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Laminations, or How to Build a Quantum-Logic-Valued Model of Set Theory

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Laminations, or How to Build a Quantum-Logic-Valued Model of Set Theory

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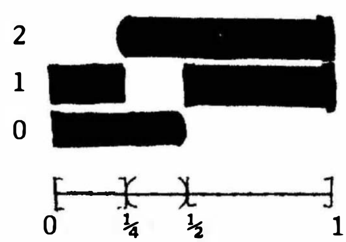
An explicit construction of the colimit of a filtered diagram in the category of topoi and logical morphisms is given and then used to construct a family of topoi with a fixed Boolean algebra of truth values but with varying amounts of cocompleteness. This same construction, when applied to the diagram of complete Boolean algebras in a quantum logic Q gives a partial topos, a noncategory which is a close to being a model of set theory with algebra of truth values Q as a noncategory can be.

This investigation concerns the construction of topoi or topos like noncategories which have specified algebras for their propositional logics. It is well known that one can construct topoi with given complete Heyting algebras as propositional logics by taking sheaves for the can onical topology, but the construction for noncomplete Boolean algebras appears in the literature only in the exercises of Johnstone's book (4) p. 331. That construction is a filtered colimit in the category of topoi. The existenc e of such colimits is asserted by Freyd (2) to follow from the essentially algebraic nature of the theory of topoi. No explicit construction exists in the literature. Freyd gives several examples of non-Grothendieck topoi constructed using colimits; for his purposes other descriptions give the desired properties more quickly.

The construction given here derives from a geometric construction for a topos with the Lebesgue measureable sets on $[0,1]$ as truth values. Objects are taken to be triples (P,Y,f) consisting of a partition P of $[0,1]$ into a countable number of measureable sets, a set Y, and a function assigning each element of the partition a subset of Y. We require the union of the values of f to be Y. Such an object may be thought of as analogous to a piece of sculpted plywood, one ply for each element of Y with sections cut out above certain elements of the partition. For example, if $P=\{[0,\frac{1}{4}],(\frac{1}{4},\frac{1}{2}), [\frac{1}{2},1]\}$, $Y = \{0,1,2\}$ and f takes $[0,\frac{1}{4}]$ to $\{0,1\}$, $(\frac{1}{4},\frac{1}{2})$ to $\{0,2\}$, and $[\frac{1}{2},1]$ to $\{1,2\}$, we may picture (P,Y,f) as



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Since the same picture results if a finer partition is used, we need to take an equivalence class to capture the idea precisely. A map from one such thing to another is a common refinement of the partitions involved and then a function from the subset above each member of the partition in the domain to the subset above the same member in the codomain. A bit of reflection shows that this is a topos with the desired algebra of truth values.

A similar construction can be done for the algebra of projections on a Hilbert space. We just define a partition to be an orthogonal family of projections with sum 1 and proceed as before. The result this time is not a category, but it comes as close to being a model of set theory as a noncategory can.

In the first section of this paper I will give the explicit construction of the filtered colimit in (Top,log) and show some of the properties it preserves. In the

second section, I will consider special cases giving Boolean topoi with varying amounts of cocompleteness. In the last section I will show how the same construction can be applied to a nonfiltered diagram to give a quantum-logic-valued model of set theory.

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I. Explicit Construction of Filtered Colimits: Logical Laminations

Let C be a filtered category and let F be a functor from C to (Top,log), the category of topoi and logical morphisms (recall that a logical morphism is a functor which preserves finite limits, powerobjects, and representation of relations). For convenience we will use upper case Greek letters for objects of C, lower case Greek letters for morphisms of C, and Roman letters for objects and morphisms of the associated topoi.

To say that C is filtered is to assert the existence of maps completing certain diagrams. In what follows we will want to refer to the objects and maps asserted to exist: hence for two objects Ξ and Ψ there is an object $\Xi\#\Psi$ and maps $\iota_\Xi:\Xi\longrightarrow\Xi\#\Psi$ and $\iota_\Psi:\Psi\longrightarrow\Xi\#\Psi$; for any diagram

$$\begin{tikzcd} & \Xi \\ \Sigma \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & \Psi \end{tikzcd}$$

there is an object $\Xi \underline{\#} \Psi$ and maps into it making the resulting square commute. This is like the assertion of the existence of colimits without universal mapping properties and the resulting uniqueness up to isomorphism. The notation does not pick out any canonical choice but rather refers to any object and maps with the desired properties.

DEFINITION 1.1: *A prelamination on* $(\mathcal{C},F)$ *is a pair* (Ξ,A) *where* Ξ *is an object of* $\mathcal{C}$ *and* A *is an object of* $F\Xi$.

This means that a prelamination specifies a topos and an object in that topos. We want to consider objects to be the same if they become the same in some later topos in the diagram. Hence we define a lamination as an equivalence class of prelaminations under the equivalence relation given by: (Ξ,A) and (Ψ,B) are equivalent if there is a diagram

$$\Xi \xrightarrow{\alpha} \Delta \xleftarrow{\beta} \Psi$$

in $\mathcal{C}$ such that $F\alpha A = F\beta B$. The equivalence classes are represented using square brackets: $[\Xi,A]$. Note that we need $\mathcal{C}$ filtered in order for this to be an equivalence.

