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Sequences, Series, and Function Approximation

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1 What's a sequence and why should you care?

Definition 1 *A sequence of real numbers is a function from the natural numbers to the reals* $a : \mathbb{N} \to \mathbb{R}$.

We typically use subscripts rather than functional notation using parentheses to indicate this so that the sequence $a : \mathbb{N} \to \mathbb{R}$ is usually written as

$$a_0, a_1, \ldots, a_n \ldots$$

Sequences are important in approximation: the usual representation of real numbers using decimals is in fact the process of giving a sequence of rational numbers approximation the real number in question successively better as more decimal places are given. These decimal approximation sequences are actually rather special: successive decimal approximations never get smaller (so the sequence is *monotone non-decreasing*) and two approximations which agree to the k^{th} decimal place differ by at most 10^{-k} (so the sequence is a Cauchy sequence: to make two values in the sequence close to each other all you need to do is take them far enough out in the sequence).

Most of the important functions from the reals to the reals which we use are actually only able to be calculated approximately. Series representations (based on sequences of real numbers) provide the means to get arbitrarily good approximations.

Sequences were also used by Cauchy to construct the real numbers from the rationals. Real numbers can be constructed as equivalence classes of Cauchy sequences of rationals under an equivalence relation with $a \sim b$ provided that for any ϵ there is an M such that if $n > M$ then $|a_n - b_n| < \epsilon$.

1.1 Limits of Sequences

Calculus deals with limits, derivatives, and integrals. For sequences the only limits of interest are those as $n \to \infty$. Derivatives don't really exist, though one can study finite differences (a whole theory for calculus of finite differences parallels the usual calculus—the major textbook in the subject is by George Boole of Boolean Algebra fame). The analog of improper integrals is given by series.

Recall the definition of the limit of a function of a real variable:

Definition 2 *If $f : \mathbb{R} \to \mathbb{R}$ then $\lim\limits_{x\to\infty} f(x) = L$ means that for any $\epsilon > 0$, there is an M such that if $x > M$ then $|f(x) - L| < \epsilon$.*

If we restrict the x's to be natural numbers we get the definition of a limit for a sequence:

Definition 3 *For a sequence $a : \mathbb{N} \to \mathbb{R}$ the limit $\lim\limits_{n\to\infty} a_n = L$ means that for any $\epsilon > 0$ there is an M such that if $n > M$ then $|a_n - L| < \epsilon$. If a sequence has a limit we say it is convergent. If not, we say it is divergent.*

The similarity of these two definitions together with the fact that you already have techniques for finding limits of functions of a real variable as $x \to \infty$ (used for finding horizontal asymptotes) makes the following lemma both useful and easy to prove:

Lemma 1 *If $a_n = f(n)$ for every n and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.*

PROOF:

> Given ϵ we get an M from $\lim_{x\to\infty} f(x) = L$ so that if $x > M$ then $|f(x) - L| < \epsilon$. Use that same M for $\lim_{n\to\infty} a_n = L$. If $n > M$ then we know $|f(n) - L| < \epsilon$, but this is the same as $|a_n - L| < \epsilon$. ∎

Notice that this lemma only goes one way: having the function have L as limit is much stronger than having the sequence have L as limit since the function of a real variable has many more values which must be made close to L. It is quite possible for the sequence a_n to converge but for a function f with $f(n) = a_n$ to have no limit. For example if $a_n = \sin(n\pi)$ then $f(x) = \sin(x\pi)$ has no limit, but the sequence does since $\sin(n\pi) = 0$ for every natural number n, so the sequence converges to 0.

Example: Using limits of real valued functions

For many limits of sequences this theorem applies quite directly. The work looks like what we are doing is just changing the variable from n (which is taken to be in $\mathbb{N}$) to x (which is taken to be in $\mathbb{R}$). This is a reminder that the kind of limit has in fact changed and we are using previous knowledge in a new situation:

1. If $a_n = \frac{n+1}{n^2+2n+3}$ then $a_n = f(n)$ where $f(x) = \frac{x+1}{x^2+2x+3}$. We know how to find
$$\lim_{x\to\infty} \frac{x+1}{x^2+2x+3} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{3}{x^2}} = 0$$
so $a_n \to 0$ as well.

2. We can find
$$\lim_{n\to\infty} \frac{n^2+1}{3n^2-2} = \lim_{x\to\infty} \frac{x^2+1}{3x^2-2} = \frac{1}{3}.$$

3. If $b_n = n \sin 1/n$ then we use $f(x) = x \sin(1/x)$. We can find the limit by
$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{t\to 0^+} \frac{\sin(t)}{t} = 1.$$
Thus $b_n \to 1$ as well.

$\diamondsuit$

Exercises: Find the limit, if it exists, of the following sequences:

1. $a_n = \dfrac{(-1)^n}{n}$
2. $a_n = \dfrac{n}{n^2-1}$
3. $a_n = \dfrac{n}{\sqrt{n^2-1}}$
4. $a_n = e^{-n}$
5. $a_n = \arctan(n)$

Several important limits make use of l'Hôpital's rule. We need to change to a functions of a real variable here since the derivative is not defined for sequences.

