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Categorical Approaches to Non-Commutative Fuzzy Logic

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Abstract

In this paper we consider what it means for a logic to be non-commutative, how to generate examples of structures with a non- commutative operation * which have enough nice properties to serve as the truth values for a logic. Inference in the propositional logic is gotten from the categorical properties (products, coproducts, monoidal and closed structures, adjoint functors) of the categories of truth values. We then show how to extend this view of propositional logic to a predicate logic using categories of propositions about a type *A* with functors giving change of type and adjoints giving quantifiers. In the case where the semantics takes place in **Set** (*L*) (Goguen's category of *L*-fuzzy sets), the categories of predicates about A can be represented as internal category objects with the quantifiers as internal functors.

Keywords: Fuzzy logic; Noncommutative residuated lattices; Quantales; Higher order fuzzy logic.

1. Introduction

Over the course of our careers several researchers in fuzzy set theory have had the opportunity to visit Ulrich Höhle at his garden house for extended intensive research in such topics as categorical foundations of fuzzy sets, fuzzy topology, sheaf theory, and generalizations to quantale based mathematics. This contribution stems from two such visits: when I was on sabbatical in 1999 and again in 2006. The shaping of my own research direction has been enriched through fruitful conversation with Ulrich in Tokyo, at several Linz seminars, and when he visited Illinois Wesleyan University when he was on sabbatical. That is not to say that he agrees with the directions taken in my work; because I am interested in generalizations of mainstream fuzzy set theory where the ⋆ operation is usually not idempotent, I have found that the otherwise fruitful development of quantale based categories using an idempotent ⋆ which parallel sheaf theory (the direction of much of Ulrich's recent work) is not the direction I want to take. I tend to prefer generalizations of the Goguen category rather than the Higgs category at this point in my career. The results there are easier, apply to more general structures, and as a result may be more shallow. With this apology, let us consider categorical approaches to non-commutative fuzzy logic.

1.1. Why non-commutative logic?

First, let us specify what we mean by non-commutative logic:

Definition 1. A non-commutative logic is a logic equipped with a form of conjunction for which $\phi \star \psi$ is not equivalent to $\psi \star \phi$.

Situations outside of mathematics giving rise to non-commutative conjunctions include

- In ordinary language a commutative *and* between clauses indicates independence, while a non-commutative *and* indicates a pragmatic dependence [19].
- Tense logics: *and then* is non-commutative.
- Semantics of parallel programming: sequential execution is a non-commutative *and* while parallel execution (where all processes are required to complete before the computation can continue) is a commutative *and*.
- Quantum mechanics makes use of non-commutative observables.
  Situations within mathematics giving rise to non-commutative conjunctions include

- Logics based on function composition.
- Lambek calculus [12] modeling sentence structure using concatenation as a non-commutative operation.
- Logics based on formal matrix multiplication.
- Fuzzy logics using pseudo-t-norms on the unit interval, and the weaker logics which result from removing the bottom of the lattice (as in [3,4,10,9,8]).
- Quantales, often based on closed linear subspaces of C*-algebras, the non-commutative *and* given by closure of the product subspace [18,16].
- Linear logics [5,23].

My object in this paper is to provide a coherent categorical picture of the variety of non-commutative logics, both propositional and predicate, and show how an internal higher order logic can be given using categories of fuzzy sets.

1.2. What does it take to be a logic?

We want logic to be about sound inference: reasoning from premises to conclusions which are at least as true as the premises were. For this we need a notion of degrees of truth, structured so we can encode what we mean by "at least as true as". One way to capture that is with a category $\mathcal{V}$ whose objects are truth values and whose morphisms indicate valid inference. A value Ψ is at least as true as a value Φ if there is a morphism $\alpha : \Phi \to \Psi$. There can be multiple morphisms from Φ to Ψ, each indicating a way to make the inference. Transitivity of inference shows up as composition of morphisms and the identities give trivial inferences. Categorical reasoning will then use universal mapping properties or adjunctions to derive morphisms giving more complicated inference.

Logic, as opposed to rhetoric, usually does not deal with the details of how individual inferences are made, only with how they get combined to give proofs. As a result we often do not want to work with the category of inferences $\mathcal{V}$ but rather with a transitive directed graph $\mathcal{G}$ with vertices given by the objects of $\mathcal{V}$ and an edge from Φ to Ψ expressing the existence of a morphism from Φ to Ψ in $\mathcal{V}$. If there is an edge in $\mathcal{G}$ we will write $\Phi \Vdash \Psi$.

Even this keeps track of more information than we usually require from degrees of truth. If we have $\Phi \Vdash \Psi$ and $\Psi \Vdash \Phi$ we will usually think of the two truth values as giving the same information and thus being equivalent. Those objects of $\mathcal{G}$ which are linked in both directions to Φ form a clique (a complete graph with edges in both directions). The graph $\mathcal{P}$ has as its vertices the cliques in $\mathcal{G}$ and an edge from $[\Phi]$ to $[\Psi]$ precisely if there is an edge in $\mathcal{G}$ from a vertex in $[\Phi]$ to a vertex in $[\Psi]$ (transitivity says if this is true for one pair of representatives it will be true for any such pair). We will usually write this without the square brackets to avoid complicating the notation unnecessarily. This will give a partial order: decliqueification gains us antisymmetry, identities give us the reflexive law, and composition and the resulting transitivity gives the transitive law. Because we have not imposed size restrictions it is possible that this is a partially ordered proper class rather than a partially ordered set. For $\mathcal{P}$ we will write $\Phi \vDash \Psi$ if there is an edge from Φ to Ψ.

Notice that all of the structures $\mathcal{V}, \mathcal{G}$, and $\mathcal{P}$ are categories and that there are obvious underlying functors $\mathcal{V} \to \mathcal{G} \to \mathcal{P}$. Notice also that the category $\mathcal{P}$ is skeletal: the only isomorphisms are identities.

Tradition in many valued ($n\geq2$) logic is for at least one of the truth values to be designated, giving a notion of "true". Often, but not always, the largest truth value is the single designated one. In situations with a classical conjunction (making $\mathcal{P}$ a $\wedge$-semilattice) the designated values form a filter: if Φ is designated and $\Phi\vDash\Psi$ then Ψ should be designated as well and the conjunction of two designated values should be designated. This makes it possible to look at tautologies, expressions which get a designated value no matter what the values of the propositional variables are. Independence of axioms in a system is often shown by constructing many valued logics where all but the axiom being shown to be independent give tautologies. A complete propositional logic is one which allows derivation of an inference from nothing (or perhaps from a designated value) to each of the tautologies. In general this tells us much less than saying which inferences will be valid, though in classical logic tautologies do capture the whole story.

If we think of many valued logics as measuring some degree of vagueness, then giving two filters $\mathfrak{F}_\top$ in $\mathcal{P}$ and $\mathfrak{F}_\bot$ in $\mathcal{P}^{\mathrm{op}}$ such that $\mathfrak{F}_\top \cup \mathfrak{F}_\bot = \mathcal{P}$ and $\mathfrak{F}_\top \cap \mathfrak{F}_\bot = \emptyset$ provides a way to pass to a crisp two valued logic. We often ask that the resulting logic be classical.

