The Comparison of Ad Valorem and Specific Taxation under Incomplete Information

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THE COMPARISON OF AD VALOREM AND SPECIFIC TAXATION UNDER UNCERTAINTY

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Abstract
The comparison between specific (per unit) and ad valorem (percentage) taxation has been one of the oldest issues in public finance. In Cournot markets, with deterministic costs structures, conventional wisdom has it that ad valorem taxation tax-revenue dominates specific. It is shown that in the presence of uncertainty, regarding firms’ cost structures, and under reasonable conditions, the conventional wisdom might not hold. The implication of this, from a policy perspective, is that the precise evaluation of the two types of taxation requires an explicit consideration of cost uncertainty.

1. Introduction
Two fundamental considerations in the design of fiscal policy, and for the success of fiscal reforms, are the stability, certainty, and predictability of tax
revenues\(^1\) and the choice between specific and *ad valorem* taxes. The interplay between these two considerations—and in particular the role of uncertainty in the comparison of specific and *ad valorem* taxes—is the focus of this paper.

Uncertainty can arise from various sources but, arguably, a dominant source relates to the volatility (and unpredictability) of firms’ input prices, and so their production costs.\(^2\) To illustrate the role of uncertainty in the welfare comparison between *ad valorem* and specific taxation, the analysis makes use of the standard Cournot model with incomplete information about own, and also rivals’, production costs. This particular information structure introduces two elements. First, the equilibrium output, and consequently price, follows a distribution. Second, while both types of taxation affect the mean of the distribution, only *ad valorem* taxation affects its dispersion (variance). The consequence of this is that all key welfare measures, which depend on the mean and the dispersion of the output distribution, differ in sign from those obtained from the classical deterministic Cournot model. Arguably, this has important implications, given that uncertainty is an inherent feature of markets, from a policy perspective.

The comparison between specific (per unit) and *ad valorem* (percentage) taxation has been one of the oldest issues in public finance, going back to Wicksell’s (1896/1959) conjecture that *ad valorem* taxes may have favorable efficiency properties relative to specific taxes in monopoly markets. This conjecture has been formally demonstrated, within a monopoly framework, by Suits and Musgrave (1953). Following the seminal contribution of Suits and Musgrave (1953), a literature has emerged investigating the desirability of one tax over the other under various market structures.\(^3\) Predominantly, most contributions, with notable exceptions to which we turn below, have focused on deterministic environments.

The results obtained can be summarized as follows. Within the deterministic model, for every specific tax one can find an *ad valorem* tax that yields the same marginal cost and, hence, the same equilibrium output. This implies that the two taxes are equivalent in terms of social welfare. Since profits, as they are discounted by the *ad valorem* tax rate, are lower under

\(^1\) Such issue has been a major concern for many developing countries (Keen 1998).

\(^2\) As in sectors that are energy-intensive. Another example of an imperfectly competitive market that is characterized by cost uncertainty is the tobacco industry. Cost uncertainty here can be—and should be—broadly defined to include not only the uncertain quality of tobacco leaves but also the costs associated with regulatory rules and legal threats (World Bank 2003).

than under specific taxation, and consumer surplus is the same, ad valorem taxation generates higher tax revenue (see, among others, Anderson, de Palma, and Kreider 2001a,b). Such a conclusion, regarding tax revenues, need not hold under cost uncertainty. The driving force behind this possibility is the fact that both consumer surplus and profits are convex in price, thereby making both consumers and firms risk lovers with respect to price. Ad valorem taxation generates higher output (and so price) variability than specific taxation and, therefore, it can lead to higher expected consumer surplus and profits. On the other hand, tax revenue under specific taxation is linear in price, while under ad valorem taxation it is concave in price, thereby making the tax authority risk-averse. Consequently, output and price variability do not affect expected tax revenue under specific taxation, but their impact on expected tax revenue is negative under ad valorem taxation. The implication of this, and contrary to the deterministic model, is that specific taxation can generate higher expected tax revenue than ad valorem. The social welfare equivalence between the two taxes, emphasized in the deterministic model, does not hold. Interestingly, under certain conditions, expected social welfare can even increase in ad valorem taxation.

Contributions that have attempted to make the link between the comparison of the two types of taxation in an uncertain environment are Fraser (1985), Dickie and Trandel (1996), and Goerke (2011).4 Like all these contributions, our paper looks at the particular link between the two types of taxation and uncertainty but it does so from a different perspective. Fraser (1985) focuses on the perfectly competitive environment with firms being risk-averse, and not on, as we do here, the standard Cournot model of output competition with firms being risk-neutral. As noted earlier, the consequence of this is that, and unlike Fraser (1985), the model generates price variability—and, hence, output variability—by assuming that costs are unknown. The modeling differences between Fraser (1985) and the present framework are too significant and, therefore, the results derived from the Cournot model cannot be anticipated from Fraser’s (1985) model. Goerke (2011) demonstrates that ad valorem taxation may be superior to the specific one, for an equal yield, in a perfectly competitive environment under uncertainty about prices (as it would also be the case in an environment without uncertainty). Our model shares a number of qualitative similarities with Goerke (2011) but it focuses on a distinctively different set of issues, including the behavior of tax revenues under the two different taxes showing, in particular, that in the presence of uncertainty the ranking of tax revenues reverses, making specific taxation the dominant tax. This has important policy

4 Perloff and Wu (2007) also emphasize the fact that tax incidence on consumers differ in markets with a price distribution, and so a tax might have an uneven effect on various parts of the price distribution. While this element appears in the model here, too, the framework of Perloff and Wu (2007) bears no resemblance to the framework that is being used here.
implications, since it suggests that some relatively high (expected) tax revenue targets set by the tax authority can only be achieved with a specific tax. Dickie and Trandel (1996) also show that the equivalence between specific and ad valorem taxes does not hold under uncertainty. They do so within a model with externalities, and mainly under demand uncertainty, focusing on the welfare comparison of Pigovian taxation, assuming, however, away any revenue requirement (and so ignoring the implications of these two types of taxation on government revenues). ⁵

The structure of the paper is as follows. Section 2 introduces the elements of the model. Section 3 describes the equilibrium and derives the equilibrium output and profits, under both types of taxation. Section 4 compares profits and tax revenues under the two taxes. Section 5 deals with consumer and social surplus. Finally, Section 6 briefly concludes.