A map of prelaminations is again a choice of topos and a choice of a map in that topos. To get a map of laminations we again need to take an equivalence class.

DEFINITION 1.2: *A morphism of prelaminations* $(h;\alpha,\Delta,\beta)$ *from* (Ξ,A) *to* (Ψ,B) *consists of a diagram*

$$\Xi \xrightarrow{\alpha} \Delta \xleftarrow{\beta} \Psi$$

in $\mathcal{C}$ *and a morphism* $h\colon F\alpha A \longrightarrow F\beta B$ *in* $F\Delta$.

Two maps of prelaminations $(h;\alpha,\Delta,\beta)$ *and* $(h';\alpha',\Delta',\beta')$ *are said to be equivalent if there is a commutative diagram*

$$\begin{tikzcd}
\Xi \arrow[dr, "\alpha"'] \arrow[drrr] & & \Psi \arrow[dl, "\beta"] \arrow[drr] & & & \Xi' \arrow[dll] \arrow[dr, "\alpha'"'] & & \Psi' \arrow[dlll] \arrow[dl, "\beta'"] \\
& \Delta \arrow[drr, "\sigma"'] & & T \arrow[d] & \Theta \arrow[dl] & & \Delta' \arrow[dlll, "\sigma'"] & \\
& & & \Sigma & & & &
\end{tikzcd}$$

in $\mathcal{C}$ *such that* $F\sigma h = F\sigma' h'$.

It is clear that equivalent maps take equivalent prelaminations to equivalent prelaminations; hence, we can call an equivalence class of prelaminations a map of laminations. Again we use square brackets to represent the equivalence classes. Again we need a filtered diagram in order for this to be an equivalence relation.

DEFINITION 1.3: *The composite of the morphism* $[h;\alpha,\Delta,\beta]$ *from* $[\Xi,A]$ *to* $[\Psi,B]$ *and* $[k;\gamma,\Theta,\delta]:[\Psi,B]\longrightarrow[\Phi,C]$ *has diagram*

$$\begin{tikzcd} \Xi \arrow[dr, "\alpha"'] & & \Psi \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & & \Phi \arrow[dl, "\delta"] \\ & \Delta \arrow[dr, "\iota_\Delta"'] & & \Theta \arrow[dl, "\iota_\Theta"] & \\ & & \Delta \mathop{\#}_{\Psi} \Theta & & \end{tikzcd}$$

and morphism $F\iota_\Theta k F\iota_\Delta h$.

The identity on $[\Xi,A]$ *is* $[id;\ id,\Xi,id]$.

PROPOSITION 1.1: *Laminations and maps of laminations form a category*, $\mathrm{Lam}(\mathcal{C},F)$.

Proof: The only non-trivial problem in showing $\mathrm{Lam}(\mathcal{C},F)$ is a category lies in showing the associativity of composition. The equivalence relation in the definition takes care of that problem by allowing us to consider representatives of both maps in the same topos: the composites $h(jk)$ and $(hj)k$ in the situation

$$[\Psi,A] \xrightarrow{[k;\alpha,\Xi,\beta]} [\Phi,B] \xrightarrow{[j;\gamma,\Delta,\delta]} [\Sigma,C] \xrightarrow{[h;\sigma,T,\tau]} [\Lambda,D]$$

live in $\Xi \mathop{\#}_{\Phi} (\Delta \mathop{\#}_{\Sigma} T)$ and $(\Xi \mathop{\#}_{\Phi} \Delta) \mathop{\#}_{\Sigma} T$, respectively, and these need not be the same. However; there is an object Γ after both in which h,j, and k each have one resulting representative. Associativity of composition in $\mathrm{Lam}(\mathcal{C},F)$ then follows because it holds in $F\Gamma$.

THEOREM 1.2: $\mathrm{Lam}(\mathcal{C},F)$ *is a topos.*

Proof: We need to show that $\mathrm{Lam}(\mathcal{C},F)$ has finite limits and powerobjects which represent relations.

The terminal relation is $[\Xi,1]$ where Ξ is any object of C and 1 is the terminal object in $F\Xi$. Note that for any two objects Ξ and Ψ the same lamination results since both $(\Xi,1)$ and $(\Psi,1)$ are equivalent to $(\Xi \# \Psi,1)$. The unique map from $[\Phi,A]$ to $[\Xi,1]$ is $[!;\iota_\Phi,\Phi \# \Xi,\iota_\Xi]$. If $[h;\alpha,\Delta,\beta]$ is another map from $[\Phi,A]$ to $[\Xi,1]$, then $[h;\alpha,\Delta,\beta]$ and $[!;\iota_\Phi,\Phi \# \Xi,\iota_\Xi]$ have the same representative in $F(\Delta \#_{(\Phi,\Xi)} (\Phi \# \Xi))$, that is, at the object making commutative the following diagram:

$$\begin{tikzcd}
\Phi \arrow[r, "\alpha"] \arrow[dr, "\iota_\Phi"] & \Delta \arrow[r] & \Delta \#_{(\Phi,\Xi)} (\Phi \# \Xi) \\
\Xi \arrow[r, "\iota_\Xi"'] \arrow[ur, "\beta"'] & \Phi \# \Xi \arrow[ur] &
\end{tikzcd}$$