Example: **If** $a_n = \left(\frac{n+1}{n}\right)^n$ **then** $a_n \to e$

$$
\begin{aligned}
\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n &= \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x \\
&= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \\
&= \lim_{x\to\infty} e^{\left(x\ln\left(1+\frac{1}{x}\right)\right)} \\
&= e^{\left(\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\right)} \\
&= e^{\left(\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)} \\
&= e^{\left(\lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}}\right)} \text{ by L'Hôpital} \\
&= e^{\left(\lim x\to\infty\frac{1}{\left(1+\frac{1}{x}\right)}\right)} \\
&= e^1 = e
\end{aligned}
$$

$\diamondsuit$

Example: $\sqrt[n]{n} \to 1$

Again we compute using l'Hôpital's rule:

$$
\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{n} &= \lim_{x\to\infty} x^{1/x} \\
&= \lim_{x\to\infty} e^{((1/x)\ln(x))} \\
&= e^{\left(\lim_{x\to\infty}\frac{\ln(x)}{x}\right)} \\
&= e^{\left(\lim_{x\to\infty}\frac{\frac{1}{x}}{1}\right)} \text{ by L'Hôpital} \\
&= e^0 = 1
\end{aligned}
$$

$\diamondsuit$

Exercises: Find the limit of the following sequences:

1. $\left(\dfrac{1-n}{n}\right)^n$
2. $\sqrt[n]{n^2}$
3. $\left(1+\dfrac{1}{n}\right)^{2n}$
4. $\left(1+\dfrac{1}{n^2}\right)^n$
5. $\left(1+\dfrac{1}{n}\right)^{n^2}$

1.2 Algebra of limits of sequences

Just as for limits of functions of a real variable we can combine limits using multiplication, addition, division, and the squeeze theorem. The proofs here closely mirror the ones for limits of functions of a real variable.

Theorem 2 *If $a_n \to A$ and $b_n \to B$ then*

1. $a_n + b_n \to A + B$
2. $a_n b_n \to AB$
3. $a_n/b_n \to A/B$ *provided* $B \neq 0$

PROOF:

Parts 1 and 2 are left as exercises. For quotients we first bound the size of $|b_n|$ away from 0. Since $b_n \to B \neq 0$ we can find an M_1 such that if $n > M_1$ then b_n is between $B/2$ and $3B/2$, thus guaranteeing that $|b_n| > |B|/2$. Next we find M_2 so that if $n > M_2$ then $|a_n - A| < \frac{|B|\epsilon}{4}$ and M_3 so that if $n > M_3$ then $|b_n - B| < \frac{B^2\epsilon}{4|A|}$. These are chosen so that the parts in the following calculation will end up with ϵ. If $n > \max(M_1, M_2, M_3)$ then

$$
\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{A}{B}\right| &= \left|\frac{a_nB - Ab_n}{b_nB}\right| \\
&= \left|\frac{a_nB - AB + AB - Ab_n}{b_nB}\right| \\
&\leq \left|\frac{a_nB - AB}{b_nB}\right| + \left|\frac{AB - Ab_n}{b_nB}\right| \\
&\leq \left|\frac{2(a_nB - AB)}{B^2}\right| + \left|\frac{2(AB - Ab_n)}{B^2}\right| \\
&= \frac{2}{|B|}|a_n - A| + \frac{2|A|}{|B^2|}|B - b_n| \\
&< \frac{2}{|B|}\frac{|B|\epsilon}{4} + \frac{2|A|}{|B^2|}\frac{|B^2|\epsilon}{4|A|} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

∎

Theorem 3 *If $a_n \leq b_n \leq c_n$ and both $A_n \to L$ and $c_n \to L$ then $b_n \to L$.*

Here again the proof, which is easy, is left as an exercise.

Exercises: Prove the remaining parts of the theorems in this section:

1. If $a_n \to A$ and $b_n \to B$ then $a_n + b_n \to A + B$ (Hint: split the allowable error between A_n and b_n.)

2. If $a_n \to A$ and $b_n \to B$ then $a_nb_n \to AB$ (Hint: you will need bound on one of a_n or b_n, then break $|a_nb_n - AB| = |a_nb_n - a_nB + a_nB - AB| \leq |a_n||b_n - B| + |a_n - A||B|$).

3. If $a_n \leq b_n \leq c_n$ and both $A_n \to L$ and $c_n \to L$ then $b_n \to L$.

1.3 Consequences of convergence

This section gives some consequences of a sequence having a limit. These results are useful sometimes to show that a sequence does not converge.

First we will need a definition.

Definition 4 *The sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if there is an upper bound b and a lower bound l such that for all $n \in \mathbb{N}$ we have $l \leq a_n \leq b$.*

Proposition 4 *Every convergent sequence is bounded.*

PROOF:

> Suppose that $a_n \to L$ is a convergent sequence. Then there is a N such that if $n > N$ we have $|a_n - L| < 1$ thus there are at most N terms not in the interval $[L-1, L+1]$. The sequence is then bounded above by the largest element of $\{a_k | k \leq N\} \cup \{L-1, L+1\}$ and bounded below by its smallest element. ∎

Because of this proposition we know that if a sequence is not bounded then it does not converge. We can also show that a sequence does not converge by demonstrating two subsequences which tend to different values.

Definition 5 *A sequence b is a subsequence of a if there is a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ with $b_k = a_{g(k)}$.*

A subsequence is thus a sequence with some of the terms left out.

Proposition 5 *Every subsequence of a convergent sequence converges to the same limit.*

PROOF:

> Suppose $a_n \to L$ and b is a subsequence of a, say using $g : \mathbb{N} \to \mathbb{N}$. Given ϵ there is an N such that if $n > N$ then $|a_n - L| < \epsilon$. Let M be the smallest natural number such that $g(M) \geq N$ then if $m > M$ we have $|b_m - L| = |a_{g(m)} - L| < \epsilon$. Thus $b_k \to L$ as well. ∎

Exercises:

1. Give an example of a sequence which is bounded but does not converge.

2. Give an example of a sequence which has two subsequences which converge to different limits.

3. The sequence
$$a_n = \frac{(-1)^n}{n}$$
converges to 0. Give upper and lower bounds for a_n.

4. The sequences
$$b_n = \frac{-1}{2n+1} \text{ and } c_n = \frac{1}{2n}$$
are subsequences of a_n in the previous problem. Give the functions $g : \mathbb{N} \to \mathbb{N}$ called for in the definition of subsequence.

1.4 Monotone Bounded Sequences

Monotone sequences are often easier to deal with than are sequences which sometimes increase and sometimes decrease.