Logic takes place at several orders: Propositional logic tells us what happens in a single one of the categories $\mathcal{V}(A)$, $\mathcal{G}(A)$, or $\mathcal{P}(A)$; quantification tells us what happens as we change type; higher order internal logic calls for representation of one of these categories of propositions about an object A as an internal category object with the functors arising from a map $f : A \rightarrow B$ giving internal functors between the category objects.

2. Propositional categorical logic

We start by seeing how categorical structures on the categories $\mathcal{V}(A)$, $\mathcal{G}(A)$, or $\mathcal{P}(A)$ give the propositional connectives and the rules of inference we need for proofs in this context.

2.1. Classical connectives in a categorical context

Identities in these categories give us the basic axioms:

In $\mathcal{V}$:	$\Phi \xrightarrow{\mathrm{id}_A} \Phi$
In $\mathcal{G}$:	$\Phi \Vdash \Phi$
In $\mathcal{P}$:	$\Phi \vDash \Phi$

The existence of composition gives a rule of inference:

In $\mathcal{V}$	$\dfrac{\Phi \xrightarrow{\alpha} \Psi \quad \Psi \xrightarrow{\beta} \varXi}{\Phi \xrightarrow{\alpha\beta} \varXi}$ composition
In $\mathcal{G}$	$\dfrac{\Phi \Vdash \Psi \quad \Psi \Vdash \varXi}{\Phi \Vdash \varXi}$ cut
In $\mathcal{P}$	$\dfrac{\Phi \vDash \Psi \quad \Psi \vDash \varXi}{\Phi \vDash \varXi}$ cut

Since it is common to allow a known truth to be inferable from anything (assuming that we are not working in a relevance logic) it makes sense to ask for a terminal and designate the terminal object in a category of truth values as designating *true*.

The usual rules for $\wedge$-introduction and $\wedge$-elimination suggest that the categorical equivalent will be a pairwise product. Similarly, the rules for $\vee$ introduction and $\vee$ elimination suggest pairwise coproducts.

A covariant functor $F : \mathcal{V} \to \mathcal{V}$ gives rise to an inference

In $\mathcal{V}$	$\dfrac{\Phi \xrightarrow{\alpha} \Psi}{F(\Phi) \xrightarrow{F(\alpha)} F(\Psi)}$ F functor
In $\mathcal{G}$	$\dfrac{\Phi \Vdash \Psi}{F(\Phi) \Vdash F(\Psi)}$ F functor
In $\mathcal{P}$	$\dfrac{\Phi \vDash \Psi}{F(\Phi) \vDash F(\Psi)}$ F functor

and similarly for a contravariant functor.

A pair of adjoint functors $F, G : \mathcal{V} \to \mathcal{V}$ with $F \dashv G$ will give inferences

In $\mathcal{V}$	$\dfrac{\Phi \xrightarrow{\alpha} G(\Psi)}{F(\Phi) \xrightarrow{\overline{\alpha}} \Psi}$ and $\dfrac{F(\Phi) \xrightarrow{\gamma} \Psi}{\Phi \xrightarrow{\underline{\gamma}} G(\Psi)}$ $F \dashv G$
In $\mathcal{G}$	$\dfrac{\Phi \Vdash G(\Psi)}{F(\Phi) \Vdash \Psi}$ and $\dfrac{F(\Phi) \Vdash \Psi}{\Phi \Vdash G(\Psi)}$ $F \dashv G$
In $\mathcal{P}$	$\dfrac{\Phi \vDash G(\Psi)}{F(\Phi) \vDash \Psi}$ and $\dfrac{F(\Phi) \vDash \Psi}{\Phi \vDash G(\Psi)}$ $F \dashv G$

Classical negation gives a contravariant functor $\neg : \mathcal{V} \to \mathcal{V}^{\mathrm{op}}$ which is adjoint to itself when viewed as $\neg : \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$. Since functors which are right adjoints must preserve products we get

$$\neg(\Phi \wedge \Psi) = (\neg \Phi) \vee (\neg \Psi)$$

(since the coproduct in $\mathcal{V}$ is the product in $\mathcal{V}^{\mathrm{op}}$). And similarly we get

$$\neg(\Phi \vee \Psi) = (\neg \Phi) \wedge (\neg \Psi)$$

2.2. Non-classical connectives

Consideration of other kinds of conjunctions will come from additional operations leading up to a monoidal structure on the category $\mathcal{V}$. Following Mac Lane [15, p. 157], we start with a bifunctor $\square : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. This makes both $A \square -$ and $- \square A$ into functors from $\mathcal{V}$ to itself. There is no requirement that they be the same functor, hence no requirement that $A \square B = B \square A$. From functoriality on both sides and bifunctoriality we get rules

In $\mathcal{V}$	$\dfrac{\Psi \xrightarrow{\beta} \Xi}{\Phi \square \Psi \xrightarrow{\mathbf{id}_\Phi \square \beta} \Phi \square \Xi}$	$\dfrac{\Psi \xrightarrow{\beta} \Xi}{\Psi \square \Phi \xrightarrow{\beta \square \mathbf{id}_\Phi} \Xi \square \Phi}$	$\dfrac{\Phi \xrightarrow{\alpha} \Psi \quad \Xi \xrightarrow{\beta} \Upsilon}{\Phi \square \Xi \xrightarrow{\alpha \square \beta} \Psi \square \Upsilon}$
In $\mathcal{G}$	$\dfrac{\Psi \Vdash \Xi}{\Phi \square \Psi \Vdash \Phi \square \Xi}$	$\dfrac{\Psi \Vdash \Xi}{\Psi \square \Phi \Vdash \Xi \square \Phi}$	$\dfrac{\Phi \Vdash \Psi \quad \Xi \Vdash \Upsilon}{\Phi \square \Xi \Vdash \Psi \square \Upsilon}$
In $\mathcal{P}$	$\dfrac{\Psi \vDash \Xi}{\Phi \square \Psi \vDash \Phi \square \Xi}$	$\dfrac{\Psi \vDash \Xi}{\Psi \square \Phi \vDash \Xi \square \Phi}$	$\dfrac{\Phi \vDash \Psi \quad \Xi \vDash \Upsilon}{\Phi \square \Xi \vDash \Psi \square \Upsilon}$

A kind of closedness condition asks for right adjoints to these two functors. Following the notation in the Lambek calculus (where these first appear in a logical, or at least proof theoretic, context for use in linguistics [12]) we will write $\Phi\backslash-$ for the right adjoint to $\Phi\square-$ and $-/\Phi$ for the right adjoint to $-\square\Phi$. If we use $\star$ for the analogous conjunction we label the right adjoints as $\Phi\searrow-$ and $-\swarrow\Phi$. The adjointness conditions then give the following rules:

In $\mathcal{V}$	$\dfrac{\Phi\square\Psi \xrightarrow{\alpha} \Xi}{\Psi \xrightarrow{\overline{\alpha}} \Phi\backslash\Xi}\backslash I$	$\dfrac{\Psi \xrightarrow{\beta} \Phi\backslash\Xi}{\Phi\square\Psi \xrightarrow{\overline{\overline{\beta}}} \Xi}\backslash E$	$\dfrac{\Phi\square\Psi \xrightarrow{\alpha} \Xi}{\Phi \xrightarrow{\overline{\overline{\alpha}}} \Xi/\Psi}/I$	$\dfrac{\Phi \xrightarrow{\beta} \Xi/\Psi}{\Phi\square\Psi \xrightarrow{\overline{\overline{\beta}}} \Xi}/E$
In $\mathcal{G}$	$\dfrac{\Phi\star\Psi\Vdash\Xi}{\Psi\Vdash\Phi\searrow\Xi}\searrow I$	$\dfrac{\Psi\Vdash\Phi\searrow\Xi}{\Phi\star\Psi\Vdash\Xi}\searrow E$	$\dfrac{\Phi\star\Psi\Vdash\Xi}{\Phi\Vdash\Xi\swarrow\Psi}\swarrow I$	$\dfrac{\Phi\Vdash\Xi\swarrow\Psi}{\Phi\star\Psi\Vdash\Xi}\swarrow E$
In $\mathcal{P}$	$\dfrac{\Phi\star\Psi\vDash\Xi}{\Psi\vDash\Phi\searrow\Xi}\searrow I$	$\dfrac{\Psi\vDash\Phi\searrow\Xi}{\Phi\star\Psi\vDash\Xi}\searrow E$	$\dfrac{\Phi\star\Psi\vDash\Xi}{\Phi\vDash\Xi\swarrow\Psi}\swarrow I$	$\dfrac{\Phi\vDash\Xi\swarrow\Psi}{\Phi\star\Psi\vDash\Xi}\swarrow E$

Strict monoidal categories are also required to have a two sided unit in the form of an object U such that $A\square U = U\square A = A$ and require that $\square$ be strictly associative. In Mac Lane's definition of a monoidal category these requirements are relaxed to ask for natural transformations α, λ, ρ giving isomorphisms $\alpha_{A,B,C} : A\square(B\square C) \longrightarrow (A\square B)\square C$, $\lambda_A : U\square A \longrightarrow A$, and $\rho_A : A\square U \longrightarrow A$ which satisfy coherence conditions making all of the diagrams which should commute. These become axiom schemes when translated into the logic.

In $\mathcal{V}$:	$A\square(B\square C) \xrightarrow{\alpha_{A,B,C}} (A\square B)\square C$	$U\square A \xrightarrow{\lambda_A} A$	$A\square U \xrightarrow{\rho_A} A$
In $\mathcal{G}$:	$A\star(B\star C)\Vdash(A\star B)\star C$	$U\star A\Vdash A$	$A\star U\Vdash A$
In $\mathcal{P}$:	$A\star(B\star C)\vDash(A\star B)\star C$	$U\star A\vDash A$	$A\star U\vDash A$

The definition of a monoidal closed category in Eilenberg and Kelly [2, p. 475] actually only asks for the adjoint for $-\square\Phi$, rather than asking for both sides to have adjoints. Some of the logics we consider later have adjoints on only one side.

2.3. Possible properties of non-commutative conjunctions

Possible axioms for a binary operation $\star$ on a poset to be considered as a possible kind of conjunction (obviously we are not asking for all of these to hold) can be grouped as follows:

Basic properties needed for a conjunction include:

- **Right functoriality**: If $a\leq b$ then $a\star c\leq b\star c$. This says that the operation $-\star c$ gives a functor; this is sometimes stated as "non-decreasing".
- **Left functoriality**: If $a\leq b$ then $c\star a\leq c\star b$.
- **Associativity**: $a\star(b\star c) = (a\star b)\star c$.

To get an implication we want

- **Autonomous**: $-\star a$ has right adjoint $-\swarrow a$ (**Right Residuation**) and $a\star-$ has right adjoint $a\searrow-$ (**Left Residuation**)

this in turn is closely related to

- **Right distributivity over** $\bigvee$: $\bigvee(a_i\star b) = (\bigvee a_i)\star b$. This also expresses right lower semi-continuity.
- **Left distributivity over** $\bigvee$: $\bigvee(b\star a_i) = b\star(\bigvee a_i)$. This also expresses left lower semi-continuity.
- **Right distributivity over** $\bigwedge$: $\bigwedge(a_i\star b) = (\bigwedge a_i)\star b$. This also expresses right upper semi-continuity.

- **Left distributivity over** $\bigwedge$: $\bigwedge(b \star a_i) = b \star (\bigwedge a_i)$. This also expresses left upper semi-continuity.
- **Preservation of** $\bot$ on right: $\bot \star a = \bot$ and on left: $a \star \bot = \bot$.

Logics arising from monoidal structures will have

- **Right unit**: There is a u with $a \star u = a$.
- **Left unit**: There is a u with $u \star a = a$.

It is often desirable for that unit to be the top of the lattice. In quantales this has a weak form given by

- **Right sided**: $a \star \top \leq a$ strictly if $a \star \top = a$.
- **Left sided**: $\top \star a \leq a$ strictly if $\top \star a = a$.

The quantale is said to be **two sided** if it is both right and left sided; again this can be strict.

Additional axioms often encountered include

- **Idempotency**: $a \star a = a$.
- **Right divisibility**: If $x > y$ then there is a z so that $y = x \star z$.
- **Left divisibility**: If $x > y$ then there is a z so that $y = z \star x$.
- **Involutivity**: Equipped with an order preserving involution $()^*$ so that $(a \star b)^* = b^* \star a^*$ (recently studied by Höhle [11]).

Some order based properties which are of interest are captured in the following possible axioms:

- **Dualizing element**: There is an element d with $d \swarrow (a \searrow d) = d = (d \swarrow a) \searrow d$. This is *cyclic* if $a \searrow d = d \swarrow a$ for every a. (This axiom is added to make a quantale a Girard quantal. If 0 is a dualizing element in a commutative BL-algebra then we have an MV algebra.)
- When the top element of the lattice is a unit for $\star$ on the relevant side and we have residuation we get
  - **Right Recovering order** $y \swarrow x = \top$ if and only if $x \leq y$
  - **Left Recovering order** $x \searrow y = \top$ if and only if $x \leq y$

Constellations of these properties have been named in the literature:

Definition 2 (*Pavelka [17]*). A **residuated lattice** is a lattice (not necessarily complete) with an operation $\star$ which is autonomous (so both residuations exist) often one assumes commutativity of $\star$ as well.

Definition 3 (*Rosenthal [18]*). A **quantale** is a complete lattice with an operation $\star$ which is associative and satisfies both right and left distributive laws over $\bigvee$. A **right Gelfand** quantale is right sided and idempotent.

Definition 4. In Goguen [6] a quantale which is a distributive lattice is called a complete lattice ordered semigroup (**closg** for short).

Most examples of closg's considered in the literature are commutative. These have both implications coming from right adjoints for both $- \wedge a$ and $- \star a$.

Definition 5. An operation $\star$ on the unit interval [0,1] is

- A **t-norm** if it is commutative, associative, functorial, and has 1 as a unit. Upper and lower semicontinuity are given by relevant distributive laws.
- A **pseudo-t-norm** [3] if it is functorial, associative, and has 1 as a two sided unit. Proper pseudo-t-norms are not commutative.

