Solution to the evolution equation for high parton density QCD

E. Levin, Tel Aviv University
Kirill Tuchin, Tel Aviv University

Available at: https://works.bepress.com/kirill_tuchin/15/
Solution to the evolution equation for high parton density QCD

E. Levin a,b, K. Tuchin a

a HEP Department, School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Science, Tel Aviv University, Tel Aviv, 69978, Israel
b DESY Theory Group, 22603, Hamburg, Germany

Received 23 August 1999; received in revised form 2 December 1999; accepted 15 December 1999

Abstract

In this paper a solution is given to the non-linear equation which describes the evolution of the parton cascade in the case of the high parton density. The related physics is discussed as well as some applications to heavy ion–ion collisions. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

During the past two decades one of the challenging problem of QCD has been to understand theoretically and to observe experimentally a new non-perturbative regime: high parton density QCD. It has been argued [1] in perturbative QCD (pQCD) that at low \( x \) (high energies) the density of parton increases in deep inelastic scattering (DIS) and reaches so high a value that partons become densely populated in a hadron. Certainly, high density system of partons cannot be treated perturbatively but pQCD leads to a new scale mean transverse momentum of partons for high parton density QCD (hdQCD) as well as to a hypothesis of parton saturation [1]. Intensive theoretical studies led to deeper understanding of physics of this system as well as to development of both perturbative [2–7] and non-perturbative [8–21] methods for such a system. However, only recently the first indications on parton saturation have appeared in HERA data on \( Q^2 \)-behaviour of the \( F_2 \)-slope and on energy behaviour of inclusive diffractive dissociation in DIS [22,23]. These data as well as their theoretical interpretation [24–31] mark a new stage of our approach which needs more quantitative methods, than it was before, to make a reliable prediction for the experimental observations.
This quantitative approach includes at least two steps:

1. a derivation of the equation which will be valid in the full kinematic region;
2. finding a solution to this equation.

There are two versions of the non-linear evolution equation at \( x \to 0 \). The first one was suggested as an obvious generalization of pQCD approach \([32,33]\) while the second has been obtained from semiclassical gluon field approach \([34,35]\). Both of them describe correctly the DGLAP \([36–39]\) evolution in the limit of low parton density as well as the GLR non-linear evolution equation \([1]\) in the region of intermediate parton densities. They give the same limit of parton density saturation at very low values of \( x \) but they are different in particular description how system reaches this saturation. Kovchegov has recently proved \([40]\) that the GLR equation itself (in the form of Eq. (2.18) in Ref. \([1]\)) is able to describe the QCD evolution in the region of high parton density or, in other words, at very low values of \( x \). His arguments, which we review briefly in the next section, are based on Mueller’s idea \([41]\) that colour dipoles rather than quarks and gluons, are correct degrees of freedom at high energies (low \( x \)) in QCD. It was also shown that the solution to the equation, suggested in Refs. \([32,33]\), coincides with the solution to the GLR equation.

The goal of this paper is to solve the GLR equation, written in the form suggested in Ref. \([40]\), in the full kinematic region including \( x \to 0 \). The paper is organized as follows: In the next section we briefly discuss the non-linear evolution equation for the high parton density QCD and we formulate our approach for searching a solution to this equation. In Section 3 we introduce the new scale \( Q_x(x) \), which appears in hdQCD as a solution to the non-linear equation, and we find the general solution to the equation for parton with \( Q^2 > Q_x^2(x) \), based on the approach developed in Ref. \([42]\). Section 4 is devoted to a solution of the hdQCD equations in the region of very low \( x \). We show that the parton density reaches the saturation limit and we find the explicit analytic expression describing how the system approaches the saturation limit. Our summary and conclusions are given in Section 5.

2. Non-linear equation for high density parton system

2.1. Kinematics, notations and definitions

In this subsection we introduce all notations and definitions as well as some kinematic relations that we will need in what follows. In general, we try to use the same notation as in Ref. \([40]\). We would like to clarify the relations between different notations and definitions that have been used in this area of activity. This task is rather complicated since different notations have been used for the same physics observables, mostly due to the different theoretical background of the authors. In this paper we explore the main physical idea of Ref. \([41]\), namely, that the correct degrees of freedom in QCD at low \( x \) (high energy) are colour dipoles rather than quarks and gluons that explicitly written in the QCD Lagrangian. This statement means that a QCD interactions
at high energy do not change the size and the energy of a colour dipole. Therefore, the majority of our variables and observables is related to distributions and interactions of the colour dipoles in a hadron.

1. \( x_{ik} = |\vec{x}_i - \vec{x}_k| \) is the size of the dipole which consists of a quark "i" at \( \vec{x}_i \) and an antiquark "k" at \( \vec{x}_k \), or, in other words, it is the transverse separation between quark and antiquark in a colour dipole;

2. \( r = \ln x_0/x^2 = \ln Q^2/Q_0^2 \) where \( Q^2 (Q_0^2) \) is the transverse momentum of quark in the dipole with size \( x(x_0) \) respectively;

3. \( x \) is the Bjorken variable for a dipole, \( x = Q^2/W^2 \), where \( W \) is the dipole energy;

4. \( y = \ln(x_0/x) \), where \( x_0 \) is defined for us the region of low \( x \). We consider that for \( x < x_0 \) all the typical features of low \( x \) physics should be seen. Practically, \( x_0 \approx 10^{-2} \);

5. \( xG(x,Q^2) \) is the gluon structure function for a nucleon;

6. \( xG_A(x,Q^2) \) is the gluon structure function for a nucleus with \( A \) nucleons;

7. \( b \) is the impact parameter for the dipole scattering;

8. The main physical observable, which we are going to discuss in this paper, is the density of dipoles \( N(x,b,y) \) with the size \( x \) and energy \( x(y) \) at the impact parameter \( b \);

9. In the parton approach (parton \( \Leftrightarrow \) dipole) the dipole density \( N(x,b,y) \) is related to the dipole scattering amplitude if we assume that this amplitude is dominantly imaginary. Therefore, \( N(x,b,y) \) can be normalized using the unitarity constraints:

\[
2N(x,b,y) = N^2(x,b,y) + G_{\text{in}}(x,b,y),
\]

where \( G_{\text{in}} \) is the contribution of the inelastic processes to the scattering of a dipole;