Definition 6 *The sequence $a : \mathbb{N} \to \mathbb{R}$ is monotone if one of the following holds:*

1. *(monotone increasing) If $n < m$ then $a_n < a_m$.*

2. *(monotone non-decreasing) If $n < m$ then $a_n \leq a_m$.*

3. *(monotone non-increasing) If $n < m$ then $a_n \geq a_m$.*

4. *(monotone decreasing) If $n < m$ then $a_n > a_m$.*

We noted earlier that decimal approximation to a real number gives a monotone non-decreasing sequence.

Proposition 6 *Every sequence has a monotone subsequence.*

PROOF:

(From Donald J. Newman and T.D. Parsons **On Monotone Sequences**, Am. Math. Monthly 95, #1 p.44-45, 1988)

Suppose our sequence is a_n. Look at the set $S = \{k \in \mathbb{N} |$ for all $j > k, a_k < a_j\}$. If S is infinite then there is a function $g : \mathbb{N} \to S$ such $g(n) < g(n+1)$ and the subsequence $a_{g(n)}$ is monotone increasing.

If S is finite then there is a smallest natural number i_0 so that for all $n \geq i$ $n \notin S$. We let $g(0) = i_0$. This is not in S, so there is some

$i_1 > f(0)$ such that $a_{i_1} \leq a_{g(0)}$. Let $g(1)$ be the smallest such. We proceed inductively letting $g(n+1)$ be the smallest k so that $a_k \leq a_{g(n)}$. This produces a subsequence $a_{g(n)}$ which is monotone non-increasing.

∎

One important theorem uses the least upper bound property to show that a limit exists even if you don't know what that limit is.

Theorem 7 *Every bounded monotone sequence converges.*

I'll give the proof for monotone increasing bounded sequences. The others are similar.

PROOF:

Let S be the set of all values of the sequence

$$S = \{a_0, \ldots, a_n \ldots\}$$

We know that S is nonempty because it contains a_0. We know it is bounded because b is an upper bound. Now the least upper bound property (which distinguishes the real numbers from the rationals) says that any non-empty set of real numbers with an upper bound has a least upper bound. Let L be the least upper bound of S. We will show that L is the limit of the sequence.

Since L is an upper bound we know that for every n, $a_n \leq L$.

Since $L - \epsilon < L$ we know that $L - \epsilon$ is **not** an upper bound since L was the least, so there must be some M with $a_M > L - \epsilon$. That is the M we are looking for. If $n > M$ then $L - \epsilon < a_M < a_n \leq L$ so $|a_n - L| < \epsilon$.∎

Exercises:

1. Prove that every bounded sequence has a convergent subsequence.

2. A proof of the Extreme Value Theorem (as given in our Analysis 1 class) uses the bisection algorithm: To find where the maximum of $f : [a, b] \to \mathbb{R}$ is achieved we set up two sequences. We start with $a_0 = a$ and $b_0 = b$. We then look at

$$c_{n+1} = \frac{a_n + b_n}{2}$$

. If every point x in $[a_n, c_{n+1}]$ has some point t in $[c_{n+1}, b_n]$ with $f(x) \leq f(t)$ then let $a_{n+1} = c_{n+1}$ and $b_{n+1} = b_n$. Otherwise let $a_{n+1} = a_n$ and $b_{n+1} = c_{n+1}$. Show that both sequences a_n and b_n are monotone and bounded and thus must converge. Then show that they must converge to the same point.

All of the tests for convergence of positive term series depend on this theorem, so it becomes increasingly important as the subject progresses.

1.5 Cauchy Sequences

In his construction of real numbers as (equivalence classes of special kinds of) sequences of rationals, Cauchy wanted a way to identify convergent sequences without having to specify the limit in advance. Noting that a sequence converges if and only if its values get arbitrarily close to the limiting value, and thus arbitrarily close to each other, he identified the sequences now named for him:

Definition 7 *A sequence $a : \mathbb{N} \to \mathbb{R}$ is a Cauchy sequence if for every ϵ there is an M such that if $n > m > M$ then $|a_n - a_m| < \epsilon$.*

We can see quickly that any convergent sequence is Cauchy: given ϵ choose M so that if $n > M$ then $|a_n - L| < \frac{\epsilon}{2}$. If both n and m are larger than M then

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The other direction, however, is far from obvious.

Theorem 8 *Every Cauchy sequence converges.*

Proof:

Let a be a Cauchy sequence. Then $\{a_n | n \in \mathbb{N}\}$ is bounded (mimic the proof that convergent sequences are bounded, but use a_{M+1} instead of L). There is also a subsequence $a_{g(n)}$ of a which is monotone, and thus must converge, say to L. Now given ϵ let M be large enough that for $g(n)$ and all $m > M$ we get both $|a_{g(n)} - L| < \epsilon/2$ and $|a_m - a_{g(n)}| < \epsilon/2$. Combining these we see that for $m > M$ we get $|a_m - L| < \epsilon$. ∎

2 Series representation of functions

2.1 Basic Definitions

A series is the analogue of an improper integral:

Definition 8 *A series* $\sum_{n=0}^{\infty} a_n$ *converges to* L *if and only if the sequence of partial sums* $S_k = \sum_{n=0}^{k} a_n$ *converges to* L.

We can make use of the convergence of monotone bounded sequences to get a condition for the convergence of series of non-negative terms:

Corollary 9 (to the theorem on convergence of bounded monotone sequences) *If each* $a_n \geq 0$ *then* $\sum_{n=0}^{\infty} a_n$ *converges if there is an upper bound on the partial sums.*

The key point here is that having each $a_n \geq 0$ makes the sequence of partial sums monotone nondecreasing.

For certain important examples we can get explicit expressions for the partial sums.

