by the family of partial sections with respect to the equivalence relation $\sim$. Since open subsheaves in $\underline{\tilde{R}}$ are determined completely by the saturated opens in $\underline{R} \times \underline{S}$, it will suffice to show that any open set in R can occur as the constant sections of a 1-slice of a saturated open subsheaf and that only opens in R can so appear.

The space X is compact and the image of any constant section in the étale space is a homeomorph of X and hence compact. If V is open in $X \times R$, then any constant section of V has a cover consisting of rectangular basic open sets. Since X is compact, this has a finite subcover. The intersection of all of the open sets in R occuring as factors in the rectangular sets in this finite subcover is an open set containing elements r such that the constant section with value r is in V. Thus the set of all r for which the constant section with value r is in V is open.

Now in $\underline{R} \times \underline{S}$ in the construction, $\underline{S}$ was given the discrete topology, so any open subsheaf of $\underline{R} \times \underline{S}$ has an open 1-slice. (The 1-slice is the intersection with $\underline{R} \times \underline{\{1\}}$; it is homeomorphic to $X \times R$.) Hence only open subsets of R can occur as sets of constant sections of the one slice of open subsheaves of $\underline{R} \times \underline{S}$.

We now construct a saturated open subsheaf of $\underline{R} \times \underline{S}$ with 1-slice equal to $X \times U$ for a given open subset $U \subseteq R$. For this we recall that the functor "pull back along $()a$" has a right adjoint $\forall_{()a}$ taking a subset B to the set of all elements of R either nondivisible by a or of the form ba with b in B (Lawvere [1]). Taking interior we get a right adjoint to $(()a)^{-1}$ as a functor from the topology of R to itself as well. Since $()a$ is continuous,

$$(()a)^{-1}(\forall_{()a} U)^{\circ} \subseteq U$$

and if V is an open subset whose pullback along $()a$ is in U, then V is contained in $(\forall_{()a}U)^{\circ}$.

Now define $\underline{U}$ to be the subset of the total space of $\underline{R} \times \underline{S}$ consisting of those (r, a, x) such that $a \notin x$ and $r \in (\forall_{()a}U)^{\circ}$. We need to show that this is an open subset of the total space with the second topology before we are justified in calling it an open subsheaf (indeed we must check that it is open in the étale space topology before we are justified in calling it a subsheaf at all). The defining statement is true for fixed r and a on the basic open set D_a in X, so the X factor can be "thickened" to an open set for fixed a and any $r \in (\forall_{()a}U)^{\circ}$. Since S was given the discrete topology the S factor does not need to be "thickened"; the R factor can be extended to all of $(\forall_{()a}U)^{\circ}$. Therefore if the point (r,a,x) is in $\underline{U}$, so is the open neighborhood of that point $(\forall_{()a}U)^{\circ} \times \{a\} \times D_a$.

Now suppose $(r, a, x) \sim (r', a', x)$ and $(r, a, x) \in \underline{U}$. Take an open set V containing r with V contained in $(\forall_{()a}U)^{\circ}$. Since only non-nilpotents occur in S, we may use the hypothesis (H) to obtain an open neighborhood N of $a'r$ such that $(()a')^{-1}N \subseteq V$. Then $(()a)^{-1}N$ contains r' by the definition of the equivalence relation and is open because $()a$ is continuous. We wish to show that $(()a')^{-1}(()a)^{-1}N$ is contained in U, since then $(()a)^{-1}N$ is contained in $(\forall_{()a'}U)^{\circ}$, and thus $(r', a', x) \in \underline{U}$. Multiplication is commutative in our rings so $aa' = a'a$ and

$$(()a')^{-1}(()a)^{-1} = (()aa')^{-1} = (()a'a)^{-1} = (()a)^{-1}(()a')^{-1}.$$

Thus $(()a')^{-1}(()a)^{-1}N = (()a)^{-1}(()a')^{-1}N$, which is contained in $(()a)^{-1}V$. But $(()a)^{-1}V$ is contained in U since $V \subseteq (\forall_{()a}U)^{\circ}$. So $(()a)^{-1}N \subseteq (\forall_{()a'}U)^{\circ}$ as needed.

It is immediate from the construction of the topology on $\underline{\tilde{R}}$ and the fact that its addition and multiplication are defined in terms of addition and multiplication in $\underline{R}$ (the fact that a discrete topology was used in the S factor makes continuity easier to satisfy) that $\underline{\tilde{R}}$ is a topological ring object in the category of sheaves on X. By restricting to a stalk $\underline{\tilde{R}}_x$ we still get a topological ring, since in the étale space associated to $\underline{\tilde{R}}_x$ each fiber is a topological space and a ring with multiplication, addition and additive inverse continuous.

Example: Consider the ring of integers Z with the topology with basis given by the set of cosets of ideals. This is a topological ring in which multiplication by a non-zero element is an open map, so the condition (H) is satisfied and our construction applies. We determine the topology on the stalk $\underline{\tilde{Z}}_p$. This is the set of all rational numbers m/n in lowest terms such that p does not divide n. Since it is a topological ring, it will suffice to give a fundamental system of neighborhoods of 0.

Consider the set N(a,k) consisting of integers divisible by ap^k. It is a subset of $\underline{\tilde{Z}}_p$. We will show that the family of all such subsets forms a fundamental system of neighborhoods of 0 in the topology on $\underline{Z}_p$. For this we need to show that any neighborhood of 0 contains one of the N(a,k) and that the N(a,k) are themselves neighborhoods. The inverse image of N(a,k) under the map into the quotient is the set of ordered pairs (s,t) with s a multiple of tap^k. For each fixed t the set of all s which so appear is an ideal, so the inverse image of N(a,k) is open, hence N(a,k) is. Observe that 0 is always a member of N(a,k), so we have a family of neighborhoods.

Now suppose that U is an open neighborhood of 0. Then the inverse image of U is a saturated open set in $Z \times Z - pZ$, which must therefore have an open l-slice. This means that the l-slice contains an ideal, which must be principle, hence of the form bZ. If p does not divide b, the open neighborhood N(b,1) is contained in U. If p does divide b, let k be the highest power of p which divides b and let a be the other factor, then N(a,k) is in U.

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