10. From Eq. (2.1) one can see that \( N(x,b,y) \leq 1 \). Therefore, saturation of the parton density means that

\[
N(x,b,y) \rightarrow 1 \quad \text{at} \quad x \rightarrow 0;
\]

11. The parameter \( \kappa \), which was introduces in many papers [2–7], (or the parameter \( W \) of Ref. [1]) is equal to

\[
\kappa(x,y) = W(x,y) = N(x,b=0,y);
\]

12. The relation between parton density \( N(x,b,y) \) and gluon structure function \( xG_A(x,1/x^2) \) is given by [32,33,40]

\[
N(x,b=0,y) = \kappa(x,y) = \frac{\alpha_s \pi^2 x^2}{2 N_c \pi R_A^2} \times xG_A(x,1/x^2),
\]

which holds only for small \( N \). We would like to stress that we do not need to know the gluon structure function for \( N = 1 \) since \( N \) itself has a clear meaning as the scattering amplitude for a dipole. It should be also stressed that Eq. (2.4) shows that \( N(x,b=0,y) \) is not a parton (colour dipole) density \( \rho = xG_A(x,1/x^2) / \pi R_A^2 \) but it is the scattering amplitude which is equal to \( \sigma_{\text{dipole}} \times \rho \);
The widely used function $\phi(k^2,y)$ (see Eq. (2.18) of Ref. [1] for example) is equal to momentum image of $N$, namely,

$$
\int d^2b_iN(x,b_i,y) = \int d^2k\{1 - e^{ik \cdot b}\} \phi(k^2,y);
$$

(2.5)

(13)

$\bar{\alpha}_s$ is used for $\bar{\alpha}_s = \alpha_s N_s/\pi$;

(14)

$R_A$ is the radius of a nucleus in the Gaussian parameterization of the nucleon density. In this parameterization the profile function in $b_i$-representation looks as

$$
S_A(b_i) = \frac{1}{\pi R_A^2} e^{-b_i^2/R_A^2}.
$$

(2.6)

(15)

$R_A$ (Gaussian) $= \frac{1}{2} R_A^2$ (Woods–Saxon), where $R_A^2$ (Woods–Saxon) $= r_0^2 A^{2/3}$ with $r_0 = 1.3$ fm (see Refs. [32,33] for more details);

(16)

All physical observables, related to the nucleus, are denoted with a subscript $A$ while the nucleon observables will be marked by a subscript $N$. Both these subscripts could be omitted if the meaning of the physical quantity is obvious;

(17)

As has been shown [1–7], the interaction between partons leads to a new scale $Q^2_{\text{cr}}(x,b_i) = 1/x^2_{\text{cr}}$ for hdQCD. The line $r = \ln(1/x^2) = \ln Q^2_{\text{cr}}(x,b_i)$ is called critical line (see Fig. 1) and all physical quantities defined or calculated on the critical line are denoted with the subscript cr. For example, we denote a new scale, which has the meaning of the average parton transverse momentum in hdQCD parton cascade, by $Q^2_{\text{cr}}(x,b_i)$ while in other papers it has a different notation: $Q^2_{\text{cr}}(x,b_i) = Q^2_{\text{cr}}(x) [1–7] = Q^2_{\text{cr}}(x) [45,46] = \mu^2(x) [8–21,34,35];$

(18)

All physical quantities defined to the right of the critical line (for $r > r_{\text{cr}} = \ln Q^2_{\text{cr}}(x,b_i)$) will carry the subscript $\rightarrow$;

(19)

All physical quantities defined to the left of the critical line (for $r < r_{\text{cr}} = \ln Q^2_{\text{cr}}(x,b_i)$) will carry the subscript $\leftarrow$.

Fig. 1. Different regions for the solution to the hdQCD evolution equation in the kinematic plot of DIS. The equation $N(x,b_i,y) = 1$ gives the critical line.
2.2. The non-linear equation for hadQCD evolution

Kovchegov in his proof [40] that the GLR equation is able to describe the whole kinematic region including very low values of \( x \), uses heavily two principle ideas suggested by Mueller [41]:

- The QCD interaction at high energy does not change the transverse separation between quark and antiquark (the colour dipole size), and, therefore, colour dipoles can be considered as the correct degrees of freedom at high energies which diagonalize the strong interaction matrix;
- The process of interaction of a dipole with the target has two clear stages:
  (1) Decaying of the dipole into two dipoles, which is described by
  \[
  |\Psi(x_{01} \to x_{02} + x_{12})|^2 = \frac{x_{01}^2}{x_{02} x_{12}^2}.
  \tag{2.7}
  \]
  (2) Interaction of each dipole with the target with amplitude \( N(x, b, y) \).

The equation is pictured in Fig. 2 and it has the following analytic form:
\[
\frac{dN(x_{01}, b, y)}{dy} = -\frac{2}{\pi} \frac{C_F \alpha_s}{\pi} \ln \left( \frac{x_{01}^2}{\rho^2} \right) N(x, b, y) + \frac{C_F \alpha_s}{\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{02} x_{12}^2} (2N(x_{02}, b, \frac{1}{2} x_{12}, y)
-N(x_{12}, b, \frac{1}{2} x_{02}, y)) \tag{2.8}
\]
Assuming that \( x_{12} \) and \( x_{20} \) are much smaller than the values of the typical impact parameter \( b \), one can see that Eq. (2.8) is the GLR equation but in the space

![Fig. 2. Pictorial representation of the non-linear evolution equation.](image)
representation. It turns out that this representation is very convenient for searching solutions to Eq. (2.8).

The first term in the r.h.s. of the equation gives the contribution of virtual corrections, which appear in the equation as a result of the normalization of the partonic wave function of the fast colour dipole (see Ref. [41]). The second term describes the decay of the colour dipole with size $x_{01}$ into two dipoles with sizes $x_{02}$ and $x_{12}$ and their interactions with the target in the impulse approximation (notice the factor 2 in Eq. (2.8)). The third term corresponds to the simultaneous interaction of the two produced colour dipoles with the target and describes the Glauber-type corrections for scattering of these dipoles.