Our object will be to use series notions to represent functions as an infinite analog to a polynomial, a power series:

Definition 9 *A power series is a function of the form*

$$\sum_{n=0}^{\infty} a_n x^n.$$

We evaluate such a function by taking the limit of the resulting series of numbers.

Our hope is that power series representations of functions will not be too hard to find, that ignoring all but the first few terms will give us good approximations, and that calculus for power series will prove to be as easy as calculus for polynomials.

2.2 Geometric Series

A basic geometric series is, perhaps, a familiar example: we start with a value a_0 and then obtain each new term of the series by multiplying by a common ratio. In keeping with our focus on functions, let us use x for the common ratio and examine the convergence of

$$f(x) = \sum_{n=0}^{\infty} a_0 x^n$$

The k^{th} partial sum of this geometric series is

$$S_k(x) = \sum_{n=0}^{k} a_0 x^k.$$

We can find its value through the following calculation:

$$\begin{aligned}
S_k(x) &= \sum_{n=0}^{k} a_0 x^n \\
xS_k(x) &= \sum_{n=1}^{k+1} a_0 x^n \\
(1-x)S_k(x) &= a_0 - a_0 x^{k+1} \\
S_k(x) &= \frac{a_0 - a_0 x^{k+1}}{1-x}
\end{aligned}$$

Now this final expression will converge to the limit

$$f(x) = \frac{a_0}{1-x}$$

when $x^{k+1} \to 0$. This in turn happens if and only if $|x| < 1$. In summary,

$$\frac{a_0}{1-x} = \sum_{n=0}^{\infty} a_0 x^n \text{ provided } |x| < 1.$$

Example: Using geometric series

Several functions can be given series expansions by decorating this result:

1. A geometric series with $a_0 = 1$ common ratio $-x$ gives series representation
$$\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x} \text{ valid for } |x| < 1.$$
2. A geometric series with $a_0 = x$ common ratio x gives series representation
$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \text{ valid for } |x| < 1.$$
3. A geometric series with $a_0 = 2$ common ratio $-3x$ gives series representation
$$\sum_{n=0}^{\infty} 2\ (-3)^n x^n = \frac{2}{1+3x} \text{ valid for } |x| < \tfrac{1}{3}.$$
4. A geometric series with $a_0 = \frac{1}{2}$ common ratio $\frac{x}{2}$ gives series representation
$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n = \frac{1}{2-x} \text{ valid for } |x| < 2.$$
5. A geometric series with $a_0 = 1$ common ratio x^2 gives series representation
$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} \text{ valid for } |x| < 1.$$

$\diamondsuit$

Exercises: As an exercise to see if you have the technique try to find a series representation of the following functions, giving the values for x for which the series converges.

1. $\dfrac{1}{1+x^2}$
2. $\dfrac{x}{1-x^3}$
3. $\dfrac{1}{1+x^2}$
4. $\dfrac{x}{1-x^3}$
5. $\dfrac{3}{4+2x^2}$
6. $\dfrac{x^2}{1+4x}$
7. $\dfrac{3x}{5-x^5}$
8. $\dfrac{x+1}{1+x^2}$
9. $\dfrac{3}{3+2x^2}$

2.3 Taylor's Theorem

There are three major meanings for the derivative $f'(a)$:

1. instantaneous rate of change of f at a
2. the slope of the tangent line to $y = f(x)$ at $(a, f(a))$
3. means to get the best linear approximation to a function f near a. This *best* has a particular meaning: the function

$$t(x) = f(a) + f'(a)(x - a)$$

gives an approximation to $f(x)$ such that

$$\lim_{x \to a} \frac{f(x) - (f(a) + f'(a)(x - a))}{x - a} = 0$$

Thus the error in using the tangent line as an approximation is (asymptotically) much smaller than the distance away from a where you are trying to use it.

This view of the derivative as giving the best approximation becomes the view which generalizes to higher dimensions. It also suggests a generalization which gives a reason for finding derivatives of higher order: perhaps using a second derivative we can get a *best* quadratic approximation, using a third derivative we can get a *best* cubic approximation, ..., using derivatives up to the n^{th} we can get a *best* n^{th} degree polynomial approximating f.

The best linear approximation is gotten by matching both $t(a)$ with $f(a)$ and $t'(a)$ with $f'(a)$.

Life will get somewhat simpler if we restrict to the specific case where we are looking for approximations near $a = 0$.

Consider a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$$

then

$$\begin{aligned}
p'(x) &= a_1 + 2a_2x + 3a_3x^2 + ...na_nx^{n-1} \\
p''(x) &= 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + ...(n-1)na_nx^{n-2} \\
p'''(x) &= 6a_3 + 2 \cdot 3 \cdot 4a_4x + 3 \cdot 4 \cdot 5a_5x^2 + ...(n-2)(n-1)na_nx^{n-3} \\
&\text{etc.}
\end{aligned}$$

This tells us that $p'(0) = 1!a_1$, $p''(0) = 2!a_2$, $p'''(0) = 3!a_3$, and in general $p^{(n)}(0) = n!a_n$ (that's the n^{th} derivative of p evaluated at 0).

This suggests that if we want the first n derivatives of $p(x)$ at 0 to match the first n derivatives of $f(x)$ at 0, we should let

$$a_k = \frac{f^{(k)}(0)}{k!}$$

for $k = 1$ to n and let $a_0 = f(0)$. This gives the n^{th} degree Taylor Polynomial for f.

Definition 10 *The n^{th} degree Taylor Polynomial (expanded around 0) for f is*

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n.$$

Taylor's theorem then tells us that if f has enough derivatives (typically we ask for $n+1$ continuous derivatives near 0) then the error involved in using $p_n(x)$ instead of $f(x)$ is asymptotically small even compared to x^n. (Note that we are working near $x = 0$, so x^n is much smaller than x).