2. The Structure of the Model

To show that the desirability of one form of tax over the other crucially depends on the information available to firms regarding own and rival costs, this paper analyzes a linear-quadratic model with cost uncertainty. This is a tractable model that gives rise to linear conditional expectations that yield closed-form solutions and, therefore, a computable (and unique) Bayesian–Nash equilibrium. ⁶ It is worth emphasizing—something that will become apparent later on—that the deterministic model (with linear demands) that has been extensively employed in the literature is a special case of the incomplete information model used here.

The model features two firms that produce a homogenous good. ⁷ Market demand is linear in output and given by

\[ p(x_1 + x_2) = A - (x_1 + x_2), \]

where \( x_i \) denotes the level of output of firm \( i = 1, 2 \). Both firms have a linear cost structure with marginal cost \( c_i \), which in the absence of taxation is given by \( c_i = \theta_i \). ⁹ It is assumed that \( (\theta_1, \theta_2) \) follow a prior distribution, with

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⁵ Keen (1998) also discusses the implications of uncertainty in the balance of ad valorem and specific taxation, showing an appropriate mixture of specific and ad valorem taxes can secure complete certainty of revenues.


⁷ The analysis extends straightforwardly to any fixed number of firms (see also footnote 20), and to differentiated products, with either Bertrand or Cournot competition but with cost uncertainty. For expositional brevity, the focus here is on the duopoly case with homogeneous products and quantity competition.

⁸ The utility function considered is of the form \( u(x, y) = Ax - x^2/2 + y \) with budget constraint \( I = px + y \) (where \( x = x_1 + x_2 \) and \( y \) is the numeraire good). Notice, for later use, that, in this case, consumer surplus (denoted by \( CS \)) is given by \( CS = (A - p)^2/2 \).

⁹ Implicitly in the analysis there is also a deterministic component of the marginal cost which is absorbed by the intercept of the demand \( A \).
the prior marginal density of \( \theta_1 \) and \( \theta_2 \) being identical and equal to \( g(\theta_i) \), and \( \mathbb{E}(\theta_i) = \bar{\theta} \geq 0 \), \( \text{var}(\theta_i) = \sigma \) (same for both firms), \( \text{cov}(\theta_1, \theta_2) = h \leq \sigma \), with \( h \geq 0 \). Each firm receives a signal \( s_i \) about \( \theta_i \). The signal \( s_i \) is an unbiased estimate of the true marginal cost, in the sense that \( \mathbb{E}(s_i|\theta_i) = \theta_i \) and \( \mathbb{E}[\text{var}(s_i|\theta_i)] = m \). Conditional on \( \theta_1 \) and \( \theta_2 \), \( s_1 \) and \( s_2 \) are independently distributed according to the conditional density function \( z(s_i|\theta_i) \). The precision of the signal \( s_i \) is \( 1/m \), whereas the precision of the prior distribution of \( \theta_i \) is \( 1/\sigma \). It is assumed that the posterior density \( z(\cdot) \) and the prior density \( g(\cdot) \) give rise to the following linear posterior expected values

\[
\mathbb{E}(\theta_i|s_i) = b + Bs_i, \quad i = 1, 2, \quad (2)
\]

\[
\mathbb{E}(s_j|s_i) = f + Fs_i, \quad i \neq j. \quad (3)
\]

Prior–posterior distribution functions that satisfy the linearity assumption in (2) and (3) include the Beta–Binomial, Gamma–Poisson, and Normal–Normal. The following lemma derives the coefficients of the posterior expectations (all proofs can be found in the Appendix section).

**LEMMA 1:**

\[
\mathbb{E}(\theta_i|s_i) = \frac{m}{\sigma + m} \bar{\theta} + \frac{\sigma}{\sigma + m} s_i, \quad i = 1, 2, \quad (4)
\]

\[
\mathbb{E}(s_j|s_i) = \frac{\sigma + m - h}{\sigma + m} \bar{\theta} + \frac{h}{\sigma + m} s_i, \quad i \neq j. \quad (5)
\]

Lemma 1 simply states that the conditional expectations are a weighted average of the prior mean \( \bar{\theta} \) and the signal \( s_i \). The weights depend on the conditional expected variance of the signal \( m \), the prior variance of the costs \( \sigma \) and, for Equation (5), the covariance between the costs \( h \). An example based on the prior–posterior Beta–Binomial distribution functions, and one that we make use of later on, is given below. This distribution function guarantees that the marginal cost is positive for all realizations of uncertainty.

**Example (Beta–Binomial).** Suppose that the marginal cost of firm \( i \) can be either high \( c_H \), with probability \( \theta_i \), or low \( c_L \), with probability \( 1 - \theta_i \). Suppose also that \( \theta_i \) is distributed on \([0, 1]\), independently between the two firms, according to a Beta distribution with parameters \( \alpha \) and \( \beta \). Thus, for any \( \theta_i \), the expected marginal cost of firm \( i \) is given by \( c_H + \theta_i \gamma \), where \( \gamma \equiv c_L - c_H \). Before choosing output \( x_i \), firm \( i \) receives a signal regarding \( \theta_i \) arising from the average of \( n \) independent Bernoulli trials with parameter \( \theta_i \).\(^{10}\)

\(^{10}\) For example, the true cost may depend on the outcome of an uncertain R&D process. If firm \( i \) observes \( k_i \) successes in \( n \) trials, the signal is \( s_i = k_i/n \).
For simplicity, and without any loss of generality, set \( \gamma = 1 \). It is, thus, the case that the likelihood of the signal \( s_i \) is \( 1/n \) Binomial \( (n, \theta_i) \), with \( \text{var}(s_i|\theta_i) = \theta_i (1 - \theta_i) / n \) and \( m = \mathbb{E}[\text{var}(s_i|\theta_i)] = [(\alpha + \beta) \text{var}(\theta_i)] / n \), where \( \text{var}(\theta_i) = \sigma = \alpha\beta / \left[ (\alpha + \beta)^2 (\alpha + \beta + 1) \right] \) and, thus, \( m = \alpha\beta / [n(\alpha + \beta) (\alpha + \beta + 1)] \).