Being a differential equation, Eq. (2.8) needs an initial condition at $y = 0$ to be solved. In Ref. [40] it was shown that the initial condition for $N(x_{01},b_{r},y)$ is the Mueller–Glauber formula [47], namely,

$$N(x_{01},b_{r},y = 0) = 2 \left( 1 - \exp \left( -\frac{\alpha_s \pi^2}{4 N_c \pi R_A} x_{01}^2 A x_0 G^{DGLAP}_N \left( x_0,1/x_{01}^2 \right) e^{-b_R^2/R_A^2} \right) \right),$$  

(2.9) 

where $x_0 G^{DGLAP}_N(x_0,1/x_{01}^2)$ is the gluon structure function obtained as a solution to the DGLAP linear evolution equations [36–39].

2.3. The strategy of searching for solutions

Eq. (2.8) is a non-linear integro-differential equation which is rather difficult to solve. However, this equation can be simplified and can be reduced to a non-linear but differential equation in partial derivatives in two different kinematic regions (see Fig. 1): to the right of the critical line ($r > r_c$) and to the left of the critical line ($r < r_c$).

2.3.1. $r > r_c$

Indeed, for $r > r_c$, the size of the two produced colour dipoles ($x_{02}$ and $x_{12}$) is much larger than the size of the initial colour dipole ($x_{01}$). Therefore, we can reduce the kernel of Eq. (2.8) to [40]

$$\int_{\rho} d^2x \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \rightarrow x_{01}^2 \pi^{1/2} \int_{\rho} d^2x_2 \frac{x_{02}^2}{x_{01}^2 (x_{02}^2)^2}. \quad (2.10)$$

Introducing a new function $n(x_{01},b_{r},y) = N(x_{01},b_{r},y)/x_{01}^2$ and using the fact that the virtual corrections do not contribute in the double log approximation (see Ref. [40] for example), one obtains the equation

$$\frac{d^2 n}{dy dr} = \frac{\alpha_s C_F}{\pi} \left( 2 n - n^2 \right),$$  

(2.11) 

where $r = \ln(1/x_{01}^2)$.

In Section 3 we will discuss the solution to this equation which satisfies the initial condition of Eq. (2.9).
2.3.2. \( r < r_c \)

As has been discussed many times (see for example Refs. [44–46]), a new scale appears in \( \text{hdQCD} \) which has a simple physical meaning of the average transverse momentum of the parton in the parton cascade \( (Q_{a}(x,b)) \). In this region the size of the initial colour dipole is larger than the typical size that we expect in \( \text{hdQCD} \) parton cascade \( (x^2 > 1/Q_{a}(x,b)) \). Therefore, the main contribution in Eq. (2.8) comes from the configuration where one of the produced colour dipoles has a size much smaller than the size of the initial colour dipole\(^1\). We anticipate that the size of the smallest colour dipole will be of the order of \( 1/Q_{a}(x,b) \). Such a configuration simplifies the kernel in Eq. (2.8), which has the form

\[
\int_{1/Q_{a}(x,b)} d^2x \frac{x^2_0}{x^2_2} \rightarrow \pi \int_{1/Q_{a}(x,b)} d^2x_0 \frac{x^2_2}{x^2_{12}} + \pi \int_{1/Q_{a}(x,b)} d^2x_0 \frac{x^2_1}{x^2_2}.
\]

The sum of two terms reflects the fact that the two different produced colour dipoles can be small. In momentum representation, Eq. (2.12) means that we sum \( \log(Q^2(x,b)/Q^2) \) which are the normal contributions for the DGLAP evolution coming from integration over transverse momentum from small transverse momentum \( (Q^2) \) to large transverse momentum \( (Q^2(x,b)) \).

Introducing a new function \( \tilde{N}(r,b,y) = \int dr' N(r',b,y) \), we reduce Eq. (2.8) to the differential equation

\[
-\frac{d^2\tilde{N}_-(r,b,y)}{dy dr} = \frac{2\alpha_s C_F}{\pi} \left\{ \tilde{N}_-(r,b,y) + \tilde{N}_-(r,b,y) \frac{d\tilde{N}_-(r,b,y)}{dr} \right\}.
\]

One can see that in the differential equation (Eq. (2.13)) we lost any dependence on the scale \( Q_a(x,b) \) and it will come back to the problem only in matching a solution to Eq. (2.13) with a solution to Eq. (2.11).

In Section 4 we will find a solution to this equation which matches the solution of Eq. (2.11) on the critical line.

2.3.3. The general solution

Let us summarize what solution we are looking for.

\begin{enumerate}
  \item We are going to solve the differential equation (see Eq. (2.11)) to the right of the critical line satisfying the initial condition given by the Mueller–Glauber formula (see Eq. (2.9)).
  \item We are going to find the solution to the differential equation (see Eq. (2.13)) to the left of the critical line.
  \item We match the solution of Eq. (2.11) with the solution of Eq. (2.13) on the critical line.
  \item We can hope that the solution of these two differential equations will be close to the solution to Eq. (2.8) in the vicinity of the critical line \( (r \approx r_c) \) since we provide the matching of these two solutions on the critical line.
\end{enumerate}

\(^1\) This idea has been suggested in Ref. [40] in one of the first versions of this paper but disappeared in the final version.
2.4. Experience of solving the GLR-type non-linear equations

During the past two decades we have learned the main properties of the solution to the GLR-like non-linear differential equations. Our experience in solving these equations is based on three different approaches that have been developed.

First, these equations were solved in the semiclassical approximation, assuming \( N = e^S \), where \( S \) is a smooth function of \( r \) and \( y \), namely \( d^2S/dydr \ll (dS/dy)^2 \) or \( (dS/dr)^2 \). In this approximation the GLR-type equation can be solved using the characteristics method which leads to the existence of a special (critical) line. No characteristics in the region of low \( x \) can cross this line but they can approach it. The non-linear term provides that the parton density \( N(r,b_0=0,y) \) is constant on the critical line. To the right of the critical line \( (r > r_c) \) the non-linear corrections turn out to be small and the solution can be found as a solution of the linear DGLAP evolution equation but with the boundary conditions \( N = \text{Const.} \) on the critical line. For our attempts to find a solution the important message from the semiclassical approach is the fact that we have a new scale \( Q_x \) and the properties of the solution looks different for \( r > r_c \) and for \( r < r_c \).

In Ref. [42] a new method was suggested for solving Eq. (2.11). A generating function was constructed, which linearizes Eq. (2.11), reducing Eq. (2.11) to a linear equation in partial derivatives but with one more variable. It is easy to find a general solution of this linear equation and all difficulties are concentrated in finding the solution which satisfies the boundary condition. We are going to use this method to find a solution to the right of the critical line.