An integral form will give Taylor's Theorem with an exact value for the error:

Theorem 10 *If f is $n+1$ times continuously differentiable, then*

$$f(x) - p_n(x) = \int_0^x (x-t)^n \frac{f^{(n+1)}(t)}{n!} dt.$$

PROOF:

We use induction and integration by parts: If $n = 0$ this says

$$f(x) - f(0) = \int_0^x f'(t)dt$$

which is just the Fundamental Theorem of Integral Calculus.

If we assume that

$$f(x) - p_k(x) = \int_0^x (x-t)^k \frac{f^{(k+1)}(t)}{k!} dt$$

then use integration by parts with $u = f^{(k+1)}(t)$ and $dv = \frac{(x-t)^k}{k!}dt$ we will get $du = f^{(k+2)}(t)dt$ and $v = -\frac{(x-t)^{k+1}}{(k+1)!}$. Thus

$$f(x) - p_k(x) = -\frac{(x-t)^{k+1}}{(k+1)!} f^{(k+1)}(t)\Big|_0^x - \int_0^x -\frac{(x-t)^{k+1}}{(k+1)!} f^{(k+2)}(t)dt$$

simplifying gives

$$f(x) - p_k(x) = \frac{x^{k+1}}{(k+1)!} f^{(k+1)}(0) + \int_0^x \frac{(x-t)^{k+1}}{(k+1)!} f^{(k+2)}(t)dt$$

so

$$f(x) - p_{k+1}(x) = \int_0^x \frac{(x-t)^{k+1}}{(k+1)!} f^{(k+2)}(t)dt$$

as needed for the induction. ∎

Lagrange gave a proof for an approximate form of the error which is easy to remember since it looks rather a lot like the next term:

Corollary 11 *If f is $n+1$ times continuously differentiable on an interval containing both x and 0, then*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between x and 0.

PROOF:

Since $f^{(n+1)}(t)$ is continuous on $(0, x)$ it assumes both its maximum and minimum, so there are values between) and x with

$$f^{(n+1)}(m) \leq f^{(n+1)}(t) \leq f^{(n+1)}(M).$$

Thus

$$f^{(n+1)}(m) \int_0^x \frac{(x-t)^n}{n!} dt \leq \int_0^x (x-t)^n \frac{f^{(n+1)}(t)}{n!} dt \leq f^{(n+1)}(M) \int_0^x \frac{(x-t)^n}{n!} dt$$

so that for some c between m and M (and thus between 0 and x) we get

$$\int_0^x (x-t)^n \frac{f^{(n+1)}(t)}{n!} dt = f^{(n+1)}(c) \int_0^x \frac{(x-t)^n}{n!} dt = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

by the Intermediate Value Theorem. ∎

For cases where a general form for the derivatives of f can be found, Taylor's Theorem gives a means of getting a series representation.

Example: Series for the exponential

A series representation for $f(x) = e^x$ is easy to obtain since

$$f^{(n)}(x) = e^x$$

for all n, so all of the derivatives at 0 are 1. This gives

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Now for any x and any n Taylor's theorem tells us that

$$e^x - p_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$$

for some c between x and 0. If we fix x and let $n \to \infty$ this error term always goes to 0. Thus the Taylor series for e^x converges to e^x for every x. ♢

Exercises: While this is the easiest example, several other functions also have Taylor series which can be found by calculating derivatives and then showing that the error term goes to 0. Try the following examples:

1. $f(x) = \sin(x)$
2. $f(x) = \cos(x)$
3. $f(x) = \dfrac{1}{1-x}$
4. $f(x) = \ln(1-x)$
5. $f(x) = \cosh(x)$

3 Finding When Power Series Converge

We have seen how to get power series representations of functions using geometric series and using Taylor's theorem, both of which give ways to tell which x's give convergent series. Let us turn next to the problem of finding what values of x make a series converge where we are given the coefficients, but we are not told what the function being approximated is.

To start with let us note that if $\sum a_n$ and $\sum b_n$ converge, then so do $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ since their partial sums can be rearranged to be the sum and difference of partial sums of a_n and b_n.

3.1 Comparison and the Ratio Test

Our most basic tests for convergence are based on the comparison test:

Theorem 12 (Comparison Test) *If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then so does $\sum a_n$.*

PROOF:

Since the terms a_n are non-negative, the sequence of partial sums

$$S_k = \sum_{n=0}^{k} a_k$$

is monotone nondecreasing. To show that $\sum a_n$ converges we need only give an upper bound for all of the partial sums. Now since $a_n \leq b_n$ for all n,

$$\sum_{n=0}^{k} a_n \leq \sum_{n=0}^{k} b_k \leq \sum_{n=0}^{\infty} b_n$$

Using the convergence of bounded monotone sequences, this tells us that $\sum a_n$ converges, though it does not tell us what it converges to. ∎

Corollary 13 *If $\sum |a_n|$ converges then so does $\sum a_n$.*

Proof:

Observe that
$$0 \leq a_n + |a_n| \leq 2|a_n|.$$
First note that if $\sum |a_n|$ converges to S then $\sum 2|a_n|$ will converge to $2S$, since each partial sum is doubled. The comparison theorem then tells us that if $\sum 2|a_n|$ converges then so will $\sum(a_n + |a_n|)$. Subtracting $\sum |a_n|$ then tells us that $\sum a_n$ converges. ∎

Definition 11 *A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges.*

Since adding a finite sum to the beginning of a convergent series does not change the convergence, though it might change the first M terms, all that matters for determining convergence of a series is what happens in the *tail*, terms from a_M on for any given M. We'll use this to provide the following comparison with geometric series:

Theorem 14 (Ratio Test) *Suppose that $a_n > 0$ for all n and*
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$
Then if $L < 1$ then $\sum a_n$ converges. If $L > 1$ then the series diverges; if $L = 1$ we don't know what happens.