Variables pertaining to specific (unit) taxation are denoted by \( u \), whereas those pertaining to \textit{ad valorem} taxation are denoted by \( v \). Denoting by \( t \) the per unit tax rate, \( \lambda \) the \textit{ad valorem} tax rate, \( q \) the producer prices, and \( p \) the consumer prices, then

\[
p^u = q^u + t; \quad p^v = q^v (1 + \lambda) .
\]  

(6)

Firm \( i \)'s marginal cost of output under specific taxation and \textit{ad valorem} taxation are given, respectively, by

\[
c^u_i = \theta_i + t; \quad c^v_i = \frac{\theta_i}{1 - \tau},
\]  

(7)

where \( 1 - \tau = 1 / (1 + \lambda) \), with \( \tau \) being the tax-inclusive rate. Given producer prices and costs, firm \( i \)'s profits are given by

\[
\pi^u_i = (p^u - c^u_i) x^u_i, \quad \pi^v_i = (1 - \tau) (p^v - c^v_i) x^v_i .
\]

(8)

The comparison between the two taxes requires a common value for one of the fundamental parameters. This, as it is typically the case—see, for instance, Anderson \textit{et al.} (2001a,b)—is taken to be the unconditional expected marginal cost.\footnote{This is not the only way to examine these issues in the presence of uncertainty. Goerke, Herzberg, and Upmann (2011) show that, in a perfectly competitive environment, the equivalence between \textit{ad valorem} and specific taxation may not hold if the point of comparison is not expected revenues, but the more stringent restriction of pathwise equivalence. This requires that tax revenues stay constant for any realization of uncertainty.} Under the two types of taxation, this cost, following (7), is the same if

\[
\frac{\tilde{\theta}}{1 - \tau} = \tilde{\theta} + t.
\]  

(9)

Attention now turns to the analysis of the equilibrium.

3. Analysis of the Equilibrium

The sequence of events is the following. Given taxes set by the government, nature draws the firms’ cost parameters \( (\theta_1, \theta_2) \), which are unobservable by all economic agents. Then, nature draws the signals \( (s_1, s_2) \). Firm \( i \), upon observing its own signal \( s_i \), updates its beliefs about cost parameters and chooses noncooperatively its level of production \( x_i \). Finally, uncertainty—that is, \( (\theta_1, \theta_2) \)—is realized.
3.1. Specific Taxation

This model, following Vives (1999), has a unique Bayesian–Nash equilibrium which is linear in the signal a firm receives and is given by

\[ x_i = \omega_i + \delta_i s_i, \]  

(10)

where \( \omega_i \) and \( \delta_i \) are parameters to be determined (we turn to this shortly later).

Central in the behavior of firm \( i = 1, 2 \), are two expectations: The first relates to the expected output of its rival, firm \( j \neq i \), conditional on the signal \( s_i \). This expected output—taking expectations in Equation (10)—is given by

\[ E \left( x_j^u(s_j) | s_i \right) = \omega_j + \delta_j E(s_j | s_i) = \omega_j + \delta_j \left( \frac{\sigma + m - h}{\sigma + m} \tilde{\theta} + \frac{h}{\sigma + m} s_i \right), \]  

(11)

where the second equality follows from (5). The second expectation relates to firm \( i \)'s own expected marginal cost, conditional on its own signal. This cost is given—following upon taking expectations in (7)—by

\[ E \left( c_i^u | s_i \right) = E(\theta_i | s_i) + t = \frac{m}{\sigma + m} \tilde{\theta} + \frac{\sigma}{\sigma + m} s_i + t, \]  

(12)

where the second equality follows from (4).

Firms, given taxes and the signal received, maximize expected profits given by

\[ E \left( \pi_i^u(x_i^u, x_j^u(s_j)) | s_i \right) = (A - x_i^u - E \left( x_j^u(s_j) | s_i \right) - E \left( c_i^u | s_i \right) ) x_i^u, \]  

(13)

with the appropriate choice of \( x_i^u \). It is straightforward to show that the level of output chosen satisfies

\[ x^u = \frac{1}{2} \left[ A - \omega - \delta \left( \frac{\sigma + m - h}{\sigma + m} \tilde{\theta} \right) - \left( \frac{m}{\sigma + m} \right) \tilde{\theta} - t - \left( \frac{h}{\sigma + m} + \frac{\sigma}{\sigma + m} \right) s \right]. \]  

(14)

(Notice that, due to symmetry, the subscripts \( i \) and \( j \) are being suppressed.)

This choice of output, for any realization of the signal, must be consistent with the hypothesized linear strategies in Equation (10). Choosing \( \omega \) and \( \delta \) in Equation (14) to be consistent with the linear strategies hypothesized in Equation (10), it is the case that

\[ \omega_u = \frac{A - t}{3} - \frac{h + 2m - \sigma}{3 \left( h + 2 \left( \sigma + m \right) \right)} \tilde{\theta}, \]  

(15)

and

\[ \delta_u = -\frac{\sigma}{h + 2 \left( \sigma + m \right)}, \]  

(16)
The Comparison of *ad Valorem* and Specific Taxation

and, thus, the equilibrium output of the typical firm, denoted by *, is given by

\[ x^* = \omega + \delta s. \]  

(17)

Clearly, the typical firm’s equilibrium output responds negatively to its own signal. The reason for this is intuitive: If a firm receives, for example, a high signal \( s \) it believes, following from (4), that own production costs are high and so it responds, for given rival’s output, by reducing the production of own output. Such a response of output depends upon \( \text{var}(\theta) = \sigma, \text{cov}(\theta_1, \theta_2) = h, \) and the conditional variance of the signal \( \text{E}[\text{var}(s|\theta)] = m. \) These are key parameters to which we turn to later on.