The product of two laminations $[\Xi,A]$ and $[\Psi,B]$ is $[\Xi\#\Psi\ ,\ F\iota_\Xi A \times F\iota_\Psi B]$ with the projections $[\pi_i;id,\Xi\#\Psi,\iota\ ]$. We need to check that this definition respects the equivalence relation and that the universal mapping property is satisfied.

Suppose that (Ξ,A) is equivalent to (Ξ',A') and (Ψ,B) is equivalent to (Ψ',B'), then we have α,Δ,β and γ,Λ,δ such that $F\alpha A = F\beta A'$ and $F\gamma B = F\delta B'$. Transporting the whole situation to $\Delta\#\Lambda$ and noting that logical morphisms preserve finite limits, we get $F\iota_\Delta\alpha A \times F\iota_\Lambda\gamma B = F\iota_\Delta\beta A' \times F\iota_\Lambda\delta B'$. Thus equivalence is respected.

Now suppose that we have maps $[h;\alpha,\Delta,\beta\ ]$ and $[k;\gamma,T,\upsilon]$ from $[\Sigma,X]$ to $[\Xi,A]$ and $[\Psi,B]$, respectively. Without loss of generality we may choose representatives so that we have a diagram of prelaminations. If we move everything to the topos $F(\Delta \#_\Sigma T)$, we get a unique map from $F\iota_\Sigma X$ to $F\iota_\Delta\beta A \times F\iota_T\upsilon B$. The equivalence class of that map is the map we need for the universal mapping property.

The equalizer of two maps $[h;\alpha,\Delta,\beta]$ and $[k;\gamma,T,\upsilon]$ from $[\Xi,A]$ to $[\Psi,B]$ is obtained by transporting all of the data to the topos $F(\Delta \underset{(\Xi,\Psi)}{\#} T)$ and taking the equalizer there. The equivalence class of this is the desired equalizer in Lam(C,F).

Before we can describe powerobject formation we need the following characterization of monomorphisms;

LEMMA 1.3: *A map* $[h;\alpha,\Delta,\beta]:[\Xi,A] \longrightarrow [\Psi,B]$ *is monic if and only if for any* Δ *there is a map* $\gamma:\Delta \longrightarrow \Sigma$ *such that the representative of* h *at* $F\Sigma$ *is monic.*

Proof: The map $[h;\alpha,$ is monic if and only if the two new maps in the pullback

$$\begin{tikzcd}
{[\Delta, A\times_{B}} \arrow[r, "{[k;id,\Delta,\alpha]}"] \arrow[d, "{[k';id,\Delta\ ,\ \alpha]}"'] & {[\Xi,A]} \arrow[d, "{[h;\alpha,\Delta,\beta]}"] \\
{[\Xi,A]} \arrow[r, "{[h;\alpha,\Delta,\beta]}"] & {[\Psi,B]}
\end{tikzcd}$$

are the same. This happens if and only if they are equivalent as maps of prelaminations. That happens if and only if there is a map $\gamma:\Delta \longrightarrow \Sigma$ such that $F\gamma k = F\gamma k'$ by the definition of the equivalence relation. Given the way that pullbacks are calculated in Lam(C,F), this is the same as saying that the representative of the diagram at $F\Sigma$ is a pullback with the two maps k and k' the same. Thus the representative of h at Σ is monic.

We can now show that the lamination $[\Xi,$ has powerobject $[\Xi,PA]$ with $\varepsilon_{[\Xi,} = [\Xi,\varepsilon_A]$. Suppose we are given a monomorphism $[h;\alpha,\quad : [T, \longrightarrow [\Xi, \quad \times$ Then for some $\gamma:\Delta \longrightarrow \Sigma$, $F\gamma h$ is monic and lands in $F\gamma\beta\iota_{\Xi}A \times F\gamma\beta\iota_{\Psi}B$. Thus it induces a unique map $\chi_{F\gamma h}$ from $F\gamma\beta\iota_{\Psi}B$ to $PF\gamma\beta\iota_{\Xi}A$, which is equal to $F\gamma\beta\iota_{\Xi}PA$ because $F\gamma\beta\iota_{\Xi}$ is logical. The universal property of the characteristic map is obtained by noting that it makes a square with ε and the characteristic map a pullback. The calcula-

tion of pullbacks in Lam(C,F) makes it clear that the universal property of the characteristic map will hold there if we use the class of $\chi_{F\gamma h}$ as the characteristic map of the class of h.

Next we may observe that what we have constructed is the colimit of the diagram in (Top,log) given by F.