Proof:

We will prove the case $L < 1$ by giving a comparison with a suitable geometric series. Since $L < 1$ we can find an r with $L < r < 1$ and for some M if $n \geq M$ then
$$\frac{a_{n+1}}{a_n} < r.$$
We will show that the sequence $\displaystyle\sum_{n=M}^{\infty} a_n$ converges by comparison with the geometric series $\displaystyle\sum_{n=0}^{\infty} a_M r^n$. Certainly $a_M \leq a_M r^0$. For $n \geq M$ we have

$a_{n+1} < ra_n$ so using an induction hypothesis we get $a_{n+1} < r^{n+1-M}a_M$. This completes the comparison, so $\sum a_n$ converges.

The proof for divergence if $L > 1$ gives a comparison the other way with a divergent geometric series. For $L = 0$ there are examples of both convergent and divergent series (p-series, which make their appearance later in these notes). ∎

Example: $\sum \frac{(n!)^2}{(2n)!}$ **converges.**

We look at

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left(\frac{((n+1)!)^2}{(2n+2)!}\right)}{\left(\frac{(n!)^2}{(2n)!}\right)} &= \lim_{n\to\infty} \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\
&= \frac{1}{4}
\end{aligned}$$

Since $L = \frac{1}{4} < 1$ this tells us that this series converges. ♢

Exercises: Try out the ratio test on the following series:

1. $\sum \frac{4^n}{n!}$
2. $\sum \frac{n}{2^n}$
3. $\sum \frac{n^{300}}{1.001^n}$
4. $\sum \frac{n^2}{n!}$
5. $\sum \frac{n^4}{4^n}$
6. $\sum \frac{4^n}{n^3}$
7. $\sum \frac{n!}{n^n} Hint: \left(\frac{n+1}{n}\right)^n \to e$

3.2 Radius of Convergence

We can apply the ratio test to get some information about where a power series converges absolutely. Trying the ratio test on

$$\sum_{n=0}^{\infty} |a_n x^n|$$

leads us to look at

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x|$$

and ask for which x the limit is strictly less than 1. To this end we take $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$ and ask that $L|x| < 1$, or equivalently, $-\frac{1}{L} < x < \frac{1}{L}$.

Definition 12 *The number $\frac{1}{L}$ is called the radius of convergence of the power series.*

Inside its radius of convergence a power series is absolutely convergent and has a little wiggle room before you get outside the radius of convergence.

It is possible for $L = 0$ in which case we will get convergence for all x. If the limit gives $+\infty$ then any x we use other than 0 will give a divergent series.

Example: All of the possibilities occur

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. We actually knew this already from Taylor's theorem, since this is the series for e^x, but it is instructive to see how the ratio test gives it:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)!}\right)}{\left(\frac{1}{n!}\right)} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

   The ratio test then gives us convergence for all x.

2. Turning the previous example upside down gives $\sum_{n=0}^{\infty} n!x^n$. Here the ratio of test gives an infinite limit, so only $x = 0$ works.

3. $\sum_{n=0}^{\infty} \frac{nx^n}{2^n}$ gives a radius of convergence $1/L$ where

$$L = \lim_{n\to\infty} \frac{(n+1)^2 2^n}{n^2 2^{n+1}} = \frac{1}{2}$$

Thus the radius of convergence is 2.

$\diamondsuit$

Exercises: Find the radius of convergence for the following power series:

1. $\sum \frac{2^n x^n}{n}$
2. $\sum \frac{n x^n}{n+1}$
3. $\sum \frac{4^n x^n}{n!}$
4. $\sum \frac{n^n x^n}{n!}$

3.3 Term by Term Integration and Differentiation

Inside the radius of convergence there are several operations we can do with power series.

Algebraic operations on power series will give power series with (possibly different) radii of convergence. There are fairly standard means for multiplying series (using convolution), adding series (term by term), composing series, and dividing series. We will not be using them in this course. We will, though, be using integration and differentiation of power series. Both of the following theorems are rather deep as they involve changing the order of two different limiting processes (like in Fubini's Theorem). The proofs are left to upper level courses in real analysis.

Theorem 15 (Term by term differentiation) *If* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *for* $|x| < r$ *then*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

for $|x| < r$.

Because of this theorem we note that if a function has a power series representation then that power series is a Taylor series. Recovering the derivatives at 0 from a power series proceeds just as the calculation for recovering the derivatives of a polynomial at 0 from the coefficients did

Theorem 16 (Term by term integration) *If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < r$ then*

$$\int_0^x f(t)\ dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

for $|x| < r$.

The theorems tell us that the easy techniques for polynomial calculus carry over to power series provided we stay inside the radius of convergence. We can apply these directly to find series for a number of functions:

3.4 Series for $\frac{1}{(1-x)^n}$

If we start with one of our standard geometric series and differentiate we will get the following series, all valid inside $|x| < 1$:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} &= \sum_{n=1}^{\infty} n x^{n-1} \\
\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\frac{1}{1-x} &= \sum_{n=2}^{\infty} n(n-1) x^{n-2} \\
\text{so } \frac{1}{(1-x)^3} &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2 \cdot 1} x^{n-2} \\
\frac{6}{(1-x)^4} = \frac{d^3}{dx^3}\frac{1}{1-x} &= \sum_{n=3}^{\infty} n(n-1)(n-2) x^{n-3} \\
\text{so } \frac{1}{(1-x)^4} &= \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} x^{n-3} \\
\frac{1}{(1-x)^k} &= \sum_{n=k}^{\infty} \binom{n}{k-1} x^{n-k}
\end{aligned}$$

3.4.1 Series for $\arctan(x)$ and $\ln(x+1)$

Integration term by term of some standard geometric series also gives useful results, valid for $|x| < 1$:

To get a series for $\arctan(x)$ we start with a geometric series

$$\frac{1}{1+x^2} \quad = \quad \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

and then integrate

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

To get a series for $\ln(1+x)$ we start with

$$\frac{1}{1+x} \quad = \quad \sum_{n=0}^{\infty} (-1)^n x^n$$

and then integrate

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

Exercises: Try using integration or differentiation term by term to obtain series representations for the following functions, specifying where your results are valid:

1. $g(x) = \dfrac{x}{(1+x^2)^2}$ using $\dfrac{x}{1+x^2}$
2. $\tanh^{-1}(x) = \displaystyle\int_0^x \frac{1}{(1-t^2)}\, dt$ using $\dfrac{1}{1-t^2}$

3.4.2 Series from Differential Equations

Term by term differentiation can also be useful for finding a series for the solution to a differential equation. We use the differential equation to provide a recurrence telling us how to get later coefficients from earlier ones. Initial conditions tell us how to get started. Since this technique generates the series without having the hypothesis that we are operating inside the radius of convergence, we need to check what that radius is when we get finished to be sure what we've done makes sense.

Example: Find a series for the function which is a solution to the differential equation

$$y' = xy \text{ with } y(0) = 1$$

We start by assuming that y has an expansion as a power series and then set about to find out what the coefficients are:

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ so } a_0 = y(0) = 1$$

Now $y' = xy$ so the series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

If we re-index the sums so that both use the power of x as index we get

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m = \sum_{m=1}^{\infty} a_{m-1} x^m$$

From which we may conclude that $a_1 = 0$ since there is no constant term in the right hand sum, and for $m \geq 1$ we have

$$a_{m+1} = \frac{a_{m-1}}{m+1}$$

Using this we see that all odd coefficients are 0 and

$$a_{2n} = \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{1}{2^n n!}$$

Applying the ratio test to this power series we see that it converges if

$$\lim_{n \to \infty} \frac{1}{2n} |x^2| = 0 < 1$$

so the series always converges.

$\diamondsuit$

Example: A series for $\sin(x)$

The sine function is the solution to the second order differential equation

$$y'' = -y \text{ with } y(0) = 0 \text{ and } y'(0) = 1$$

If we guess that there is a series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

then the initial conditions tell us that

$$a_0 = y(0) = 0 \text{ and } a_1 = y'(0) = 1.$$

The differential equation tells us that

$$\begin{aligned}
\frac{d^2}{dx^2}\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= -\sum_{n=0}^{\infty} a_n x^n \\
\text{re-indexing gives} & \\
\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n &= -\sum_{n=0}^{\infty} a_n x^n \\
\text{so} & \\
a_{n+2} &= -\frac{a_n}{(n+2)(n+1)}
\end{aligned}$$

Using this we get the series

$$sin(x) = 0 + x - 0x^2 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} \cdots$$

Again checking the radius of convergence we see that this converges everywhere. $\diamondsuit$

Exercises: Here again, you should try this technique for generating some important series:

1. The function $y = (1 + x)^p$ satisfies the differential equation

$$(1 + x)y' = py \text{ with the initial condition } y(0) = 1$$

   for all p for which we can make sense of the power. Use this to generate a series expansion for $(1 + x)^p$. This is the **Binomial Series** originally found by Newton.)

2. The hyperbolic sine $y = \sinh(x)$ satisfies the second order differential equation

$$y'' = y \text{ with initial conditions } y(0) = 0 \text{ and } y'(0) = 1$$

   Use this to find a series for $\sinh(x)$. Compare your answer to the series for $\sin(x)$.

3. The hyperbolic cosine $y = \cosh(x)$ satisfies the second order differential equation

$$y'' = y \text{ with initial conditions } y(0) = 1 \text{ and } y'(0) = 0$$

   Use this to find a series for $\cosh(x)$. Compare your answer to the series for $\cos(x)$.

4 What about endpoints?

We obtained the radius of convergence using a comparison with a geometric series. Such a test will not tell us what happens at the endpoints, so we need tests which tell us when some non-geometric series converge.

4.1 Integral test and p-series

One good way to get an upper bound for the partial sums of a series is with an integral:

Theorem 17 (Integral Test) *If $a_n = f(n)$ for every $n \in \mathbb{N}$ and if f is a positive, decreasing function, then $\sum a_n$ converges if and only if $\int_0^\infty f(x)\, dx$ converges.*

PROOF:

By constructing step functions based on the series we can see that

$$\sum_{n=1}^{k} a_n < \int_0^k f(x)\ dx < \sum_{n=0}^{k-1} a_k$$

Now if $\int_0^\infty f(x)\ dx$ converges then we get

$$\sum_{n=0}^{k} a_n < a_0 + \int_0^\infty f(x)\ dx$$

giving a bound on the partial sums. Thus $\sum a_n$ converges.

If, on the other hand, $\int_0^\infty f(x)\ dx$ diverges then the numbers $\int_0^k f(x)\ dx$ get arbitrarily large, and thus so do the partial sums. This tells us that the series also diverges. ∎

Corollary 18 (p-series) *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proof:

The integral

$$\int_1^\infty \frac{1}{x^p}\ dx = \lim_{U\to\infty} \frac{U^{-p+1}}{-p+1} - \frac{1}{1-p}.$$

For $p > 1$ this converges to $\frac{1}{p-1}$. For $p < 1$ it diverges. If $p = 1$ then the integral gives a natural logarithm $\ln(U)$ which diverges as $U \to \infty$. ∎

Example: Use of the p-series test

1. $\sum \frac{1}{n^2}$ converges
2. $\sum \frac{1}{\sqrt{n}}$ diverges

3. $\sum \frac{1}{n}$ diverges. This is called the **harmonic series**.

$\diamondsuit$

Usually we use the limit comparison theorem with p-series when checking end-points of intervals of convergence:

Theorem 19 (Limit Comparison Theorem) *If $0 < a_n$ and $0 < b_n$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

then if $0 < L < \infty$ then either both of $\sum a_n$ and $\sum b_n$ converge or both diverge. If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$.