Since the cost of production (and the signal received) is a random variable, so will be the production decision of the typical firm, with, in particular, mean and variance. To see this, take expectations in Equation (17) to arrive at

\[ \text{E}(x^*_u) = \frac{\Lambda - t - \bar{\theta}}{3}, \]  

(18)

whereas the variance of equilibrium output is given by

\[ \text{var}(x^*_u) = \delta^2 \text{var}(s) = \frac{\sigma^2 (\sigma + m)}{(h + 2(\sigma + m))^2}, \]  

(19)

where the second equality follows from Equation (16) and the fact that \( \text{var}(s) = \sigma + m. \)

Equations (18) and (19), and the corresponding for the *ad valorem* case ones, are central for the intuition underlying the main results. It is clear that specific taxation affects expected output but it *does not* affect the variance of the output distribution. The reason for this is the additive nature of the specific tax. When a firm receives a high signal \( s \) it lowers its output because a high \( s \) indicates that the marginal cost \( \theta + t \) is high. Nevertheless, the change in the marginal cost, due to a high signal (or equivalently high \( \theta \)), does not depend on \( t \), as the specific tax \( t \) enters the marginal cost additively. This, too, suggests that the slope of the equilibrium function \( \delta_u \), given in (16) and which measures the responsiveness of output to the signal a firm receives, does not depend on \( t \) and so the variance of output is independent of the level at which the specific tax is set. This, as will be shown shortly later, is not the case under *ad valorem* taxation.

If there is no uncertainty regarding own production costs, and so \( \sigma = 0 \), then, following from (19), the variance of output is zero, and the model reduces to the standard Cournot duopoly outcome where each firm produces the “certainty equivalent” level of output given in (18).

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12 This follows from the law of iterated expectations that is, \( \text{var}(s) = \text{var}[\text{E}(s|\theta)] + \text{E}_\theta[\text{var}(s|\theta)] = \sigma + m. \)
Making use now of the equilibrium strategy in (17) into Equation (13), expected profits for the typical firm, after some manipulations, can be shown to be

$$E(\pi^*_u) = \left( \frac{A - t - \bar{\theta}}{3} \right)^2 + \text{var}(x^*_u).$$  \hspace{1cm} (20)

Equation (20) reconfirms a standard result—see Vives (1999)—and one that will play an important role in the interpretation of the results later on: Firms’ profits are convex in their own marginal cost and, thus, they prefer high uncertainty about these costs.\textsuperscript{13}

We turn now to \textit{ad valorem} taxation.

### 3.2. Ad Valorem Taxation

The details for \textit{ad valorem} taxation are similar to those under specific taxation, and in order to avoid repetition they are omitted. It suffices here, however, to state that the typical firm’s equilibrium output strategy is

$$x^*_v = \omega_v + \delta_v s,$$  \hspace{1cm} (21)

where

$$\omega_v = \frac{A}{3} - \frac{(h + 2m - \sigma) \tilde{\theta}}{3 (1 - \tau) (h + 2 (\sigma + m))},$$  \hspace{1cm} (22)

and

$$\delta_v = -\frac{1}{(1 - \tau) h + 2 (\sigma + m)}.\hspace{1cm} (23)$$

A firm’s output is, then, distributed with mean

$$E(x^*_v) = \frac{A}{3} - \frac{\tilde{\theta}}{3 (1 - \tau)},$$  \hspace{1cm} (24)

and variance

$$\text{var}(x^*_v) = \delta^2_v \text{var}(s) = \frac{1}{(1 - \tau)^2} \frac{\sigma^2 (\sigma + m)}{(h + 2 (\sigma + m))^2},$$  \hspace{1cm} (25)

where, again, \text{var}(s) = \sigma + m. As in the case of specific taxation, when there is no uncertainty regarding own production costs, the equilibrium reduces to the standard Cournot outcome, where each firm produces the level of output given in Equation (24). With uncertainty, higher signal implies lower output. The most crucial difference with specific taxation is that \textit{ad valorem} taxation affects not only the mean of the distribution but also the variance (as can be seen from (25)). This is because the change in the marginal cost

\textsuperscript{13} Or, to put it differently, firms exhibit risk-loving behavior (see also Oi 1961).
\[ \theta/(1 - \tau) \] when \( s \) changes does depend on \( \tau \), simply because ad valorem taxation, unlike specific, enters in a nonadditive way. This dependence affects the slope \( \delta_v \) of the typical firm’s equilibrium output response to own signal and, consequently, the variance of output. Close inspection of the typical firm’s equilibrium output response to own signal under the two types of taxation reveals that—for any ad valorem tax rate \( \tau \) and for given signal \( s \)—ad valorem taxation gives rise to a more responsive output than specific taxation. To see this, it is easy to verify—following from (16) and (23)—that

\[
\delta_v = \frac{1}{1 - \tau} \delta_u, \tag{26}
\]

and so for any given signal \( s \), and for \( \tau < 1 \), \( \partial x^*_v/\partial s = \delta_v > \partial x^*_u/\partial s = \delta_u \). This in turn implies, following from (19) and (25), that the variance of output under ad valorem taxation is higher than the variance of output under specific taxation. To emphasize:

**PROPOSITION 1:** (Higher output variance under ad valorem): The variance of output under ad valorem taxation, all other things being equal, is higher than under specific.

It is also straightforward to verify, following from (15) and (22), that when expected effective costs are the same (and so Equation (9) holds), expected output is the same under both ad valorem and specific taxation, and equal to

\[
\omega_u = \omega_v = \frac{A}{3} - \frac{\hat{\theta}}{3(1 - \tau)}, \tag{27}
\]

and so will be expected prices. This will be the basis of comparison for the two types of taxation.

Notice, to complete the description, the typical firm’s expected profits are given by

\[
E(\pi^*_v) = (1 - \tau) \left[ \left( \frac{A(1 - \tau)}{3(1 - \tau)} - \frac{\hat{\theta}}{3(1 - \tau)} \right)^2 + \text{var}(x^*_v) \right], \tag{28}
\]

and, thus, in this case, too, expected profits depend positively on output variability.\(^{14}\)

\(^{14}\) When \( m = 0 \), the signal firm \( i \) receives is perfectly informative. Price and output variability, however, are still positive because each firm faces uncertainty about its own cost (which is resolved completely when \( m = 0 \)) and uncertainty about rival costs (which is not resolved completely if the correlation is less than one, that is \( h < \sigma \)). Hence, even with \( m = 0 \) and \( h = \sigma \) price and output do depend on the variance and our main results do not change qualitatively. Output variability under specific taxation, see Equation (19), becomes \( \sigma/9 \). On the other hand, when the signal is uninformative, \( m \to \infty \), output and price variability tend to zero. Firms, in this case, rely exclusively on the average cost \( \hat{\theta} \). Hence, the driving
4. Profit and Tax Revenue Comparison

In the deterministic Cournot model *ad valorem* taxation, compared to specific, is associated with lower profits and, therefore, higher tax revenue; a point emphasized in, among others, Delipalla and Keen (1992) and Anderson *et al.* (2001a,b). Such comparisons, however—and this is an issue at the heart of this paper—fail to hold when production cost is not deterministic. We turn to this next, starting from the comparison of profits.