In the region of very low \( x \), a new idea for searching a solution of the GLR-type non-linear equation was suggested in Ref. [43]. In this paper it was argued that the solution of Eq. (2.11), which satisfies all physics restrictions, is a function of one variable: \( z = 4\pi y - r \). We will find this solution in an explicit way in Section 4 and will show how to match this solution with the solution to the right of the critical line.

2.5. Theory status of the equations

In this subsection we recall the main assumptions that have been made to obtain the non-linear evolution equations for hdQCD (see Eqs. (2.8), (2.11) and (2.13)).

(1) For the derivation of all equations we used the leading log(1/x) approximation of pQCD. Only in this approximation we can neglect the energy recoiled during emission of the colour dipole (the linear term in the equations) or during rescattering of this dipole (the non-linear term). Formally speaking, we have to assume that \( \alpha_s \log(1/x) \gg 1 \) while \( \alpha_s \ll 1 \).

(2) To obtain the differential equations (see Eqs. (2.11) and (2.13)) we have to assume the existence of a more restricting parameter \( \alpha_s r y \gg 1 \), but \( \alpha_s r \ll 1 \) and \( \alpha_s y \ll 1 \), which corresponds to the double log approximation of pQCD in the linear evolution equation.

(3) The number of colours should be large \( (N_c \gg 1) \). It has been shown [50–52] that for \( N_c = 1 \) the correlations between colour dipoles should be taken into account which breaks the simple physical picture of Fig. 2. In the double log approximation these correlations can be taken into account [42] but they lead to rather complicated equations. On the other hand, in Ref. [42] it was demonstrated that all these complications lead to corrections of the order of \( 1/N_c^2 \) and can be neglected even for \( N_c = 3 \).
Fig. 3. (a) “Fan” and (b) enhanced diagrams.

4 The master equation (see Eq. (2.8)) sums the so-called “fan” diagrams (see Fig. 3a), which reflect the fact that the fast colour dipole decays into two colour dipoles. However, the recombination of two colour dipoles into one has been omitted (see Fig. 3b). For \( r > r_c \) it has been proven that such diagrams give only small corrections [1] but in the kinematic region to the left of the critical line such enhanced diagrams can be neglected only in the case of scattering off the heavy nucleus. It turns out (see Refs. [32,33,40,53]) that in the case of a heavy nucleus the contributions of “fan” diagrams are proportional to \( \gamma A^{1/3} \), while the enhanced diagrams of Fig. 3b are of the order of \( \gamma \) without a large factor \( A^{1/3} \). Therefore, our approach is valid only for heavy nuclei and can be considered only as a model for the interaction with a hadron.

5 We assume that (i) there are no correlations between different nucleons in a nucleus and (ii) the average \( b_r \) for colour dipole–nucleon interaction is much smaller than \( R_N \). Both these assumptions are customary in the treatment of nucleus scattering.

3. Solution to the right of the critical line

In this kinematic region we use the generating function method, proposed in Ref. [42], which linearizes Eq. (2.11). Following Ref. [42], we introduce a generating function

\[
g_{\lambda} (r, y, b, \eta) = \sum_{n=1}^{\infty} n_{N, \lambda} (r, y, b) e^{\eta n},
\]

where \( n_{N, \lambda} (r, y, b) = N(r, y, b) e^{\eta (\alpha_s \pi^2 / 2 N_c) p(r, y, b)} \) is proportional to the density of colour dipoles in the target. \( n_{N, \lambda} \) is proportional to the probability to have \( n \) colour dipoles with the same size \( e^{-\eta} \) in the parton cascade. In the case when we neglect correlations between colour dipoles in the parton cascade, \( n_{N, \lambda} = C_\lambda (A n_{N}^{(1)})^n \). The coefficients \( C_\lambda \) should be found from the initial condition of Eq. .

For \( n_{N, \lambda} (r, y, b) \) we can write the evolution equation in the form of Eq. (2.11), noticing that every colour dipole with size \( r \) at rapidity \( y \) and impact parameter \( b \) can either propagate to rapidity \( y + dy \) changing its size from \( r \) to \( r + dr \) (see the first two
terms in Fig. 2) or decay into two colour dipoles (see the last term in Fig. 2 and Fig. 3a). This observation leads to the equation [42]

$$\frac{\partial^2 n_{(n)}^{(m)}(r,b_1,y)}{\partial y \partial r} = \bar{\alpha}_s \{ n_{(n)}^{(m)}(r,b_1,y) - n e^{-r} n_{(n+1)}^{(m)}(r,b_1,y) \},$$  

(3.2)

where the coefficient $\bar{\alpha}_s n^2$ corresponds to $\omega \gamma_y(\omega)$ where $\gamma_y$ is the anomalous dimension of $n_{(n)}^{(m)}$ and $\omega$ is the variable conjugated to $y$. In Ref. [54] the limit of small $\omega$ (low $x$) for $\gamma_y$ has been calculated.

Comparing Eq. (3.1) with Eq. (2.9) one can obtain the following information on the generating function of Eq. (3.1):

$$C_n = \frac{(-1)^{n+1}}{2^n n!},$$  

(3.3)

$$N(r,b_1,y) = 2 g_A(r,y,b_1,\eta = -r + \eta_\Lambda),$$  

(3.4)

$$n_{(n)}^{(m)}(r,y = 0(x = x_0),b_1) = C_n (n_{(n)}^{DGLAP}(r,y = 0(x = x_0),b_1))^n,$$  

(3.5)

$$n_{(n)}^{DGLAP}(r,y = 0(x = x_0),b_1) = \frac{\alpha_s \pi^2}{2 N_c \pi R_N^2} x_0 G_N(x_0,\eta') S_N(b_1).$$  

(3.6)

$$\eta_\Lambda = \ln(AR_N^2 S_A(b_1)).$$  

(3.7)

Eqs. (3.5) and (3.6) mean that we consider $x = x_0$ to be so small that we can neglect correlations between the produced colour dipoles.

Eq. (3.2) can be rewritten as a linear equation for the generating function $g_A$ of Eq. (3.1),

$$\frac{\partial^2}{\partial y \partial r} g_A(r,y,b_1,\eta) = \bar{\alpha}_s \frac{\partial^2}{\partial \eta^2} g_A(r,y,b_1,\eta) - \bar{\alpha}_s \gamma e^{-r-\eta} \left( \frac{\partial g_A(r,y,b_1,\eta)}{\partial \eta} \right) - g_A(r,y,b_1,\eta),$$  

(3.8)

with $\gamma = 1$.