Definition 6. A **BL-algebra** [7] is a bounded lattice with an operation $\star$ which is functorial, commutative, associative, two sided, autonomous, prelinear, and has $\wedge$ expressible in terms of $\star$ and $\rightarrow$.

A **pseudo-BL-algebra** [8] is a bounded lattice with an operation $\star$ which is functorial, associative, two sided, autonomous (both $p \searrow q$ and $q \swarrow p$ exist), prelinear using either implication, and has $\wedge$ expressible in terms of $\star$ and either implication. If we only ask for $\swarrow$ we get a post-BL algebra. If we only ask for $\searrow$ we get a pre-BL algebra.

Definition 7. A **flea** [10] is a lattice with a top and an operation ⋆ which is functorial, associative, two sided, autonomous, and prelinear.

In general fuzzy logics assume two sidedness and prelinearity because the model in mind is based on operations on the unit interval. Logics with values in a quantale usually do not assume two sidedness, though they may assume left sidedness, and do not assume that the unit is the top of the lattice. We will see later that for predicate calculus we will want completeness of the lattice.

When ⋆ satisfies a distributive law over $\bigvee$ and preserves the bottom of the lattice we get a residuation giving an implication. Hájek uses & for the conjunction and $\to$ and $\rightsquigarrow$ for the two implications (his $p \to q$ is our $q \swarrow p$ and his $p \rightsquigarrow q$ is our $p \searrow q$). I have trouble distinguishing his implication symbols visually in the text of his paper [8]. He notes that subscripted arrows have been used elsewhere for other purposes. Furthermore, there is not an agreed upon convention about whether which side the ⋆ is on or which divisibility property one has determines what is left and what is right. My hope is that the notation used here (with its historical roots in Lambek's work in linguistics) will avoid some of the possible confusions and will be clearer for the mildly visually impaired.

Hájek gives his logics [7,10,9,8] for Fuzzy Logic in terms of implication with only a few references to ⋆. His versions of the axioms are numbered A1–A7 in [7]. He works in a setting where the top element of the lattice is a two sided unit, so properties of implication which give tautologies describe what happens in the order. He allows an inference from $\top \Vdash p \searrow q$ to $\top \Vdash q \swarrow p$ (and vice versa) which holds because both are equivalent to $p \Vdash q$ when the top is a two sided unit. Note that this is not the same as allowing the inferences $p \searrow q \Vdash q \swarrow p$ and $q \swarrow p \Vdash p \searrow q$.

Several of the non-commutative logics we consider do not have this property. As a result we will not have the rule for replacing one implication with another and we will state further properties in inference form. If $\top$ is the unit for ⋆ then the central inference can be replaced with implication and we require that the result be true, that is to say, equal to $\top$.

- **Transitivity of inference**: If we have $p \Vdash q$ and $q \Vdash r$ we want to conclude $p \Vdash r$ This is built into the categorical semantics, but its expression as in terms of implications is rather problematic in the non-commutative case. In the commutative case Hájek [7] uses A1:

  $$(p \to q) \to ((q \to r) \to (p \to r))$$

  which actually mixes in some commutativity with the transitivity. For a non-commutative ⋆ in [8] he uses the tautology form of
    ○ A1br: $(q \swarrow p) \Vdash ((q \swarrow r) \swarrow (p \swarrow r))$
    ○ A1bl: $(p \searrow q) \Vdash ((r \searrow p) \searrow (r \searrow q))$
- **⋆ Removal** $p \star q \Vdash p$ and $p \star q \Vdash q$. These are the inference forms of Hájek's A2. These follows from one-sidedness and functoriality. Together they will give $p \star q \Vdash p \wedge q$.
- **Expression of** $\wedge$: These are also forms of divisibility using implication. On the left side: $p \star (p \searrow q) = p \wedge q$ and right side: $p \wedge q = (q \swarrow p) \star p$ in Hájek's terms this becomes
    ○ A4l: $p \star (p \searrow q) \Vdash q \star (q \searrow p)$
    ○ A4r: $(q \swarrow p) \star p \Vdash (p \swarrow q) \star q$
- **Currying**: These express the adjointness in the residuation:
    ○ A5ar: $(r \swarrow q) \swarrow p \Vdash r \swarrow (p \star q)$
    ○ A5al: $(q \searrow (p \searrow r)) \Vdash ((p \star q) \searrow r)$
    ○ A5br: $r \swarrow (p \star q) \Vdash (r \swarrow q) \swarrow p$
    ○ A5bl: $((p \star q) \searrow r) \Vdash ((q \searrow (p \searrow r))$
- **Prelinearity**: $(x \swarrow y) \vee (y \swarrow x) = \top = (x \searrow y) \vee (y \searrow x)$ which Hájek expresses using
    ○ A6r: $r \swarrow (q \swarrow p) \Vdash r \swarrow (r \swarrow (p \swarrow q))$ and
    ○ A6l: $((p \searrow q) \searrow r) \Vdash (((q \searrow p) \searrow r) \searrow r)$
- **Existence of** $\bot$: A7: $\blacktriangle \Vdash p$
- **Expression of** $\vee$: A8:
    ○ $p \vee q = ((q \swarrow p) \searrow q) \wedge ((p \swarrow q) \searrow p)$ and
    ○ $p \vee q = (q \swarrow (p \searrow q)) \wedge (p \swarrow (q \searrow p))$

We can use non-commutative logics to demonstrate the independence of axioms:

Example 1. Consider the linear order $0 < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < 1$ with 1 as the designated value. Define the non-commutative operation $\star$ by the table

$\star$	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
0	0	0	0	0	0
1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

A direct calculation shows that this gives an associative operation which distributes over max.

Since the linear order is a complete lattice we can produce tables for the right and left residuation using the formulas

$$b \swarrow a = \max\{x | x \star a \leq b\} \text{ and } a \searrow b = \max\{x | a \star x \leq b\}$$

This gives the implications

$\swarrow$	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
0	1	1	1	1	1
1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{1}{2}$	0	1	1	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	0	1	1	1	$\frac{1}{3}$
$\frac{1}{4}$	0	1	1	1	1

and

$\searrow$	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
0	1	1	1	1	1
1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{1}{2}$	0	1	1	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{1}{3}$	0	1	1	1	$\frac{1}{2}$
$\frac{1}{4}$	0	1	1	1	1

In [1] a proof of A3 from the other axioms of BL is given. This depends heavily on using the form of axiom A1 in Hájek's book [7] rather than the form used in [8]. Using $\searrow$ this example shows the independence of commutativity from Hájek's other axioms for his logic BL with the alternate form for A1 given by A1bl: it satisfies axioms A1bl, A2l, A4l, A5al, A5bl, A6l, A7l, and quasi-linearity, and does not satisfy A3l.

3. New logics from old

Let us suppose that we start by giving names to elements and operation which may be available to us in our starting structure: $(L, \leq, \wedge, \vee, \bigwedge, \bigvee, \star, \swarrow, \searrow, \Rightarrow, \top, \bot, u, ()^*, \neg)$. Typical starting places might ask that $(L, \top, \bot, \wedge, \vee)$ be a bounded lattice or that $(L, \leq, \star, \swarrow, \searrow)$ be an ordered semigroup with residuation on both sides. Reference to $\bigvee$ or $\bigwedge$ assumes that our structure is cocomplete or complete.