4.1. Profits

The difference in expected profits under the two types of taxation, for any $t$ and $\tau$ and evaluated at the same effective cost using (9), is given—following from (19), (20), (25), and (28)—by

$$E(\pi_v^*) - E(\pi_u^*) = -\frac{\tau}{9(1-\tau)^2} \left( A (1-\tau) - \tilde{\theta} \right)^2 + \frac{\tau}{(1-\tau)} \frac{\sigma^2(\sigma + m)}{(h + 2(\sigma + m))^2}.$$  

(29)

It is, thus, the case that the balance of the comparison between *ad valorem* and specific taxation depends on the two terms on the right-hand side of (29). The first one reflects the difference in profits when the variance of the costs $\sigma$ is zero. The second term captures the difference in profits when taxation (*ad valorem* and specific) affects the output and price distributions. Clearly, the first term is negative and, thus, points to a dominant (in terms of profits generated) specific taxation. The sign of the second term is positive and points towards a dominant *ad valorem* (in terms of profits generated) tax. Interestingly, the latter term dominates the former if the variance of the costs $\sigma$ is sufficiently high. To emphasize:  

**Proposition 2:** (*Profits comparison*) If, all other things being equal, the variance of the costs $\sigma$ is sufficiently high, (expected) profits are higher under *ad valorem* than specific taxation.

Proposition 2 emphasizes that, contrary to the deterministic models, profits can be higher under *ad valorem* taxation when the variance of the costs $\sigma$ (and so the variance of output) is sufficiently high. The intuition for this comparison is as follows. As we alluded to previously, profits are convex in the marginal cost and so the signals firms receive. This implies that the loss force behind our qualitative results is uncertainty and not the precision of the signal (as long as it is informative).

\[\text{It is not difficult to find examples for which this holds. Suppose, to see such possibility, that costs (between firms) are uncorrelated and so } h = 0. \text{ In this case, Equation (29) is strictly positive if } \frac{\sigma^2}{4(\sigma + m)} > \frac{(A (1-\tau) - \tilde{\theta})^2}{9(1-\tau)}. \text{ Clearly, given that the right-hand side of the latter condition is independent of the variance parameters } \sigma \text{ and } m, \text{ one can find parameter values such that the above inequality holds.}\]
of profits when the signal is high (and, hence, the marginal cost is high) is less than the gain when the signal is low. As a result, higher variance increases expected profits. Since, as showed earlier, only \textit{ad valorem} taxation affects the variance of output, it is the \textit{ad valorem} tax that generates higher variance and makes higher expected profits possible, even when the two taxes yield the same expected marginal cost. We turn now to the comparison of tax revenues.

4.2. Tax Revenues

Expected tax revenues differ under both types of tax. The difference hinges upon the way the signals (a noisy proxy for marginal costs) received by firms affect revenues. Under specific taxation, tax revenues are linear in the signals.\(^16\) Therefore, expected tax revenues do not depend on the variance of output, as it is indeed shown below (evaluated at equal effective costs using (9)):

\[
E\left( r_u^* \right) = \frac{2\bar{\theta} \tau}{3 (1-\tau)^2} \left[ A (1-\tau) - \bar{\theta} \right]. \tag{30}
\]

Under \textit{ad valorem} taxation a lower signal raises output, but decreases consumer price \(p^v\) and, consequently, producer price \(q^v\). The consequence of this is that tax revenues are concave in the signals. The implication of this is that the tax authority, under \textit{ad valorem} taxation, becomes risk-averse, and, therefore, higher variance lowers expected revenues. This latter observation can be seen from the expected tax revenues under \textit{ad valorem} taxation

\[
E\left( r_v^* \right) = \frac{2\tau (A (1-\tau) + 2\bar{\theta} (A (1-\tau) - \bar{\theta}))}{9 (1-\tau)^2} - \frac{2\tau \sigma^2}{(1-\tau)^2} \left( h + \sigma + m \right) \frac{h + 2 (\sigma + m)}{(h + 2 (\sigma + m))^2}, \tag{31}
\]

which is decreasing in the variance (that appears in the second term on the right-hand side). If the variance \(\sigma = 0\) and the effective cost is the same, then, \textit{ad valorem} dominates in terms of tax revenues. Indeed, setting \(\sigma = 0\) and taking the difference between tax revenue under the two types of taxation, making use of (30) and (31), one arrives at

\[
E\left( r_v^* \right) - E\left( r_u^* \right) = \frac{2\tau (A (1-\tau) - \bar{\theta})^2}{9(1-\tau)^2} > 0. \]

This confirms the result in Anderson et al. (2001a).

Interestingly, however, the dominance of \textit{ad valorem} taxation might not hold in the presence of cost uncertainty.\(^17\) Comparing expected total

\(^{16}\) Under specific taxation, revenues are given by (B1) in the Appendix. Since output is linear in the signal, it follows that revenues are also linear in the signal.

\(^{17}\) The higher volatility of tax revenue under \textit{ad valorem} taxation can be highly undesirable by a tax authority, especially when financial markets are illiquid and borrowing—in case the realized tax revenue is low—to finance budget deficits is not feasible.
revenues in this case we have that

\[ \text{E}(r_v^*) - \text{E}(r_u^*) = \frac{2\tau}{(1 - \tau)^2} \left[ \frac{(A - \hat{\theta})^2}{9} - \frac{\sigma^2 (\sigma + m + h)}{(h + 2(\sigma + m))^{\frac{3}{2}}} \right] \]  \hspace{1cm} (32)

Close inspection of (32) reveals that specific taxation is more likely to generate higher expected revenues if the variance \( \sigma \) is high enough. To demonstrate the possibility that specific taxation revenue dominates \textit{ad valorem}, consider the following numerical example.