This equation is a linear equation which can be solved by just going to the Mellin transform with respect to $y$ and $\xi = r + \eta$,

$$g_A(\xi,y,b_1,\eta) = \int \frac{d\omega dp}{(2\pi i)^2} g_A(\xi,\omega,b_1,p) e^{\omega y + p \eta}.$$  

(3.9)

In Eq. (3.9) the contours of integration over $\omega$ and $p$ lie along the imaginary axis to the right of all singularities in $\omega$ and $p$.

Substituting Eq. (3.9) into Eq. (3.8) we obtain the following equation for the Mellin image $g_A(\xi,\omega,b_1,p)$:

$$\frac{dg_A(\xi,\omega,b_1,p)}{d\xi} = \frac{\bar{\alpha}_s}{\omega} \left( p^2 - \gamma(p - 1) e^{-\xi} \right) g_A(\xi,\omega,b_1,p).$$  

(3.10)
which has an obvious solution,
\[
g_A(\xi, \omega, b, p) = g_A(\omega, p, b) e^{\frac{\pi s}{\omega} (p^2 - \xi^2)} e^{\frac{\pi s}{\omega} (p - 1) (e^{\gamma} - e^{-\gamma})}.
\]
(3.11)
The function \( g_A(\omega, p, b) \) should be found from the initial condition of Eq. (2.9) (see also Eqs. (3.3)–(3.6)). Our statement is that
\[
g_A(\omega, b, p) = \Gamma(-f(p)) \left( \frac{\alpha_s \pi^2}{4 \pi R_N^2} \right)^f \frac{df(p)}{dp} \left( \frac{\pi s}{\omega} (p - 1) e^{\gamma} - \frac{\pi s}{\omega} e^{\gamma} \right)
\]
satisfies the initial conditions, with \( f(p) = p + \alpha_s p^2 / \omega \). Substituting Eq. (3.12) in Eq. (3.11) and changing the variables of integration from \( p \) to \( f \), we obtain
\[
g_A(\xi, b, \eta, \eta') = \int \frac{d\omega d\eta}{2\pi i} \Gamma(-f) \left( \frac{\alpha_s \pi^2}{\pi R_N^2} \right)^f e^{-p(f) + f \eta + \omega \eta + \pi s (p(f) - 1) / \omega},
\]
where \( p(f) \) is determined by
\[
f = p(f) + \frac{\alpha_s}{\omega} f^2(f).
\]
(3.14)
Indeed, at \( y = y_0 \) \( (x = x_0) \), \( |\xi| \ll r, \xi = \eta, r \gg 1 \), we have
\[
g_A(r, b, y = y_0; \eta = -r + \eta) = \int \frac{d\omega d\eta}{2\pi i} \Gamma(-f) e^{-p(f) + f \eta + \omega \eta - \pi s / \omega^2}
\]
\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} e^{-\eta r} \left( \frac{\alpha_s \pi^2}{N_c \pi R_N^2} \right)^n \int \frac{d\omega}{2\pi i} e^{\omega \eta - \pi s r / \omega}
\]
\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{\alpha_s \pi^2}{N_c \pi R_N^2} \right)^n \gamma_n^2 A x_0 G^{DGLAP}(x_0, 1/x_0^2)^n.
\]
(3.15)
Taking into account Eq. (3.4), one can see that Eq. (3.15) reproduces correctly the initial condition of Eq. (2.9).
It should be stressed that the contour of integration over \( f \) was taken along the positive real axis in such a way that all positive and integer \( f \) are encircled by it. The contour over \( f \) is well defined due to the rapid decrease of \( \Gamma(-f) \) (see Ref. [42]). To study the asymptotic behaviour of the parton densities at low \( x \) we have to move the contour towards the imaginary axis. The asymptotic stems from the rightmost singularities in \( f \) as well as from possible saddle points in this integral. The first dangerous region is the vicinity of \( f \rightarrow 1 \) where we expect a saddle point. To investigate the
situation near \( f = 1 \), we introduce a new variable \( f = 1 + t \), hoping that \( t \) will be very small. For \( t \to 0 \) our solution of Eq. (3.13) has the form

\[
g_A(b, y, \eta) = \int \frac{d\omega d\xi}{(2\pi i)^2} \Gamma(-f) e^{\omega}(p-1)e^{-\alpha_1 + f(\eta_1 + \bar{\eta}) + p_1 + w_1} \]

\[
= \int \frac{d\omega d\omega dt}{(2\pi i)^2} e^{\phi(u, r, t, b)} e^{\eta_1 + \bar{\eta}},
\]

(3.16)

where \( \bar{\eta} = \ln(\alpha_s \pi^2/N, \pi R^2) \).

In Appendix A we calculate the integral of Eq. (3.16) using the saddle point approximation and the answer that we obtain in this appendix (see Eq. (A.5)) can be written in the form

\[
g_A(y, \xi) = \eta_1, r, b) = \frac{\alpha_s^2}{2\pi} \frac{\alpha_s^2 \pi^2 A}{e^{b_1/r^2} A e^{u_1 - r_1 + \pi_1/r}} e^{b_1/r^2} A e^{u_1 - r_1 + \pi_1/r},
\]

(3.17)

with \( \omega_{ci} \) given by Eq. (A.3) and \( t_0 \) by Eq. (A.4).

One can see that Eq. (3.17) leads to \( g_A(y, \xi) = \eta_1, r, b) \), which is constant on the critical line given by the equation

\[
\Psi = \omega_{ci} y_{ci} - r + \frac{\alpha_s}{\omega_{ci}} r + \eta_1 + \bar{\eta} - \ln \frac{2\pi s r t_0}{\omega_{ci}} = \text{Const.}(\alpha_s),
\]

(3.18)

where \( \text{Const.}(\alpha_s) \) can be a function of \( \alpha_s \) but it does not depend on either \( y \), \( r \) or \( \eta_1 \).

We anticipate that the value of \( t_0 \) will be small on the critical line and, because of this, the last term in Eq. (3.18) has been taken into account in the phase \( \Psi \).

The solution of Eq. (3.18) gives

\[
y_{ci} = \frac{1}{4\pi s} (r - 2(\eta_1 + \bar{\eta}) + 2).
\]

(3.19)

This line gives the minimal value for \( \Psi \), namely, \( \Psi = 1 \).