PROPOSITION 1.4: Lam(C,F) is $\lim\limits_{\to C}$ FΞ.

Proof: First we note that there is a logical morphism from each FΞ into Lam(C,F) given by $A \longmapsto [\Xi,A]$ and $h: A \to B \longmapsto [h;id,\Xi,id]$. That this is logical is clear from the construction of limits and powerobjects. Furthermore, for each $\alpha:\Xi \to \Delta$ in C we have $[\Delta,F\alpha A] = [\Xi,A]$ and $[F\alpha h;\alpha,\Delta,\alpha] = [h;id,\Xi,id]$, so these injections from FΞ into Lam(C,F) form a cocone. We need to show that it is the universal cocone.

Suppose $(E,\{e_\Xi | \Xi \ \ C\})$ is another cocone. Then the functor taking $[\Xi,A]$ to $e_\Xi A$ and $[h;\alpha,\Delta,\beta]$ to $e_\Delta h$ is the unique logical functor making all of the triangles commute.

This proposition makes it possible for us to identify the NNO, the reals, and Ω in Lam(C,F) and to describe the global sections thereof. The NNO is $[\Xi,N]$; the reals are $[\Xi,\mathbb{R}]$; Ω is $[\Omega,\Xi]$, where in each case Ξ is an arbitrary object of C. In general, it will only make sense to talk about the global sections of objects for small C.

COROLLARY1.5: If C is small, then $\Gamma\Omega = \lim\limits_{\to C} \Gamma\Omega_\Xi$.

Proof: Since each Fα is logical, each takes global sections of Ω to global sections of Ω; thus F induces a diagram in *Sets* with Ξ going to $\Gamma\Omega_\Xi$. The colimit of this diagram is the coproduct modulo the same equivalence relation used in defining a map of laminations.

A similar characterization can be obtained for the global sections of any structure defined logically-- the natural numbers and the reals are good examples.

The next question we can ask is what about generating sets? We will show in section II that completeness properties fail to be preserved by this construction (fortunately) so the next proposition tells us that the construction of laminations on a diagram of Grothendieck topoi comes as close as possible to being a Grothendieck topos, given that it need not be complete.

PROPOSITION 1.6: *If $\mathcal{C}$ is small and each $F\alpha$ is faithful and if $\{G_{i,\Xi}\}$ is a set of generators for $F\Xi$ given for each object of $\mathcal{C}$, then the union of the sets $\{[\Xi, G_{i,\Xi}]\}$ is a set of generators for $\mathrm{Lam}(\mathcal{C},F)$.*

Proof: The proposed set of generators is small because $\mathcal{C}$ is and the sets for each Ξ are. Two maps $[h;\alpha,\Delta,\beta]$ and $[k;\gamma,T,\upsilon]:[\Xi,A] \longrightarrow [\Psi,B]$ can be distinguished by a generator in $F(\Delta \mathbin{\#_{(\Xi,\Psi)}} T)$. Since each $F\omega$ is faithful, the class of that generator will distinguish the maps of laminations.

To investigate the axiom of choice we will need a characterization of the epimorphisms in $\mathrm{Lam}(\mathcal{C},F)$.

LEMMA 1.7: *A morphism $[h;\alpha,\Delta,\beta]$ is epic if and only if there is a map $\gamma;\Delta \to \Sigma$ with $F\gamma h$ epic.*

Proof: Dualize the proof of Lemma 1.3 by characterizing epic maps in terms of pushouts. This cannot be done before we know $\mathrm{Lam}(\mathcal{C},F)$ is a topos unless we give a construction for colimits.

PROPOSITION 1.8: Lam(C,F) will satisfy AC if for any Ξ in C there is a map $\gamma:\Xi \to \Psi$ such that FΨ satisfies AC.

Proof: By the previous lemma an epi in Lam(C,F) has a representative which is epi, say in FΞ. Since Fγ is logical, it preserves epis, so if $\gamma:\Xi \to \Psi$ with FΨ having epis split (AC) takes our epi to a split epi. The class of the splitting map in FΨ is the splitting map in Lam(C,F).

EXAMPLE: This proposition does not give a necessary and suffiecient condition. If we let C be the category of initial segments of N with inclusion maps and let F assign each segment the topos $\mathit{Sets} \times \mathit{Sets}^{Z_2}$ and each inclusion to the logical morphism

$$\mathit{Sets} \times \mathit{Sets}^{Z_2} \xrightarrow{\pi_1} \mathit{Sets} \xrightarrow{(id,\mathit{Sets}^!)} \mathit{Sets} \times \mathit{Sets}^{Z_2}.$$

Lam(C,F) is just *Sets* since any difference in the second factors is ignored by the equivalence relation. In this case Lam(C,F) satisfies AC even though none of the FΞ do.