3.1. Composition logics

3.1.1. All functions

First let us consider the set L^L of all functions from L to L under composition. This set has order and any operations on L all defined pointwise. The following proposition summarizes the direct consequences of the definitions:

Proposition 1. *With no assumptions L^L*

1. *has an associative composition which is rarely commutative;*
2. *has an order with the constant function with value $\top$ as largest element and the constant $\bot$ as smallest element;*
3. *both $\top \circ f = \top$ and $\bot \circ f = \bot$, though the similar equations with the composition on the other side require preservation properties for the functions being considered;*
4. *$\circ$ is right functorial: If $f \leq g$ then $f \circ h \leq g \circ h$. Left functoriality will require restriction to order preserving maps from L to L instead of all maps;*
5. *has the identity function id_L as two sided unit;*

6. *composition satisfies several right distributive laws:*
   - $(\bigvee f_i) \circ g = \bigvee (f_i \circ g)$
   - $(\bigwedge f_i) \circ g = \bigwedge (f_i \circ g)$
   - $(f \odot g) \circ h = (f \circ h) \odot (g \circ h)$ *for any operation $\odot$ defined on L*

   The related left distributive laws all require that the functions in question preserve the relevant operation on L. In addition left functoriality requires that the functions in question preserve the order;
- *there is a right residuation giving $g \swarrow f$ given by*

$$g \swarrow f(\lambda) = \bigvee \{h(\lambda) | h \circ f \leq g\}$$

the value of this at x is given by $\bigvee \{z \leq x | f(x) \leq g(z)\}$.

Absence of left functoriality makes all of the forms of A1 fail. In general the resulting structure does not satisfy quasilinearity, recovery of the order, or recovery of $\wedge$ even in the case where L is a chain.

This lattice ends up having the unit somewhere in the middle. That makes the transition between natural deduction or categorical style proof theory and a Hilbert style deductive system problematic since knowing that $\phi \leq \psi$ is distinct from knowing that $\psi \swarrow \phi = \top$. The asymmetry between what happens on the left (generally nothing nice) and what happens on the right (in the previous proposition) is also rather ugly. To correct for these deficiencies it is useful to restrict the functions being considered.

3.1.2. Structure preserving maps below the identity

Suppose we have a complete lattice L. Let S be the set of functions $f : L \to L$ such that

1. f preserves order;
2. $f \leq \mathrm{id}_L$;
3. $\bigvee (f(l_i)) = f(\bigvee l_i)$.

Such a structure will have composition as an operation which is functorial on both sides, is associative, satisfies both distributivity axioms over $\bigvee$, is strictly two sided (since we have forced the two sided unit to be the top of the lattice), is autonomous (both residuations exist), has both right and left order recovery from the implication.

Proposition 2. *If L is linearly ordered then the composition operator on the lattice of $\bigvee$-preserving functions less than the identity satisfies both right and left quasilinearity:*

$$(f \searrow g) \vee (g \searrow f) = \mathrm{id}_L$$

and

$$(f \swarrow g) \vee (g \swarrow f) = \mathrm{id}_L$$

Proof. This follows from the calculations

$$(f \searrow g)(x) = \left(\bigvee \{h | f \circ h \leq g\} \right)(x) = \bigvee \{z \leq x | f(z) \leq g(x)\}$$

and

$$(g \swarrow f)(x) = \left(\bigvee \{h | h \circ f \leq g\} \right)(x) = \bigvee \{z \leq x | f(x) \leq g(z)\}$$

Notice that if $f(x) \leq g(x)$ both of these give x. In a linear order L for each x we either have $f(x) \leq g(x)$ or $g(x) \leq f(x)$, so one of $(f \searrow g)(x) = x$ or $(g \searrow f)(x) = x$ holds and one of $(f \swarrow g)(x) = x$ or $(g \swarrow f)(x) = x$ holds. $\square$

A small example of this structure which we can consider explicitly is

Example 2. Consider the set of increasing functions $f : \{0, 1, 2\} \to \{0, 1, 2\}$ which have the property that $f(x) \leq x$. This forms a lattice using the operations

$$(f \wedge g)(x) = \min(f(x), g(x)) \text{ and } (f \vee g)(x) = \max(f(x), g(x))$$

The identity function is the largest element in this lattice. Composition of functions gives us a non-commutative operation which distributes over $\wedge$ and thus has residuations on both sides, one adjoint to composition on the right and one adjoint to composition on the left. This is a small example (there are only five such functions) so we can give tables for the operations:

The Hasse diagram for the lattice is

```
             012
          ↗       ↖
    002               011
          ↖       ↗
             001
              ↑
             000
```
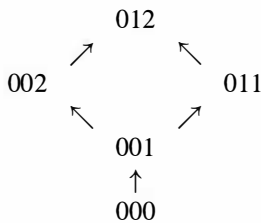
The non-commutative monoidal structure is given by

∘	000	001	011	002	012
000	000	000	000	000	000
001	000	000	000	001	001
011	000	001	011	001	011
002	000	000	000	002	002
012	000	001	011	002	012

The residuations giving adjoints to $- \circ a$ and $a \circ -$ are

↙	000	001	011	002	012
000	012	002	002	000	000
001	012	012	002	011	001
011	012	002	012	011	011
002	012	012	002	012	002
012	012	012	012	012	012

and

↘	000	001	011	002	012
000	012	012	012	012	012
001	011	012	012	012	012
011	000	002	012	002	012
002	011	011	011	012	012
012	000	001	011	002	012

A direct computation from these tables then shows that each of the following forms of the axioms gives a tautology (i.e. the value is always 012, the top of the lattice): On the right we get A1br, A2r, A5ar, A5br, A6r, and A7r. On the left we get A1bl, A2l, A5al, A5bl, A6l, and A7l.

The following do not give tautologies: A1r, A1l, A4r, and A4l.

3.2. Matrix logics

Matrix multiplication over $\mathbb{R}$ is the second example most students see of a non-commutative operation in ordinary mathematics (though one could claim that as seen in linear algebra this is just another coding of function composition, hence really the first example they see taken in another light). The formal operation of matrix multiplication can be carried out with other operations than addition and multiplication of numbers to get associative, functorial operations useable in logic.

Given L, form $L[n,n]$, the set of n by n matrices with entries in L. Then using the relations and operations on L we can define many operations and relations componentwise:

- Order: $M \leq P$ if and only if $m_{i,j} \leq p_{i,j}$ for all i,j.
- $M \wedge P$ and $M \vee P$ are computed componentwise.
- $\bigvee\{M_i\}$ and $\bigwedge\{M_i\}$ are computed componentwise
- The matrix with all entries $\bot$ is the smallest in $L[n.n]$ and the matrix with all entries $\top$ is the largest.
- Any additional operation on L induces an operation on $L[n,n]$ componentwise satisfying the same equations and thus the same axioms.