**Numerical example.** To illustrate that expected revenues under specific taxation can be higher than expected revenues under \textit{ad valorem}, consider the Beta–Binomial example introduced earlier. Assume that the marginal cost \( \theta_i, i = 1, 2 \), is distributed according to a Beta distribution with parameters \( \alpha = 1/2 \) and \( \beta = 2 \), that the intercept of demand is \( A = 3/2 \), and that \( h = (1/2)\sigma \). In this case, it is straightforward to verify that the expected revenue functions under specific and \textit{ad valorem} taxations are both inverse U-shaped with respect to \( t \) and \( \tau \), respectively. Figure 1 depicts expected tax revenue when the variance \( \sigma = 0 \) (left panel) and when \( \sigma > 0 \) according to the Beta–Binomial parameters (right panel). When the variance is \( \sigma = 0 \), left panel, \textit{ad valorem} taxation (dashed curve) revenue dominates specific taxation (solid curve), but with variance \( \sigma > 0 \), right panel, the reverse can be true. It is straightforward, then, to show that the maximum expected revenues under specific taxation are attained at \( t \approx 0.58 \) (or \( \tau \approx 0.64 \), using (9)) whereas under \textit{ad valorem} taxation it is attained at \( \tau \approx 0.48 \). The maximum expected tax revenues under these taxes are \( \text{E}(r_v) \approx 0.23 > \text{E}(r_u) \approx 0.19 \). It can be easily verified that if \( \tau \geq 0.504 \), then unit taxation dominates \textit{ad valorem} in tax revenues collected. To complete the example, one now

![Figure 1: Expected tax revenues, Beta–Binomial.](image-url)
needs to show that for any $\tau \geq 0.504$, output produced is nonnegative for any realization of the signal $s$. This is indeed the case. Output under specific taxation is positive even if the signal attains its highest value, $s = 1$, if $\tau \leq 0.58$. Therefore, for any $\tau \in (0.504, 0.58)$ specific taxation yields higher expected tax revenue than those under \textit{ad valorem}. In this numerical example, any tax target in $(0.19, 0.23)$ can only be attained through specific taxation. To summarize the preceding discussion:

\textbf{PROPOSITION 3: (Tax revenue comparison): If, all other things being equal, the variance of the costs $\sigma$ is sufficiently high, (expected) tax revenue under specific taxation yields higher (expected) tax revenues than \textit{ad valorem} taxation.}

We now turn to consumer and social surplus.

\section*{5. Comparison of Consumer and Social Surplus}

\subsection*{5.1. Consumer Surplus}

For any realizations of the signals $(s_1, s_2)$, expected consumer surplus under specific taxation is given by\textsuperscript{18}

\begin{equation}
E(CS_u^*) = \frac{2}{9} \left( A - \bar{\theta} - t \right)^2 + \frac{\sigma^2 (h + \sigma + m)}{(h + 2 (\sigma + m))^2},
\end{equation}

and

\begin{equation}
E(CS_v^*) = \frac{2}{9 (1 - \tau)^2} \left( A (1 - \tau) - \bar{\theta} \right)^2 + \frac{1}{(1 - \tau)^2} \frac{\sigma^2 (h + \sigma + m)}{(h + 2 (\sigma + m))^2},
\end{equation}

under \textit{ad valorem} taxation. Clearly, if the variance $\sigma = 0$ and expected effective cost is the same under both taxes, then $E(CS_u^*) = E(CS_v^*)$. This, however, does not hold in the presence even of a \textit{small} degree of uncertainty. In this case, \textit{ad valorem} taxation yields higher consumer surplus than specific. To emphasize:

\textbf{PROPOSITION 4: (Consumer surplus comparison): Expected consumer surplus under \textit{ad valorem} taxation, all other things being equal, is strictly higher than that under specific taxation, for any positive level of variance $\sigma$.}

The intuition behind Proposition 4 is that consumers like price variability and \textit{ad valorem} taxation does precisely this;\textsuperscript{19} it generates higher price

\textsuperscript{18} The derivations are tedious and are, for brevity, omitted.

\textsuperscript{19} For an algebraic derivation see footnote 8. There is a simple intuition behind this. Suppose the price of output changes. Utility changes linearly in price, holding output constant. However, the consumer can change his/her consumption when price changes and
(output) variance than specific taxation. Furthermore, *ad valorem* taxation can even increase expected consumer surplus. To see this, differentiate Equation (34) with respect to $\tau$ and evaluate the derivative at $\tau = 0$ to obtain

$$
dE(CS_u^*) \frac{d}{d\tau} = -\frac{4\bar{\theta} (1 - \bar{\theta})}{9} + \frac{2 (h + \sigma + m) \sigma^2}{(h + 2 (\sigma + m))^2}. \tag{35}
$$

The first term in the right-hand side of (35) is negative, pointing to the adverse effect of taxation on consumer welfare (a consequence, as emphasized earlier, of a high expected price). The second term, however, is positive pointing to the beneficial effect of taxation on consumer welfare (a consequence of price variability). It is, thus, the case that for high enough variance, the second term dominates the first, suggesting that higher *ad valorem* taxation can increase expected consumer surplus, at least for low $\tau$. Specific taxation, on the other hand, always affects expected consumer surplus negatively. The reason for this is intuitive: Price variability is independent of the specific tax and, hence, the second term in (35) vanishes. This will also hold next when we examine the effects of the two types of taxes on social surplus.

The above analysis begins to suggest that *ad valorem* may dominate specific when it comes to social surplus. We turn to this next.

### 5.2. Social Surplus

The two taxes are not equivalent in terms of expected social surplus. More importantly, *ad valorem* taxation can even *increase* expected social surplus when the variance of the costs is strong enough. Expected social surplus is the sum of expected consumer surplus, expected profits and expected revenues. Expected social surplus under specific taxation is given—making use of (20), (30), and (33)—by

$$
SS_u^* = \frac{2 \left(2 (A - \bar{\theta}) + t\right) (A - \bar{\theta} - t)}{9} + \frac{\sigma^2 (h + 3 (\sigma + m))}{(h + 2 (\sigma + m))^2}, \tag{36}
$$

whereas—making use of (28), (31), and (34)—expected social surplus under *ad valorem* taxation is given by

$$
SS_v^* = \frac{2 \left( (1 - \tau) - \bar{\theta} \right) \left( 2A (1 - \tau) - (2 - 3\tau) \bar{\theta} \right)}{9 (1 - \tau)^2} + \frac{(h + 3 (\sigma + m) - 2\tau (h + 2 (\sigma + m))) \sigma^2}{(1 - \tau)^2 (h + 2 (\sigma + m))^2}. \tag{37}
$$

moreover he/she can only become (weakly) better off in doing so, suggesting that maximized utility must be convex in price.
The expected social surplus under specific taxation when evaluated at same expected effective costs, following from Equation (9), becomes

\[ SS_u^* = \frac{2 (A (1 - \tau) - \bar{\theta}) (2A (1 - \tau) - (2 - 3\tau) \bar{\theta})}{9 (1 - \tau)^2} + \frac{\sigma^2 (h + 3 (\sigma + m))}{(h + 2 (\sigma + m))^2} . \] (38)

It is clear that if the variance \( \sigma = 0 \), then, the second term both in (37) and (38) vanish, leaving \( SS_u^* = SS_v^* \). This reconfirms the point made by Anderson et al. (2001a). The intuition here is the same as the one offered in Anderson et al. (2001a): With effective costs being the same output is the same under both ad valorem and specific taxation and so is the price.