II. Laminations on a Boolean Algebra and Other Examples

In [3] Freyd used filtered colimits of topoi to construct a nonsolvable topos; he also used then in the proofs of the embedding theorems for small topoi in section 5 of [2] and [3]. His claim there is that since the theory of topoi is essentially algebraic (i.e. a partially algebraic theory in which the domain for each operation is equationally defined in terms of previously defined operations) filtered colimits are easy to construct. Johnstone [4] p.331 states that Freyd has used the construction to obtain topoi with arbitrary Boolean algebras as algebra of truth values. Personal communication with Freyd indicates that his construction gives the free

Freyd indicates that his construction gives the free Boolean topos with the given algebra of truth values. One must be a little careful about the sense in which one means free, however. Lambek [5] and Mitchel[6] both note that (Top,log) does not have an initial object and hence does not have arbitrary colimits. If one uses orthodox functors (i.e. one makes choices for all limits, subobjects, and exponentials and asks that these choices be preserved) then the initial objects and free topoi exist. In this section we will first note another example of the use of colimits due to Andreas Blass and then investigate a family of topoi LamB_π with a given Boolean algebra B of truth values and differing amounts of cocompleteness.

At the meeting of the New York Topos Seminar, April 16,1978, Andreas Blass gave the following example:

Let G be the group $Z^{\aleph}$ where $\aleph$ is uncountable. If $E \subseteq \aleph$ then $G_E = \{g \in G \mid \text{For all } e \text{ in } E, g_e = 0\}$. If a subgroup H of G contains G_E for some countable E, we call H large; if it contains G_E for a finite E, we call it immense. If a subgroup H acts on X then fix x = $\{g \mid g(x)=x\}$ and ker X is the intersection of all the fix x with x in X. Blass considers the topos of laminations on $(\mathcal{C},F)$ where $\mathcal{C}$ is the category of immense subgroups of G and F takes a subgroup to the topos of H-sets such that for each x, fix x is immense and such that ker X is large. The inclusion map $H' \longrightarrow H$ induces a logical morphism from H'-sets to H-sets which restricts to a logical morphism between the topoi with size restrictions. Blass showed that the category $\mathrm{Lam}(\mathcal{C},F)$ has no nontrivial injective abelian group objects.

For the next example let us assume that B is a Boolean algebra which has all unions and intersections of cardinality less than κ. If π is a cardinal less than κ,

then a π-partition is a disjoint family with cardinality less than or equal to π which has supremum 1. We say that a partition P refines a partition Q if every element of Q is the supremum of elements of P. Refinement orders the set of all π-partitions. Any two π-partitions P and Q have a least common refinement P # Q consisting of all the nonempty intersections $p \cap q$ with p in P and q in Q. The singleton {1} is an initial element in the category of partitions. The category of π-partitions is thus co-complete and hence filtered.

For each π-partition P we can take the topos of presheaves on P thought of as a discrete category. If P refines Q then there is a logical morphism from Sets^Q to Sets^P taking a presheaf F to the presheaf G whose value at p is Fq where q is the unique element of Q containing p.

This gives a filtered diagram in (Top,log); call its colimit LamB_π. The sculpted plywood image described in the introduction results from a sort of étale space representation of the presheaf categories. From Lemma 1.3 and Lemma 1.7 we obtain the following:

LEMMA 2.1: *A map [P,f] is monic (resp. epic) if and only if it has a representative which is monic (resp. epic).*

In fact for LamB_π it will be the case that every representative is monic (resp. epic).

PROPOSITION 2.2: *In* LamB_π, $\Gamma(\Omega) = B$.

Proof: Any subobject of 1 consists of a partition and a choice of 0 or 1 on each element of the partition. An equivalent representative can always be found by taking the union of the elements on which the subobject is 1 and a union of the elements on which the subobject is 0,

getting a partition with two elements. The subobjects of 1 then are in 1-1 correspondence with elements of B with the correspondence given by taking a subobject to the element on which it is 1.

If B is complete we may remove the cardinality condition on the partitions to obtain LamB. This also has $\Gamma(\Omega) = B$.

We may note that since each of the presheaf categories is a Boolean Valued model of set theory in the sense of Tierney [10] (each satisfies AC and has subobjects of 1 generate), so are LamB_{π} and LamB. In both cases the NNO is $[N,\{1\}]$.

Note that if B is not complete then LamB_{π} cannot be cocomplete since otherwise $\Gamma\Omega$ (which is B by proposition 2.2) would be complete. Somewhat more can be said about cocompleteness using the following lemma:

LEMMA 2.3: *For both LamB and LamB_{π}, if the copower of 1 indexed by the cardinal λ exists, then it is isomorphic to $[\{1\},\lambda]$.*

Proof: For each element a of λ there is a map from 1 to $[\{1\}, \lambda]$ given by $[* \to a, \{1\}]$. This family of maps is jointly epic, so the induced map from the copower of 1 to $[\{1\},\lambda]$ is epic. Let (f,P) be a representative of the induced map. We will show that the component of f at p is monic for each p in P. Suppose not, then for some p there are values x and y both of which are taken to the same value by f at p. The injection map from 1 to the copower at p corresponding to the index b, $i_{b,p}$, takes * to something which maps to b. If $b' \neq b$, then

$i_{b',p}$* will not be x or y if both of them are mapped to b by f. Since $i_{b,p}$ is single valued it must miss one of x and y, say y. If we redefine f at p so that it takes y to b' $\neq$ b, we will not have changed any of the compositions $f_p i_{a,p}$. This violates the uniqueness of f. Thus if LamB (or LamB_π) has the copower of 1 the induced map f must be monic as well as epic, hence an isomorphism.