PROOF:

Suppose that $L < \infty$ and $\sum b_n$ converges. Then for large enough n we know that

$$\frac{a_n}{bn} < L + 1$$

thus $a_n < (L+1)b_n$. Now if $\sum b_n$ converges then so does $\sum (L+1)b_n$ so the comparison theorem will tell us that $\sum a_n$ converges.

If $L > 0$ we can invert to see that

$$\lim_{n\to\infty} \frac{b_m}{a_n} \to \frac{1}{L} < \infty$$

so if $\sum a_n$ converges, so does $\sum b_n$. ∎

Example: Use of comparisons

1. $\displaystyle\sum \frac{n-1}{n^3+n^2-n}$ converges by limit comparison with $\displaystyle\sum \frac{1}{n^2}$. Here

$$\lim_{n\to\infty} \frac{\left(\frac{n-1}{n^3+n^2-n}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3+n^2-n} = 1$$

2. $\displaystyle\sum \frac{n-1}{n^2-n}$ diverges by limit comparison with $\displaystyle\sum \frac{1}{n}$. Here

$$\lim_{n\to\infty} \frac{\left(\frac{n-1}{n^2-n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^2-n}{n^2-n} = 1$$

$\diamondsuit$

Exercises: Here are some for you to try:

1. $\sum \frac{n}{\sqrt{n^3 - n}}$
2. $\sum \frac{n^2}{n^3 - n}$
3. $\sum \frac{\sqrt{n}}{n^3 - n}$
4. $\sum \frac{\sqrt{n}}{\sqrt{n^3 - n}}$
4. $\sum \frac{3n}{n^3 - n^2}$

4.2 Alternating series and good approximations

While the several tests we have given so far for convergence give us useful information, they do not, in general, tell us what our error is if we truncate the series. The alternating series test is particularly nice because it does give us an error estimate:

Theorem 20 (Alternating Series Test) *Suppose a_n is a monotone decreasing sequence which converges to 0, then*

$$S = \sum_{n=0}^{\infty} (-1)^n a_n$$

converges and has $|S_k - S| < a_{k+1}$.

PROOF:

The partial sums of an alternating series whose terms monotonically go to 0 form a Cauchy sequence. To see this we note that the even partial sums form a decreasing sequence since

$$S_{2n} - S_{2n+2} = a_{2n+1} - a_{2n+2} > 0$$

and the odd partial sums form an increasing sequence since

$$S_{2n+1} - S_{2n+3} = -a_{2n+2} + a_{2n+3} < 0.$$

Furthermore, each even partial sum is larger than the odd partial sums right before it and right after it. Thus the partial sum S_m with $m > n$ lies between S_n and S_{n-1}. If both m and k are larger than n then $|S_m - S_k| \leq a_n$. Since $a_n \to 0$ this tells us S_n is a Cauchy sequence and thus converges, say to S. Since all of the S_m are in the closed interval bounded by S_n and S_{n-1} we must have S in that interval as well. Since the limit is between the $k+1$st and kth partial sums, either endpoint will give an error less than a_{k+1}. ∎

Example: Alternating series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges since it is an alternating series and $\frac{1}{n} \to 0$ is monotone decreasing. To get the sum to within .001 will take 999 terms! This is called the **alternating harmonic series**.
2. $\sum (-1)^n n$ diverges, even though it is an alternating series because n does not go to 0.
3. $\displaystyle\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.
4. To use the series for $\sin(x)$ to get $\sin(1)$ to within $\pm.0001$ we need only take enough terms in the series

$$1 - \frac{1}{3!} + \frac{1}{5!} + \cdots (-1)^n \frac{1}{(2n+1)!} \cdots$$

so that the first term omitted is less than .0001. If we take 2 terms we get to within $\frac{1}{7!} \approx 1.98 \times 10^{-3}$ which is not quite good enough. Three terms will give an error less than $\frac{1}{9!} \approx 2.76 \times 10^{-6}$. So $\sin(1) \approx 1 - \frac{1}{120} + \frac{1}{7!} \approx .991865$.

♢

Exercises: Find the interval of convergence for the following power series. First find the radius of convergence using the ratio test, then use limit comparison with a p-series or the alternating series test to see what happens at the endpoints.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{2^n}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+2}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n^3+2}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{10^n}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{n!x^n}{n^n}$

4.3 Conditional Convergence and Its Pathologies

Definition 13 *A series is said to be conditionally convergent if* $\sum a_n$ *converges but* $\sum |a_n|$ *does not converge.*

The alternating harmonic series $\sum \frac{(-1)^n}{n}$ is an example of this kind of behavior.

Conditionally convergent series will have a a subsequence of positive terms and a subsequence of negative terms which both give series which diverge. The individual terms $a_n \to 0$ since otherwise the series would be divergent. These can be used to produce a rearrangement of the series which converges to any value we wish. To make the limit L what you do is take positive terms until the first time that the sum exceeds L, then take negative terms until you first get a sum which is less than L. Each time you step across L you will be taking smaller steps so the resulting series will converge to L.

A similar argument will show that you can rearrange a conditionally convergent series to diverge as well.

Since the rearrangement of series is an infinite analog of the commutative law, we must consider this behavior pathological. Fortunately, inside the radius of convergence of a power series we get absolute convergence, which does not share this behavior.

Exercises:

1. Under what conditions on p is $\sum(-1)^n/(n^p)$ conditionally convergent?

2. Convergence of conditionally convergent series can be extraordinarily slow: compare the number of terms needed to approximate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

to within $\pm 5 \times 10^{-5}$ (i.e. to get four correct decimal places).

3. Prove that the terms of a conditionally convergent series can be rearranged to give a series whose partial sums go to ∞.

4. Prove that the terms of a conditionally convergent series can be rearranged to give a series whose partial sums oscillate wildly.