The difference in the expected social surplus between the two taxes is given by

\[ SS_v^* - SS_u^* = \frac{(2 (m + \sigma) - \tau (h + 3 (\sigma + m))) \sigma^2 \tau}{(1 - \tau)^2 (h + 2 (\sigma + m))^2} . \] (39)

It can be shown that this difference is concave in \( \tau \) achieving a maximum at \( \tau = (\sigma + m)/(h + 3m + 3\sigma) < 1 \). It is also the case that for \( \tau \in (0, 2 (\sigma + m)/(h + 3m + 3\sigma)) \) the difference is positive, whereas for \( \tau > 2 (\sigma + m)/(h + 3m + 3\sigma) \) the difference is negative.

An increase in specific taxation \( t \)—where \( t \) changes according to \( \bar{\theta} + t = \bar{\theta}/(1 - \tau) \), and for \( \tau \leq 1 \)—lowers expected social surplus in the sense that

\[ \frac{\partial SS_u^*}{\partial \tau} = -\frac{2 (A (1 - \tau) - \bar{\theta} (1 - 3\tau)) \bar{\theta}}{9 (1 - \tau)^3} \leq 0 . \] (40)

Interestingly, this might not be the case under ad valorem taxation. To see this, differentiate (37) with respect to \( \tau \), and evaluate the derivative at \( \tau = 0 \), to obtain

\[ \frac{\partial SS_v^*}{\partial \tau} = -\frac{2 (A - \bar{\theta}) \bar{\theta}}{9} + \frac{2\sigma^2 (\sigma + m)}{(h + 2 (\sigma + m))^2} . \] (41)

Inspection of (41) reveals that for high enough variance \( \sigma \) a tax increase leads to higher (expected) social surplus. The reason for this, unlike specific taxation, is that, as noted earlier, ad valorem taxation affects the variance and as a consequence an increase in the tax has two effects on expected social welfare. On the one hand, expected marginal cost increases which lowers welfare, but, on the other hand, the increase in variance increases welfare. It is plausible—and indeed this is the case—that for high levels of variance \( \sigma \) the second effect may dominate. To summarize the preceding discussion:

**PROPOSITION 5:** (Social surplus): Expected social surplus is decreasing in the specific tax \( t \). However, expected social surplus under ad valorem taxation is increasing in the ad valorem tax \( \tau \), for low levels of \( \tau \), provided that the variance of the costs \( \sigma \) is high enough. In particular, it is inverse U-shaped in \( \tau \). When the variance of the costs is low, then expected social surplus under ad valorem is decreasing in \( \tau \).
The intuition behind Proposition 5 follows from the responses of the individual components of the expected social surplus with respect to the variance of the cost $\sigma$. Under specific taxation expected profits and expected consumer surplus depend positively on this variance, while expected tax revenue is independent of the variance. Under \textit{ad valorem} taxation, on the other hand, expected tax revenue depends negatively on the variance. For low levels of $\tau$, \textit{ad valorem} yields higher expected social surplus because it generates higher output variance, but for high levels of $\tau$ the negative effect of variance on expected tax revenue dominates and specific taxation is more efficient.\footnote{The analysis can be extended to $n \geq 1$ firms in the market. The number of firms does not change the structure of the expected profits (as given by (20) and (28)) and, hence, the main results, qualitatively. Therefore, we can conclude from this that uncertainty is the driving force behind our main insights. Market structure affects the results only quantitatively.}

6. Concluding Remarks

The analysis has focused on the explicit comparison of all key welfare measures under \textit{ad valorem} and specific taxation. It has been shown that, contrary to the deterministic model, specific taxation might dominate \textit{ad valorem} in terms of tax revenues, while \textit{ad valorem} taxation might dominate in terms of profits and consumer surplus. While the two taxes yield the same social welfare in a deterministic model (a special case of the model considered here), this is not the case under uncertainty. Interestingly, social surplus can even increase in the \textit{ad valorem} tax.

A question that, naturally, arises is which tax dominates if the government solves the Ramsey problem, in the sense that it wishes to maximize consumer welfare (consumer surplus plus profits) subject to some given tax revenue requirement. Unfortunately, while it is easy to demonstrate—as the analytics here have shown—the possibility that, compared to the deterministic model, all key welfare measures change sign in the presence of uncertainty, obtaining an easily interpretable solution for the Ramsey problem that shows the possibility that specific taxation can dominate \textit{ad valorem} taxation has not been possible.\footnote{We have obtained a number of numerical solutions to the Ramsey problem. In all the specifications we have tried, \textit{ad valorem} dominated specific taxation, provided that the revenue target can be attained by both types of taxes.} Although we cannot offer a comprehensive solution to the Ramsey problem, we can say the following about how our results can be used to create a new view to this problem when uncertainty is an important factor. Our most important result from a theoretical and policy perspective is that, under uncertainty, specific taxation can yield higher expected revenue than \textit{ad valorem} taxation. This implies that there are tax revenue targets that can only be attained using specific tax instruments. If the expected tax revenue in the Ramsey problem is high enough, then the only solution is to use specific taxes.
The Comparison of \textit{ad Valorem} and Specific Taxation

We hope to have shown that the results obtained are instructive and could serve as stepping stones to future explorations of the comparison between \textit{ad valorem} and specific taxation under uncertainty in more general settings.