<u>THEOREM 2.4</u>: <u>LamB_π is λ-cocomplete if and only if B satisfies a (λ,π) distributive law.</u>

Proof: By Sikorski 20.3 [8], (λ,π)-distributivity is equivalent to the statement that any family of at most λ π-partitions has a common refinement. Suppose B is (λ,π)-distributive, then we can construct a colimit of a diagram involving λ pieces of data by choosing a representative of each piece of data to obtain a family of at most λ partitions, then take the common refinement guaranteed by distributivity and then take colimits on each member of the common refinement.

If B does satisfy a (λ,π)-distributive law, then there is a family of partitions which does not have a common refinement, say $\{P_a \mid a \in \lambda\}$. We will show that the copower of 1 indexed by λ cannot exist. By Lemma 2.3 if the copower exists we may as well assume that it is $[\{1\},\lambda]$. For each a in λ we obtain a map from 1 to $[\{1\},B]$ using the partition P_a and taking * to p on p. Thus we get an induced map from $[\{1\},\lambda]$ to $[\{1\},B]$. The partition for this map must refine all of the P_a since the element a must go different places on different elements of P_a. But such a common refinement does not exist. Thus the copower cannot exist.

We can use the same tools to specify when LamB is the category of sheaves on B. We know from Barr [1] that a topos over *Sets* has Δ logical if and only if the topos is atomic. For LamB we know Δ is logical, the following theorem says that it is adjoint to Γ if and only if B is atomic.

THEOREM 2.5: LamB <u>is isomorphic to</u> Sh(B) <u>if and only if</u> B <u>is atomic</u>.

Proof: If B is atomic, then the functor from *Sets* to LamB given by $\Delta A = [\{1\},A]$ is logical. If B is atomic, it is also adjoint to the global sections functor. We need B to be atomic so that the adjunction map from $\Delta\Gamma$ to the identity on LamB can be defined. The partition it uses is the set of all atoms. By 5.39 in Johnstone [4], a topos over *Sets* satisfies AC if and only if it is isomorphic to cannonical sheaves on $\Gamma\Omega$. Since LamB satisfies AC and is a topos over *Sets* when B is atomic, this tells us that LamB is isomorphic to Sh(B) when B is atomic.

If B is not atomic then LamB is not cocomplete. In particular it does not have the copower of 1 indexed by B. If that copower exists, by Lemma 2.3, it is $[\{1\},B]$. For each β in B there is a map from 1 to $[\{1\},B]$ which takes * to β on β and 0 on the complement of β. This would induce a map from $[\{1\},B]$ to itself. The partition involved in this map must be a disjoint set of elements of B which generates all of B. The existence of such a set is precisely the statement that B is atomic.

If B is not atomic LamB is not cocomplete, hence it cannot be Sh(B), since Sh(B) is cocomplete.

Categories of laminations on a Boolean algebra have another nice property: if B' is a π-complete subalgebra of B then any π-partition of B' is also a π-partition of B; thus there is a natural inclusion $\mathrm{LamB}'_\pi \to \mathrm{LamB}_\pi$. This inclusion is a logical morphism.

III: <u>Laminations on a Quantum Logic</u>

The construction in section I runs into difficulties if *C* is not filtered-- the equivalence relation used to define laminations is no longer transitive so we must take its transitive closure; composition of morphisms

need not be defined; limits cause a few problems because we needed to beable to put all of the data in the same topos. Still, we would like to be able to paste together Boolean topoi to obtain Quantum Logic-valued models of set theory since this would give us a setting in which to do quantum valued mathematics. Since a quantum logic is typically an orthomodular lattice and not a Heyting algebra, we should not expect to get a topos. Indeed, since we want the lattice of truth values to be non-distributive, we cannot use a category since logical categories always have distributive lattices of truth values (Reyes [7]). We will aim for a structure which is as close to being a topos as a noncategory can be.

To do this we will introduce a notion of compatibility corresponding to commutativity in the case where the orthomodular lattice is the lattice of all projection operators on a Hilbert Space. We will ask that limits of compatible data exist, that compatible data gives exponentiation, and that subobjects be representable in a nice compatible fashion.

In this section we will study the situation in which C has an initial object, but need not be filtered, F is a functor from C to (Top,log). A construction parallel to the one in section I leads to a partial topos-- a partial category which is as close to being a topos as a noncategory can be.