Proposition 3. *If $(L, \top, \bot, \wedge, \vee, \ddagger, \odot)$ is a lattice with binary operations $\ddagger$ and $\odot$ such that $\ddagger$ is associative and commutative and $\odot$ is associative and distributes over $\ddagger$ on both sides then matrix multiplication defined as usual with $\ddagger$ replacing + and $\odot$ replacing multiplication gives an operation which is associative. If both $\ddagger$ and $\odot$ preserve order, $\wedge, \bigwedge, \vee,$ or $\bigvee$, so will matrix multiplication.*

3.3. Words with concatenation

Let L be a partially ordered set. We consider three kinds of words with lexicographical order:

- All words of non-zero finite length L^+. Concatenation of two such words is again such a word. If L has a least element $\bot$ then the word of length 1 given by $\bot$ is the smallest element of L^+. There is no largest word even if L has a top. If L is totally ordered (satisfying trichotomy) then so will L^+ be.
- All words of finite length. This is L^*. The empty word it will be the smallest. There is no largest word. Concatenation of two words of finite length is again a word of finite length, so no modifications of common notions are needed. If L is totally ordered, so is L^*.
- All words of either finite or infinite length: $L^{\ddagger}$. Concatenation works for words of finite length with no problem. If σ is of infinite length, then $\sigma\tau = \sigma$ no matter what τ is. If σ is finite and τ is infinite then $\sigma\tau$ puts σ at the head of τ. The empty word is the bottom of $L^{\ddagger}$. If L has a top $\top$ then $L^{\ddagger}$ also has a top, given by the infinite word $\top\top \ldots$.
- All words of length n or less: L^{*n} Concatenation involves truncation to achieve the constraint on length. Here again we get the empty word as the bottom of L^{*n} and the n-tuple $\top\top \ldots \top$ as the top.

   Now in all of these cases concatenation is left functorial but not right functorial. The problem with right functoriality ($\sigma \leq \tau \Vdash \sigma\pi \leq \tau\pi$) comes when the smaller word is a truncation of the larger word and the first letter in the tail of the longer word is smaller than the first letter in π. The empty word is a two sided unit in all of these except L^+ (which does not have an empty word).

   If L is a lattice then L^{*n} and $L^{\ddagger}$ are also lattices. If σ and τ are comparable we already know how to find $\sigma \wedge \tau$ and $\sigma \vee \tau$. If they are not comparable there is a first place in which they differ. To find $\sigma \wedge \tau$ take the common head, then the minimum of the next entries, then pad with $\top$. To find $\sigma \vee \tau$ take the common head, then the maximum of the next entries and then nothing. This approach to finding $\sigma \vee \tau$ also works for L^* and L^+, though in those cases $\sigma \wedge \tau$ is not always defined. Concatenation distributes on the left over both $\wedge$ and $\vee$, but need not distribute on the right (here again the problem comes if τ is a truncated part of σ and the first letter of π has the wrong relationship with the first letter of the relevant tail of σ).

   If L has arbitrary suprema then so will L^{*n} and $L^{\ddagger}$ but not L^* or L^+. If the family can be turned into a chain it is clear what to do, if not then there is a first place where the family of entries does not include a maximum. Take the supremum there and quit. Here again concatenation on the left will preserve arbitrary sups though concatenation on the right may not.

3.4. Infinite sequences with shuffle

Here we take L a lattice and look at $L^{\mathbb{N}}$ with the shuffle of two functions given by

$$(f \odot g)(n) = \begin{cases} f(n/2) & \text{if } n \text{ is even} \\ g((n-1)/2) & \text{if } n \text{ is odd} \end{cases}$$

Notice that $f \odot g$ and $g \odot f$ are distinct and that there is no unit. The only idempotent elements are constant sequences.

If we use the lexicographic order, then $L^{\mathbb{N}}$ is a lattice: $\wedge$, $\vee$, and any large max or min are found as in $L^{\ddagger}$. The shuffle is both right and left functorial and both right and left distributive over any suprema that exist.

Now, $L^{\mathbb{N}}$ is also a lattice with order, $\wedge$, $\vee$, and any large max or min found as in pointwise. The shuffle is both right and left functorial and both right and left distributive over any suprema that exist.

3.5. Categorical semantics of non-commutative propositional logic

Categorical semantics for first order logic is covered extensively in [22], so I will be brief here and concentrate on a sequent calculus related to the logic given by the rules for deduction given above (which have more of the character of natural deduction).

The rather wide variety of choices available for properties of a (usually non-commutative) conjunction $\star$ means that we will need to specify what rules we allow when we explore semantics of non-commutative logics. In addition, study of tautologies is not sufficient for capturing inference in cases where $\top$ is not a two sided unit. As a result our semantics will be more complex than the usual semantics for propositional logic.

In order to define well formed formulas we need first to specify the available connectives:

- In $\mathcal{V}$ we allow $\times$, $+$, $()^{\bullet}$, $()^{\perp}$, 0, 1, $\square$, $/$, $\backslash$, U when we have products, coproducts, exponential adjoints, self-adjoint negation, an initial object, a terminal object, a tensor, a right adjoint to $-\square a$, a right adjoint to $a\square-$, and a unit for the monoidal structure, respectively.
- In $\mathcal{G}$ and $\mathcal{P}$ we allow $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\perp$, $\top$, $\star$, $\swarrow$, $\searrow$, U as underlying structures.

The rules of inference of the logic then arise from universal mapping properties, functoriality, and adjointness.

Definition 8. We can then define **wffs** in $\mathcal{V}$ as

1. Propositional variables: a, b, c, ….
2. 0-ary connectives $0,1,U$ if they are in the language of the logic.
3. For any wff w we get a wff $(w)^{\perp}$ if $()^{\perp}$ is in our logic.
4. For any wffs w and v we get the wffs $(w \times v)$, $(w+v)$, (w^{v}), $(w\square v)$, (w/v), and $(w\backslash v)$ if the logic has the desired connective.

Similarly, in $\mathcal{G}$ and $\mathcal{P}$ we get wffs as

1. Propositional variables: a, b, c, ….
2. 0-ary connectives $\perp$, $\top$, U if they are in the language of the logic.
3. For any wff w we get a wff $\neg w$ if $\neg$ is in our logic.
4. For any wffs w and v we get the wffs $(w \wedge v)$, $(w \vee v)$, $(v \Rightarrow w)$, $(w\star v)$, $(w \swarrow v)$, and $(w \searrow v)$ if the logic has the desired connective.

Definition 9. An **interpretation** of a wff w in

- a category $\mathcal{W}$ is the object constructed by assigning objects to each variable and then computing connectives in $\mathcal{W}$:
  1. variables a,b,c get assigned objects $[a],[b],[c]$;
  2. 0-ary connectives $0,1,U$ if they are in the language of the logic get assigned the initial object $[0] = \perp$, the terminal object $[1] = \top$ and the unit for the $\square$ operation $[U]$, respectively;
  3. for any wff w we get a wff $[(w)^{\perp}] = [w]^{\perp}$;
  4. for any wffs w and v we get the wffs $[(w \times v)] = [w] \times [v]$, $[(w+v)]=[w]+[v]$, $[(w^{v})]=[w]^{[v]}$, $[(w\square v)] = ([w]\square[v])$, $[(w/v)]=[w]/[v]$, and $[(w\backslash v)] = [w]\backslash[v]$;
- a graph $\mathcal{H}$ or poset $\mathcal{Q}$ is the vertex obtained by assigning vertices to each variable and then computing the connectives using the operations on the appropriate poset or graph.