The value of \( g_A(y, \xi) = \eta_1, r, b) \) is equal to

\[
g_A(y, \xi) = \frac{e^{b_1/r^2} A e^{u_1 - r_1 + \pi_1/r}}{\pi}.
\]

(3.20)

Solving Eq. (3.19) with respect to \( r \) we obtain the typical momentum (colour dipole size) on the critical line

\[
Q_A^2(x; b) = Q_0^2 \left( \frac{\alpha_s}{N, R^2} e^{b_1} \left( \frac{AR^2}{R^2} \right) e^{\pi s/r} e^{-2b_1/r^2},
\]

(3.21)

At \( b = 0 \) \( Q_A^2(x; b = 0) \propto A^{1/3} \), as can be seen easily from Eq. (3.21).

Comparing Eq. (3.20) with Eq. (2.4) one sees that \( xG_A(x, 1/x^2) \) on the critical line is equal to

\[
xG_A(x, 1/x^2) = \frac{2N, R^2 Q_0^2}{\alpha_s \pi^2} N(x^2, b = 0, y_{ci}) = \frac{4N^2 e}{\pi^2 - R^2} Q_0^2(x) \propto A^{1/3}.
\]

(3.22)
It is interesting to notice that such an $A$-dependence can be understood directly from the Mueller–Glauber formula of Eq. (2.9). Indeed, the shadowing corrections become essential when

$$\frac{x^2 A x G(x, 1/x^2)}{4N_c \pi R_A^2} \approx 1.$$  

(3.23)

Recalling that the anomalous dimension in the vicinity of the critical line is equal to $\gamma(\omega) = \bar{\omega}_A / \omega_A = \bar{\omega}$, one can find from Eq. (3.23) that $x^2 = 1 / Q^2_A \propto A^{-2/3}$, which is the same $A$-dependence that we obtained from more advanced calculations (see Eq. (3.22)).

### 4. Solution to the left of the critical line

In this section we are going to solve Eq. (2.13) using the idea suggested in Ref. [43], namely, $\tilde{N}_- = \tilde{N}_- (z)$ where

$$z = 4\bar{\omega}_A y - r - \beta(A; b_i).$$  

(4.1)

Function $\beta(A; b_i)$ can be chosen in a such way that the equation for the critical line in terms of new variable $z$ looks as $z = 0$. One can see from Eq. (3.19) that $\beta(b_i)$ is equal to

$$\beta(b_i) = -2\bar{\eta} - 2\frac{\pi R_N^2}{\pi R_A^2} + \frac{2b_i^2}{R_A^2} + 2.$$  

(4.2)

Assuming that $\tilde{N}_- (r, b_i, y)$ is a function of $z$ and $b_i$ only, we can rewrite Eq. (2.13) in the form:

$$\frac{d^2 \tilde{N}_- (z, b_i)}{dz^2} = \frac{1}{4} \left( 1 - \frac{d \tilde{N}_- (z, b_i)}{dz} \right) \tilde{N}_- (z, b_i).$$  

(4.3)

Here we used the explicit equation for variable $z$, namely, $z = 4\bar{\omega}_A y - r - \beta(b_i)$.

#### 4.1. A solution to Eq. (4.3)

First, we find a solution to Eq. (4.3) introducing function $\xi(z)$ in the following way:

$$\tilde{N}_- (z, b_i) = \int^z d\xi (1 - e^{-\xi(z, b_i)}).$$  

(4.4)

For function $\tilde{N}_- (z, b_i)$ Eq. (4.3) can be reduced to

$$\frac{d\xi (z, b_i)}{dz} = \frac{1}{4} \int^z d\xi' (1 - e^{-\xi(z', b_i)}).$$  

(4.5)

Changing variable in the integral from $\xi'$ to $\xi''$ we have

$$\frac{d\xi (z, b_i)}{dz} = \frac{1}{4} \int^\xi \xi'' (b_i) d\xi'' / \xi'' (1 - e^{-\xi''}).$$  

(4.6)
Eq. (4.6) can be easily solved and the solution is

\[ z = \sqrt{2} \int_{\xi_0(b)}^{\xi} \frac{d\xi'}{\xi_0(b) + (e^{-\xi'} - 1)} . \]  

(4.7)

The numerical solution to Eq. (4.7) is given in Fig. 4. This solution depends on the initial conditions on the critical line. There are two of them, namely,

\[ N_+ (z = 0) = N_+ (\xi = 0), \]  

(4.8)

\[ \left. \frac{d\ln N_+}{dz} \right|_{\xi = 0} = \left. \frac{d\ln N_+}{dz} \right|_{\xi = 0} = \frac{1}{2} . \]  

(4.9)

We can assume that \( \zeta \gg 1 \) at large \( z \). In this case we can neglect \( e^{-\xi'} \) in the integrand and we obtain an explicit analytic solution:

\[ \zeta (z) = \left( \sqrt{\xi_0(b)} + \frac{z}{2\sqrt{2}} \right)^2 , \]  

(4.10)

or in terms \( N(z,b) \) we have

\[ N_+ (z,b) = 1 - \exp \left( - \left( \sqrt{\xi_0(b)} + \frac{z}{2\sqrt{2}} \right)^2 \right) . \]  

(4.11)

where \( z_0(b) \) should be found from matching of this solution with the solution of the previous section on the critical line.