<u>DEFINITION 3.1</u>: <u>A partial category consists of a family I of indices of compatibility, a family F of subsets of I giving compatible sets of indices, a class of objects, a class of morphisms, and an assignment of an index (indicated by anI with an appropriate subscript) to each object and morphism such that the following conditions are satisfied</u>:

1. If $(h,I_h):(A,I_A)\longrightarrow(B,I_B)$ is a morphism, then $\{I_h,I_A,I_B\}$ is in F.
2. There is an identity on (A,I_A) with index I_A.
3. For any pair of maps

$$(A,I_A)\xrightarrow{(h,I_h)}(B,I_B)\xrightarrow{(k,I_k)}(C,I_C)$$

   with $\{I_A,I_h,I_B,I_k,I_C\}$ in F, there is a composite (kh,I_{kh}) such that if $S\cup\{I_h,I_k\}$ is in F, then so is $S\cup\{I_{kh}\}$.
4. Composition is associative when defined.
5. Composition with the identity on either side is always defined and changes neither the map composed with nor its index.
6. F contains all singletons.
7. F is closed under subsets.

As before, a prelamination on (C,F) will consist of a pair (Ξ,A) where Ξ is an object of C and A is an object of $F\Xi$. To define laminations we will take equivalence classes of prelaminations where the equivalence relation is the transitive closure of the one used in section II. Thus (Ξ,A) is equivalent to (Ξ',A') if and only if there is a finite chain of the form

$$\begin{tikzcd}[column sep=small] & \Delta_1 & & \Delta_2 & & \Delta_n & \\ \Xi \arrow[ur, "\alpha_1"] & & \Xi_1 \arrow[ul, "\beta_1"'] \arrow[ur, "\alpha_2"] & & \cdots\cdots & & \Xi \arrow[ul, "\beta_n"'] \end{tikzcd}$$

such that for each i, $F\alpha_{i+1}A_i = F\ {}_{i+1}A_{i+1}$.

In a similar fashion a map of laminations h from $[\Xi,A]$ to $[\Psi,B]$ consists of a family of maps of prelaminations such that

1. the domain of each representative $(h;\alpha,\Delta,\beta)$ is in $[\Xi,A]$
2. the codomain of each representative is in $[\Psi,B]$

3. if $(h;\alpha,\Delta,\beta)$ is a representative taking (Ξ,A) to (Ψ,B) and $(k;\alpha',\Delta',\beta')$ is a map of prelaminations taking (Ξ',A') to (Ψ',B'), where (Ξ',A') is in $[\Xi,A]$ and (Ψ',B') is in $[\Psi,B]$, then $(k;\alpha',\Delta',\beta')$ is in h if and only if there is a finite chain of maps of prelaminations $(h_i;\alpha_i,\Delta_i,\beta_i)$: $(\Xi_i,A_i) \to (\Psi_i,B_i)$ connected as illustrated below:

$$\begin{tikzcd}
 & \Delta \arrow[rr, "\delta_1"] & & \Delta_1 & & \Delta_2 \arrow[ll] & \cdots & \Delta_n & & \Delta' \arrow[ll, "\delta_{n+1}"'] \\
\Xi \arrow[ur, "\alpha"] \arrow[rr] & & \Xi_1 \arrow[ur] & & \Xi_2 \arrow[ll] \arrow[ur] & \cdots & \Xi_n \arrow[ur] & & \Xi' \arrow[ll] \arrow[ur, "\alpha'"] & \\
 & \Psi \arrow[uu, "\beta"'] \arrow[rr] & & \Psi_1 \arrow[uu] & & \Psi_2 \arrow[ll] \arrow[uu] & \cdots & \Psi_n \arrow[uu] & & \Psi' \arrow[ll] \arrow[uu, "\beta'"']
\end{tikzcd}$$

in which each of the squares commutes and each $F\delta_i . h_{i-1} = h_i$ for odd i and $F\delta_i . h_{i+1} = h_i$ for even i.

Note that if the "only if" part of condition 3 is satisfied we can always extend the family of maps to satisfy the "if" part of the condition.

The family of indices of compatibility for C is the set of all nonempty classes of objects of C such that

1. if Ξ is in I and there is a map $\alpha:\Xi \to \Psi$ then Ψ is in I
2. any two objects in I are connected by a finite chain of maps in the full subcategory of C on the objects in I similar to the chain used in the definition of a lamination.

A family of indices is compatible if every subset of it has an intersection which is an index.

The index of a lamination $[\Xi,A]$ is the class of objects at which it has a representative. The index of a map of laminations is the class of all Δ such that there is a representative $(h;\alpha,\Delta,\beta)$.

PROPOSITION 3.1: For any category C equipped with a functor F to the category of topoi with logical morphisms, the structure with objects laminations, morphisms maps of laminations, and indices of compatibility as defined above is a partial category.

Proof: Properties 1,2,6,and 7 are essentially trivial--to show that Definition 3.1 is satisfied we need only give the composite for compatible composable maps. Our definition of compatibility guarantees that the class of objects for which two maps of laminations have composable representatives forms an index. This means that if we define the composite to be the family of maps of prelaminations consisting of the composites of representatives at objects in this index we will almost have a map of laminations. All that can fail is the "if" part of condition 3 in the definition of a map of laminations. By the note after the definition, we can extend to get a unique map of laminations. This composition is clearly associative when defined and behaves properly with respect to identities.