Using underlying structure functors an interpretation at the category level gives rise to interpretations at the graph and poset levels.

The notion of semantic entailment of sequents comes from

Definition 10. If Λ and Δ are sets of wffs then for any interpretation of all of the wffs in Λ and Δ we get a notion of truth of the sequent

- in $\mathcal{V}$: $\Lambda \rightsquigarrow \Delta$ is true in the interpretation if whenever there are maps in $\mathcal{V}$ taking $f_\lambda : \alpha \to \lambda$ for every $\lambda \in \Lambda$ and $g_\delta : \delta \to \beta$ for every $\delta \in \Delta$ then there is a map $h : \alpha \to \beta$. Note that if the category has both products and coproducts this says there is a map $\prod \Lambda \to \coprod \Delta$;
- in $\mathcal{G}$: $\Lambda \Vdash \Delta$ is true in the interpretation if whenever $\alpha \Vdash \lambda$ for all $\lambda \in \Lambda$ and $\delta \Vdash \beta$ for every $\delta \in \Delta$ then $\alpha \Vdash \beta$;
- in $\mathcal{P}$: $\Lambda \vDash \Delta$ is true in the interpretation if whenever $\alpha \vDash \lambda$ for all $\lambda \in \Lambda$ and $\delta \vDash \beta$ for every $\delta \in \Delta$ then $\alpha \vDash \beta$.

We say that these sequents are **valid** if they are true in every interpretation.

Notice that if Λ and Δ consist of single wffs then the sequent entailment is just inference. If the we have strict two sidedness then whenever all of the expressions in Λ are tautologies (expressed as $\top \Vdash \lambda$) then the supremum of the truth values of the δ would also have to be $\top$. For a $\Delta = \{\delta_0\}$ this would say that δ_0 is a tautology as well.

An interpretation will make a sequent valid if the desired map or edge does in fact exist in the category, graph, or poset which is the target of the interpretation.

Next we need to say what a proof in the categorical logic looks like. For simplicity I will work in $\mathcal{G}$.

Definition 11. A proof of an inference $\phi \Vdash \psi$ is a tree with root $\phi \Vdash \psi$ the leaves of which are axioms (of the form $\alpha \Vdash \alpha$ or from associativity or commutativity axioms) and each branch of which is an instance of a rule of inference.

Example 3. In the logic of quantales we have connectives $\wedge$, $\vee$, $\star$, $\searrow$, and $\swarrow$, axioms from the associativity of $\star$ and inference rules from the universal mapping properties, functoriality, and adjointness relations.

Here is a proof of $(q \swarrow p) \Vdash r \swarrow (p \star (q \searrow r))$:

$$
\cfrac{
\cfrac{(q \swarrow p) \star (p \star (q \searrow r)) \Vdash ((q \swarrow p) \star p) \star (q \searrow r) \quad P}{(q \swarrow p) \star (p \star (q \swarrow r)) \Vdash r} \; cut
}{q \swarrow p \Vdash r \swarrow (p \star (q \searrow r))} \; \swarrow I
$$

where P is

$$
\cfrac{
\cfrac{\cfrac{q \swarrow p \Vdash q \swarrow p}{(q \swarrow p) \star p \Vdash q} \; \swarrow E}{((q \swarrow p) \star p) \star (q \searrow r) \Vdash q \star (q \searrow r)} \; \text{functoriality} \qquad \cfrac{\cfrac{q \searrow r \Vdash q \searrow r}{q \star (q \searrow r) \Vdash r} \; \searrow E}{}
}{((q \swarrow p) \star p) \star (q \searrow r) \Vdash r} \; cut
$$

Here the axioms

$$q \searrow r \Vdash q \searrow r \text{ and } q \swarrow p \Vdash q \swarrow p$$

come from the identities and the axiom

$$(q \swarrow p) \star (p \star (q \searrow r)) \Vdash ((q \swarrow p) \star p) \star (q \searrow r)$$

is associativity of $\star$.

With the notion of a proof within a categorical logic we can define what it means for a sequent to be provable:

Definition 12. We say $\Lambda \vdash \phi$ if there is a proof of $\alpha \Vdash \phi$ when leaves of the proof are allowed to either be axioms or inferences of the form $\alpha \Vdash \lambda$ for $\lambda \in \Lambda$.

We say $\Lambda \vdash \Delta$ if whenever $\delta \Vdash \beta$ for all $\delta \in \Delta$ we have $\Lambda \vdash \beta$.

3.6. Soundness and completeness

Soundness of our propositional logic says that if $\Lambda \vdash \Delta$ then for any interpretation $\Lambda \models \Delta$ is true. Completeness says that if every interpretation makes $\Lambda \models \Delta$ true then $\Lambda \vdash \Delta$.

Theorem 4 (*Soundness*). *Any categorical logic defined using universal mapping properties, functoriality, monoidal structure, and adjointness is sound.*

Proof. This is mostly a matter of checking that the maps posited in an interpretation in a category give us the axioms and inferences in our proof technique. For axioms of the form $\phi \Vdash \phi$ this is the identity required by the definition of a category. Axioms of the form $\alpha \Vdash \lambda$ are part of the definition of the interpretation of the sequent. Our inference rules reflect exactly the morphisms guaranteed to exist by the categorical structures specified. If we have terminal objects, initial objects, associativity of $\star$, or units these will give additional axioms as starting places for our proofs; axioms which are guaranteed to be true by the definition of an interpretation. $\square$

Completeness will also follow as in [13]. What we need to do is produce the free category on the symbols in our sequent which has the desired structure.

4. Predicate logic with non-commutative $\star$

As mentioned earlier and developed at length in [22] a categorical form of semantics for predicate calculus takes this picture and spreads it out over a category of types:

Let $\mathbf{T}$ be a category with objects represented by capital letters A, B, C, ... and morphisms given by lower case letters f, g, h, ... We assume that $\mathbf{T}$ has a terminal object T. $\mathbf{T}$ will be our category of types. We then give a contravariant functor from $\mathbf{T}$ to the category of categories:

For each object A in $\mathbf{T}$ we have a category $\mathcal{V}(A)$ and graphs $\mathcal{G}(A)$ and $\mathcal{P}(A)$. For each $f : A \to B$ we have functors f^* making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{V}(A) & \overset{f^*}{\longleftarrow} & \mathcal{V}(B) \\
U \downarrow & & \downarrow U \\
\mathcal{G}(A) & \overset{f^*}{\longleftarrow} & \mathcal{G}(B) \\
U \downarrow & & \downarrow U \\
\mathcal{P}(A) & \overset{f^*}{\longleftarrow} & \mathcal{P}(B) \\
A & \overset{f}{\longrightarrow} & B
\end{array}
$$

When we impose additional structures on our categories of predicates about A we will ask for the functors f^* to preserve the additional structure.