4.2. Matching with the solution to the right of the critical line

We cannot use the simple solution of Eq. (4.10) for matching with the solution of Eq. (3.17) since \( \zeta \) is expected to be rather small in the vicinity of the critical line and,

![Fig. 4. Dipole number density \( N(z) \) as a function of the critical line parameter \( z \). The solid line is the numerical solution with \( \hat{N}(0) = 0.01 \), while the dashed line corresponds to \( \hat{N}(0) = 0.4 \). The thick dashed line shows the asymptotic solution of Eq. (4.11) with \( \hat{N}(0) = 0.4 \).](image-url)
therefore, we cannot neglect $e^{-\xi}$ in Eq. (4.7). However, for small $\xi \ll 1$ we easily obtain a simple solution to Eq. (4.7), namely,

$$\ln \frac{\xi}{\xi_0(b_i)} = \frac{1}{2} \tilde{z}. \quad (4.12)$$

Eq. (4.12) leads to Eq. (4.9) of the matching condition. This equation allows us to find $\xi_0(b_i)$, namely,

$$1 - e^{-\xi_0(b_i)} = N_{\xi, \rightarrow}. \quad (4.13)$$

Eq. (4.12) and Eq. (4.13) provide matching as well as determine the parameters of the asymptotic behaviour for $N_{\xi}$ given by Eq. (4.11). Using Eq. (3.20) one can find from Eq. (4.13) that

$$\xi_0(b_i) = 2 \frac{e^{\tilde{n}_s}}{\pi}. \quad (4.14)$$

Fig. 4 illustrates the behaviour of $N(z)$ versus $z$. One can see that the asymptotic is reached only at very large $z \approx 7 - 10$ or for very low $x (x \approx 10^{-6} - 10^{-7})$. At HERA we have $z \approx 4 - 5$ and we are far away from the asymptotic solution. However, we can penetrate the region of large $z$ using a nuclear target, as one can see in Eq. (3.19). Indeed, for heavy nuclei we can easily have $2n_A = 3.5 - 4$. It gives a possibility to have, even in the HERA kinematic region, sufficiently large values of $z \approx 7.5 - 9$.

4.3. Stability of the solution

The arguments of the last section show that the solution given by Eq. (4.7) has a big chance to be the solution to our equation. To prove this statement we only need to show that the solution of Eq. (4.7) is stable with respect to small perturbations of the initial conditions on the critical line. In other words, let us consider a small variation of the initial conditions $|\delta \zeta_{\alpha}(y, r, b_i)| < \delta$, where $0 < \delta \ll 1$. The solution is stable if for every $\delta$ we can find a small $c$ such that $c \rightarrow 0$ for $\delta \rightarrow 0$. We can find a linear equation for the function $\delta \zeta(y, r, b_i)$ substituting $\zeta(y, r, b_i) \rightarrow \zeta_i + \delta \zeta(y, r, b_i)$, where $\zeta_i$ is given by Eq. (4.7). Assuming that $\delta \zeta_{\alpha}(y, r, b_i)$ is small we obtain the linear equation

$$\frac{\partial^2 \delta \zeta(y, r, b_i)}{\partial y \partial r} = \overline{\alpha} \delta \zeta(y, r, b_i) e^{-\xi_{\alpha}(z)}. \quad (4.15)$$

In the variables $z$ and $r$ this equation has the form

$$\frac{\partial^2 \delta \zeta(z, r, b_i)}{\partial z \partial r} = \frac{1}{4} \delta \zeta(z, r, b_i) e^{-\xi_{\alpha}(z)}. \quad (4.16)$$

which can be solved using the Mellin transform with respect to the variable $r$. Indeed, for the Mellin image we have

$$\frac{d \delta \zeta(z, \nu, b_i)}{dz} = \frac{1}{4 \nu} \delta \zeta(z, \nu, b_i) e^{-\xi_{\alpha}(z)}. \quad (4.17)$$
The general solution of Eq. (4.17) looks as follows:

$$\delta \xi(z, v, b) = \int_a^{a+i\pi} \frac{d\nu}{2\pi i} \delta \xi(\nu) \exp \left( \nu r + \frac{1}{\nu} \int_0^z dz' e^{-\zeta(z', b)} \right).$$  \hspace{1cm} (4.18)

The integral over $z'$ in Eq. (4.18) does not depend on $z$ for large $z$. It means that the solution given by this equation is actually a function of $r$ only. It is obvious that such a solution cannot be tolerated by the initial conditions on the critical line.

### 4.4. Low-$x$ (high-energy) asymptotic

As we discussed, the total dipole cross section is equal to

$$\sigma_{\text{dipole}} = \int d^2 b \, 2N(r, y, h) = 2\pi \left[ \int_0^{b_0 \gamma} db z^2 N_{\gamma} + \int_{b_0 \gamma}^{\infty} db z^2 N_{\gamma} \right]$$

$$= 2\pi \int_0^{b_0 \gamma} db z^2 - 2\pi b_0 \gamma(y, r),$$  \hspace{1cm} (4.19)

since $N_{\gamma}(r, y, h) \to 1$ at low $x$ (see Eq. (4.11)) and $N_{\gamma}(r, y, h)$ gives a small contribution to the integral in Eq. (4.19).

The value for $b_0 \gamma(y, r)$ can be evaluated by recalling that the solution of Eq. (4.11) is

$$b_0 \gamma(y, r) = \frac{R_z^2}{2} \left( 4\pi \gamma y - r + 2\ln \left( \frac{AR_z^2}{R_A^2} \right) \right),$$  \hspace{1cm} (4.20)

which gives

$$\sigma_{\text{dipole}} = \pi R_A^2 \left( 4\pi \gamma y - r + 2\ln \left( \frac{AR_z^2}{R_A^2} \right) \right).$$  \hspace{1cm} (4.21)

For the gluon structure function, Eq. (4.21) leads to

$$xG(x, Q^2) \to Q^2 R_A^2 \left[ 4\pi \gamma \ln(1/x) - \ln Q^2 + 2\ln \left( \frac{AR_z^2}{R_A^2} \right) \right].$$  \hspace{1cm} (4.22)

Eq. (4.19) can be rewritten as an integral over $z$, namely

$$\sigma_{\text{dipole}} = \int d^2 b \, 2N(r, y, h) = 2\pi \left[ \int_0^{b_0 \gamma} db z^2 N_{\gamma} + \int_{b_0 \gamma}^{\infty} db z^2 N_{\gamma} \right]$$

$$= \pi R_A^2 \int_0^{4\pi \gamma y - r + 2\ln(AR_z^2/R_A^2)} dz \left( 1 - e^{-z} \right) + 2\pi \gamma R_A^2.$$  \hspace{1cm} (4.23)

It is interesting to note that $\sigma_{\text{dipole}}^A$ for a nucleus target manifests itself in a remarkable scaling in the kinematic region to the left of the critical line. Indeed, from Eq. (4.23) one can conclude that

$$\frac{\sigma_{\text{dipole}}^A}{\pi R_A^2} = \frac{Q^2 \left( R_A^2 \right)^2}{A^2 \left( R_A^2 \right)^2}.$$  \hspace{1cm} (4.24)
Fig. 5 shows the $Q^2$ behaviour of the ratio $\sigma_{A dipole}^A / \pi R_A^2$ at $x = 10^{-4}$ with $y = \ln(0.01/x)$ for different nuclei. The full line describes the nucleon target and other lines correspond to nuclei with $A = 40$ (Ca), 150, 296 (Au) going from the bottom to the top. One can see that the cross sections are considerably larger than the geometrical estimates.