The notion of a limit in a partial category must carry a compatibility condition. A limit is a compatible cone on a diagram which is universal among all compatible cones. The partial category Lam(C,F) has such limits and also has powerobject formation.

DEFINITION 3.2: A partial category is a partial topos if

1. Any finite diagram of compatible data has a limit with index I_{lim} such that if $S \cup \{I \mid I$ is an index involved in the data$\}$ is in F then so is $S \cup \{I_{lim}\}$.

2. <u>For any object A there is an object PA and a subobject ε_A of $PA \times A$, both with the same index as A, such that subobjects of $A \times B$ correspond 1 to 1 with maps $B \longrightarrow PA$ through a pullback</u>

$$\begin{tikzcd} C \arrow[rr] \arrow[d, "h"'] & & \varepsilon_A \arrow[d] \\ A \times B \arrow[rr, "1 \times \chi_h"] & & PA \times A \end{tikzcd}$$

<u>THEOREM 3.2:</u> Lam(C,F) <u>is a partial topos.</u>

Proof: the existence of limits of finite compatible diagrams is easy: since the diagram is compatible, the intersection of the indices of the data is an index at each member of which we have a diagram. The limits of these diagrams fit together to form a lamination (extending if necessary to get a full equivalence class) which is the limit. The terminal is $[I,1]$ where I is the initial object of C and 1 is the terminal object of FI.

The objects PA and ϵ_A are obtained by taking $[\Xi,PA]$ and $[\Xi,\epsilon_A]$. In order to show that these classify relations into A, we need a characterization of monos in Lam(C,F).

Now in any category m is monic iff the square

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, "="'] & D \arrow[d, "m"] \\ D \arrow[r, "m"] & C \end{tikzcd}$$

is a pullback. For this to be true for a map of laminations there must be an index on which each representative is a pullback. Hence, each representative must be a mono. With this information it is clear that the characteristic map of a relation is the map of laminations consisting of the characteristic maps of representatives of the relation at each level in the index of the relation.

This theorem tells us that Lam(C,F) is as close to being a topos as a partial category can be. To pinpoint its internal propositional calculus we may ask what the subobjects of the terminal are. In a manner analogous to the technique used when C has colimits we can show that $\Gamma\Omega = \varinjlim_{C} \Gamma\Omega_{\Xi}$. Investigating the axiom of choice yeilds the result that if each of the topoi $F\Xi$ satisfies AC then so does Lam(C,F). Thus if we use this construction to piece together models of set theory we will obtain a structure very like a model of set theory, only with a highly nonstandard propositional logic.

In particular, if we let C be the category with objects complete Boolean algebras of projections on a Hilbert space H and morphisms inclusions as complete subalgebras, then we can define F taking B to LamB as defined in section II (the topos of arbitrary partition laminations on B) and taking an inclusion morphism to the logical morphism described at the bottom of p.17.

If we apply the construction of this section to this situation, we will obtain a partial topos which has the lattice of all projections on H as truth values and which has all epimorphisms split. It has an NNO given by [I,N] where I is the complete Boolean algebra consisting of the identity on H and the zero map. This means that Lam(C,F) has precisely the kind of structure that we would like a Quantum-logic valued model of set theory to have. In particular, we can model higher order theories in Lam(C,F) without having to change too many of our logical thought processes. We can do mathematics in a quantum - logic valued world.

It would be more desirable to use a functor which gave the Boolean-valued models studied by Takeuti [9] instead of LamB, but it is not known how (or whether) the inclusion of complete Boolean algebras induces a logical morphism between the Boolean -valued models he considers.

REFERENCES

[1] BARR, M., Atomic Sites, manuscript, 1976

[2] FREYD, P., Aspects of Topoi, Bul.Austr. Math.Soc. 7 (1972) #2

[3] FREYD, P., Aspects of Topoi, Corregia and Acknowledgements, Bul.Austr.Math.Soc. 7 (1972) #3, p.467-485

[4] JOHNSTONE, P. T., Topos Theory, Academic Press, London New York, San Francisco, 1978

[5] LAMBEK, J., From Types to Sets, to appear in Advances in Mathematics

[6] MITCHELL, W., Categories of Boolean Topoi, J.Pure and Applied Algebra 3(1973) p.193-201

[7] REYES, G., From Sheaves to Logic, in Studies in Algebraic Logic, MAA Studies in Mathematics, Vol 9 (1974), p.143-204

[8] SIKORSKI, R., Boolean Algebras, Springer Verlag New York, 1969

[9] TAKEUTI, G., Boolean Valued Analysis I, II, III, manuscript, University of Illinois, 1976

[10] TIERNEY, M., Sheaf Theory and the Continuum Hypothesis, in Lecture Notes in Mathematics 274, Springer Verlag, New York, Heidelberg, Berlin, 1972