Appendix

\textit{Proof of Lemma 1:} Given the linearity of the posterior expectations, following Ericson (1969), it is the case that

\[ E(\theta_i|s_i) = \left( \frac{R}{R+r} \right) E(\theta_i) + \frac{r}{R+r}s_i, \quad (A1) \]

where \( r = 1/E(\text{var}(s_i|\theta_i)) \) and \( R = 1/\text{var}(\theta_i) \).

Since \( E(\text{var}(s_i|\theta_i)) = m, \text{var}(\theta_i) = \sigma \) and \( E(\theta_i) = \tilde{\theta} \), it follows from (A1) that

\[ E(\theta_i|s_i) = \frac{m}{\sigma + m} \tilde{\theta} + \frac{\sigma}{\sigma + m}s_i, \quad i = 1, 2. \quad (A2) \]

Following from the linear posterior expected values given in Equations (2) and (3), it is the case that

\[ E(s_1s_2) = EE(s_1s_2|s_1) = E(s_1E(s_2|s_1)) = E(s_1 \left[ d + Ds_1 \right]), \quad (A3) \]

\[ = dE(s_1) + DE(s_1^2), \quad (A4) \]

\[ E(s_2) = EE(s_2|s_1) = d + DE(s_1). \quad (A5) \]

Since

\[ E(s_1s_2) = h + \tilde{\theta}^2, \quad (A6) \]

\[ E(s_1^2) = \sigma + m + \tilde{\theta}^2, \quad (A7) \]

\[ E(s_1) = \tilde{\theta}, \quad (A8) \]

solving now for the parameters (simultaneously) one obtains (for \( i \neq j \))

\[ E(s_j|s_i) = \frac{\sigma + m - h}{\sigma + m} \tilde{\theta} + \frac{h}{\sigma + m}s_i, \quad (A9) \]

as required. \( \blacksquare \)
Below, we offer the steps taken for the derivation of the expected revenues under both taxes. Since the derivations of the other welfare measures follow analogous steps, they are omitted for brevity.

**Derivation of Equation (30)**

Under specific taxation, revenues are given by

\[ r^u = t \sum x^*_u, \]  

(B1)

which upon using the definition of equilibrium output (10) it becomes

\[ r^*_u = t \left( 2\omega_u + \delta_u \sum s_i \right). \]  

(B2)

Substituting the equilibrium definitions of \( \omega_u \), following from (15), and the fact that

\[ t = \frac{\bar{\theta} \tau}{1 - \tau}, \]  

(B3)

it is the case that

\[ r^*_u = \frac{2\bar{\theta} \tau}{1 - \tau} \left[ A (1 - \tau) - \bar{\theta} \tau \frac{2m + h - \sigma}{3 h + 2 (\sigma + m)} + \frac{\delta_y}{2} \sum s_i \right]. \]  

(B4)

Taking expectations—making use of the fact that \( \text{E}(s_i|\theta_i) = \bar{\theta} \) but also the equilibrium definition of \( \delta_u \) from (16)—and after some straightforward manipulations, one obtains

\[ \text{E}(r^*_u) = \frac{2\bar{\theta} \tau}{3 (1 - \tau)^2} \left[ A (1 - \tau) - \bar{\theta} \right], \]  

(B5)

as required.

**Derivation of Equation (31)**

Under *ad valorem* taxation, tax revenues are given by

\[ r^v = q^v \lambda \sum x^*_v, \]  

(C1)

\[ = \frac{\lambda}{1 + \lambda} \left( A - \sum x^*_v \right) \sum x^*_v, \]  

(C2)

where the second equality follows from (6) and the definition of \( p^v \) in (1). Substituting the definition of equilibrium output given in (21), and using the fact that \( 1 - \tau = 1/ (1 + \lambda) \), (C2) becomes

\[ r^v = \tau \left( a - \delta_v \sum s_i \right) \left( b + \delta_v \sum s_i \right), \]  

(C3)
where \( a \equiv A - 2\omega_v \) and \( b \equiv 2\omega_v \). Taking expectations in (C3) it is the case that

\[
E (r^*_v) = \tau \left\{ ab + (a - b) \delta_v E \left( \sum s_i \right) - \delta_v^2 E \left[ \left( \sum s_i \right)^2 \right] \right\},
\]

(C4)

\[
= \tau \left\{ ab + 2 (a - b) \delta_v \tilde{\theta} - \delta_v^2 E \left[ \left( \sum s_i \right)^2 \right] \right\},
\]

(C5)

where the second equality follows from the fact that \( E \left( \sum s_i \right) = 2\tilde{\theta} \). Following from the proof of Lemma 1 and, in particular (A6) and (A7), it is also the case that

\[
E \left[ \left( \sum s_i \right)^2 \right] = v + h + 4\tilde{\theta}^2,
\]

(C6)

where

\[
v \equiv h + 2 (\sigma + m).\]

(C7)

Following now from (C7), (22), and (23) can be written, respectively, as

\[
\omega_v = \frac{1}{3} \left( A - \frac{(v - 3\sigma) \tilde{\theta}}{(1 - \tau) v} \right); \quad \delta_v = -\frac{1}{(1 - \tau)} \frac{\sigma}{v}.
\]

(C8)

It is straightforward now to verify that

\[
ab + 2 (a - b) \delta_v \tilde{\theta} = \frac{2 \left( A (1 - \tau) + 2\tilde{\theta} \right) \left( A (1 - \tau) - \tilde{\theta} \right)}{9 (1 - \tau)^2} + \frac{4\tilde{\theta}^2 \sigma^2}{(1 - \tau)^2 v^2},
\]

(C9)

and

\[
\delta_v^2 E \left[ \left( \sum s_i \right)^2 \right] = \frac{\sigma^2 \left( v + h + 4\tilde{\theta}^2 \right)}{(1 - \tau)^2 v^2}.
\]

(C10)

Substituting now (C9) and (C10) into (C5)—and upon using the definition in (C7)—one arrives at

\[
E (r^*_v) = \frac{2\tau \left( A (1 - \tau) + 2\tilde{\theta} \right) \left( A (1 - \tau) - \tilde{\theta} \right)}{9 (1 - \tau)^2} - \frac{2\tau \sigma^2}{(1 - \tau)^2} \frac{h + \sigma + m}{(h + 2 (\sigma + m))^2},
\]

(C11)

as required.

References