Following Lawvere [14] and the common practice in topos theory we get quantifiers by asking for adjoints to these functors: $\exists_f \dashv f^* \dashv \forall_f$. Since we have not specified how we get the f^* functors, only that they are present and preserve the structures of interest, the existence of these adjoints is an additional condition. It frequently imposes completeness and cocompleteness constraints on the logical structures.

Once we have the adjoints we get rules for each $f : A \to B$:

1. Since these are functors we get rules of inference

$$
\frac{\phi \Vdash_A \psi}{\exists_f(\phi) \Vdash_B \exists_f(\psi)} \quad \frac{\phi \Vdash_A \psi}{\forall_f(\phi) \Vdash_B \forall_f(\psi)} \quad \frac{\phi \Vdash_B \psi}{f^*(\phi) \Vdash_A f^*(\psi)}
$$

2. The adjointness gives rules of inference:

$$
\frac{\exists_f \phi \Vdash_B \psi}{\phi \Vdash_A f^\star(\psi)} \exists_f \dashv f^\star \quad \frac{\psi \Vdash_B \forall_f \phi}{f^\star(\psi) \Vdash_A \phi} \ f^\star \dashv \forall_f
$$

where the double line indicates a reversible inference, giving both rules for introduction and elimination of quantifiers.

Example 4. As an example of such a situation we let **T** be the category of sets and let **C** be a category. We can construct a categories of **C**-valued fuzzy sets and get $\mathcal{V}(A)$ to be a category with much of the structure of **C** defined pointwise. An object of $\mathcal{V}(A)$ is an assignment to each element of A an object of **C**. Morphisms are then A-indexed families of morphisms in **C**. Essentially all of the kinds of structure considered in this paper will be inherited on $\mathcal{V}(A)$ from those of **C**.

Given a function $f : A \to B$ we get the functor $f^* : \mathcal{V}(B) \to \mathcal{V}(A)$ by taking the object with b-coordinate $h(b)$ to the object of $\mathcal{V}(A)$ with a-coordinate $h(f(a))$. The action on morphisms is componentwise.

In order to get a left adjoint to f^* we need for the category **C** to be cocomplete. The value of $\exists_f$ at an object $g : A \to$ **C** of $\mathcal{V}(A)$ will take b to the coproduct of all of the objects $g(a)$ which have $f(a)=b$. The right adjoint to f^* will have value $\forall_f(g)(b)$ given by the product of those objects $g(a)$ which have $f(a)=b$.

Example 5. Another approach using fuzzy sets would take a complete lattice with a second conjunction $\star$ and construct the Goguen category **Set** (L) with objects pairs $(A, \alpha : A \to L)$ and maps $f : (A, \alpha) \to (B, \beta)$ given by functions $f : A \to B$ with $\alpha(a) \leq \beta(f(a))$ for all $a \in A$ to use as the category of types. We can then take $\mathcal{G}(A, \alpha)$ to be the subcategory of **Set**$(L)/(A, \alpha)$ consisting of those maps into (A, α) with underlying set map an isomorphism. We can then think of $\mathcal{P}(A\alpha)$ as consisting of the fuzzy sets (A, α') with $\alpha'(a) \leq \alpha(a)$ for all $a \in A$ (the *unbalanced subobjects* of (A, α)). As noted in [20,21] these lattices inherit many of the structures from L, though the unit, the top, and thus one sidedness may be lost. If $f : (A, \alpha) \to (B, \beta)$ then

$$f^*(B, \beta')(a) = \beta'(f(a)) \wedge \alpha(a)$$
$$\exists_f(A, \alpha')(b) = \bigvee_{\{a | f(a)=b\}} \alpha'(a)$$

and

$$\forall_f(A, \alpha')(b) = \begin{cases} \bigwedge_{\{a | f(a)=b\}} \alpha'(a) & \text{if there is such an } a \\ \beta(b) & \text{otherwise} \end{cases}$$

5. Representation using internal categories

If we want to do higher order logic in a category we need to find a way to represent all of this structure internally. For logics based in **Sets** where the categories of predicates about a set are small everything is automatically internal. In topoi the subobject lattices are represented internally as category objects using exponentials of the subobject representer Ω.

5.1. Predicate representation

If we take L to be a quantale with $\top$ as left sided unit which satisfies left recovering order, then we can represent the lattice of unbalanced subobjects of a fuzzy set (A, α) as a category object in **Set** (L). The object of objects will be

$$\mathcal{U}(A, \alpha) = \left(\{f : A \to L\}, \xi : f \mapsto \bigwedge_{a \in A} (f(a) \searrow \alpha(a)) \right)$$

and the object of morphisms will be

$$\mathcal{M}(A, \alpha) = \left(\{(f, g : A \to L)\}, \zeta : (f, g) \mapsto \bigwedge_{a \in A} (f(a \searrow g(b)) \wedge \xi(f) \wedge \xi(g)) \right)$$

with the projections giving the domain and codomain maps, diagonal giving the inclusion of identities, and composition given by projection. Global sections (i.e. maps from $(\{s\}, \top)$ to $\mathcal{U}(A, \alpha)$ in **Set** (L)) recover the category of predicates about (A, α).

Given a map $f : (A, \alpha) \to (B, \beta)$ we can internalize the functors f^*, $\exists_f$, and $\forall_f$ as maps $\overline{f^*} : \mathcal{U}((B, \beta)) \to \mathcal{U}((A, \alpha))$, $\overline{\exists_f} : \mathcal{U}((A, \alpha)) \to \mathcal{U}((B, \beta))$ and $\overline{\forall_f} : \mathcal{U}((A, \alpha)) \to \mathcal{U}((B, \beta))$ which are internal functors. Here

$$\overline{f^*}(h : B \to L)(a) = h(f(a))$$

and

$$\overline{\exists_f}(g : A \to L)(b) = \bigvee_{\{a \mid f(a)=b\}} g(a), \quad \overline{\forall_f}(g : A \to L)(b) = \bigwedge_{\{a \mid f(a)=b\}} g(a)$$

making the whole first order logic internal to **Set** (L).

5.2. Subobject representation and internal second order logic

Unbalanced subobjects of a fuzzy set (A, α) have only a weak form of subobject representation. For any unbalanced subobject there is a characteristic map such that

$$
\begin{array}{ccc}
(A,\alpha') & \longrightarrow & (L,\mathrm{id}_L) \\
\downarrow & & \downarrow \\
(A,\alpha) & \xrightarrow{\chi} & (L,\top)
\end{array}
$$

is a pullback, but such characteristic maps need not be unique. This means that we cannot use the exponential adjoint in the category **Set**(L) to get an internal higher order logic uniquely represented.

We need a way to represent membership as an object in $\mathcal{P}((A, \alpha) \times \mathcal{U}(A, \alpha))$. Fortunately there is an obvious candidate:

$$\varepsilon_{(A,\alpha)} : (a,f) \mapsto f(a)$$

which will allow us (in further work) to express an internal higher order logic using the representation of predicates on (A, α) as internal category objects and the quantifiers as internal functors.

Also left for further investigation is the connection between the approaches given here and those developed by Meseguer. The author thanks the anonymous referee for pointing out that large literature.

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