It should be stressed that Eq. (4.20) is an artifact of the oversimplified Gaussian parameterization for the nucleus profile function (see Eq. (2.6)). In a more realistic approach to the nucleus profile function (for example the Woods–Saxon one [55]) instead of $\exp(-b_r^2/R^2_A)$ in Eq. (2.6) we have a function which is equal to 1 for $b_r \leq R_A$ and falls down as $\exp(-b_r/h)$, where $h$ does not depend on $A$, for $b_r > R_A$. Such a function gives $b_0(y,r)$,

$$b_0(y,r) = R_A + \frac{h}{2} \left( 4\alpha_s y - r + 2\ln\left(\frac{AR^2}{R_A^2}\right) \right),$$

instead of Eq. (4.20).

Therefore, we expect the following asymptotic behaviour for $\sigma_{A dipole}$ and $xG(x,Q^2)$:

$$\sigma_{A dipole} \to 2\pi \left( R + \frac{h}{2} \left( 4\alpha_s y - r + 2\ln\left(\frac{AR^2}{R_A^2}\right) \right) \right)^2,$$

$$xG(x,Q^2) \to Q^2 \left( R + \frac{h}{2} \left( 4\alpha_s y - r + 2\ln\left(\frac{AR^2}{R_A^2}\right) \right) \right)^2.$$  

5. Summary

In this paper we found the solution to the evolution equation for high parton density QCD (see Eq. (2.8)). This solution is given by two equations: Eq. (3.13) to the right of the critical line (see Fig. 1) and Eq. (4.4) to the left of the critical line described by Eq. (3.19).

---

We are very grateful to Al Mueller for pointing out to us a model feature of Eq. (4.20).
This solution gives the colour dipole density \( \rho = xG(x, 1/x^2)/\pi R_A^2 \) which is small to the right of the critical line, reaches a value of the order of unity on the critical line and increases up to a value \( p = 1/\alpha_s \) at very small values of \( x \).

Such a behaviour of \( \rho \) reveals itself in the following properties of the gluon structure function:

1. To the right of the critical line \( xG \) is given by the DGLAP evolution equations;
2. On the critical line \( xG(x, Q^2_A) = Q^2_A(x) R_A^2 \) (see Eq. (3.22)) where \( Q^2_A(x) \) is defined by Eq. (3.21);
3. To the left of the critical line \( xG(x, Q^2) = Q^2 [R_A^2 + \frac{2}{7}(4\alpha_s y - r + 2\ln(A R_S^2/R_A^2))] \).

As far as the \( A \)-dependence is concerned, in the kinematic region to the right of the critical line we have \( xG_A \propto xG_N \), while on the critical line \( xG_A(x, Q^2_A(x)) \propto A^{4/3} \) and only to the left of the critical line we have \( xG_A \propto Q^2 R_A^2 \propto A^{2/3} \).

Our solution reproduces the saturation of the gluon density but due to the sufficiently mild dependence on the impact parameter the saturation leads to the dipole-target total cross section proportional to \( \ln(1/x) \) in the region of extremely low \( x \) (see Eq. (4.21) for the exact answer).

Both the exact solution near the saturation limit (see Eq. (4.11)) and the proportionality to \( xG \propto Q^2 R_A^2 \) (see Eq. (4.22)) are a manifestation of the fact that the non-linear equation leads to a parton distribution which peaks at the size \( x^2 = 1/Q^2_A(x) \). In other words, in the region of low \( x \) the parton distribution has a definite mean transverse momentum, namely, \( \langle p_t^2 \rangle = Q^2_A(x) \). Therefore, this solution supports the more intuitive approach that has been discussed earlier for the parton distribution at low \( x \) [44–46].

We also want to draw the reader’s attention to the fact that in the region of small \( x \) our solution leads to a new scaling between DIS with a nucleon and DIS with a nucleus given by Eq. (4.24). This equation allows us to calculate how nucleus DIS approaches the \( A^{2/3} \) dependence in the region of low \( x \).

We hope that our solution will stimulate a more detailed study of the properties of the high density parton system and, in particular, that it will lead to a more quantitative development of the ideas suggested in Refs. [45,46].

Appendix A

In this appendix we calculate the phase \( \psi(\omega, r, t, \eta, b) \) in Eq. (3.16) for \( t \ll 1 \). Using Eq. (3.14), we can find \( p \) for \( \alpha_s \ll 1 \) and \( t \to 0 \)

\[
p = 1 + t - \frac{\bar{\alpha}_s}{\omega} (1 + 2t),
\]

which leads to

\[
\psi(\omega, r, t, \eta, b) + t(\eta_t + \bar{\eta}) = \frac{\bar{\alpha}_s y}{\omega} e^{-\eta_t (p - 1)} + pr + \omega y + t(\eta_t + \bar{\eta}) \approx \omega y + r - \frac{\bar{\alpha}_s}{\omega} r - \ln \left( -\frac{\bar{\alpha}_s y}{\omega} e^{-\eta_t} - 1 + \frac{2\bar{\alpha}_s}{\omega} + \frac{1}{r} (\eta_t + \bar{\eta}) \right). (A.2)
\]
Integration over \( t \) gives a \( \delta \)-function which defines the value of \( \omega = \omega_{ct} \),

\[
\omega_{ct} = \frac{2 \bar{\alpha}_s + \frac{\bar{\alpha}_s y}{r} e^{-n_i}}{1 - \frac{1}{r}(\eta_h + \bar{\eta})} = \frac{2 \bar{\alpha}_s s}{1 - \frac{1}{r}(\eta_h + \bar{\eta})},
\]

(A.3)

where \( s = 1 + \frac{2}{r} e^{-n_i} \).

We can carry out the integration over \( \omega = \omega_{ct} + \Delta \), considering \( \Delta / \omega_{ct} \ll 1 \),

\[
\psi(\omega, r, t, b_i) \equiv -r + \frac{\bar{\alpha}_s r}{\omega_{ct}} + \omega_{ct} y + \Delta \left( y - \frac{\bar{\alpha}_s r}{\omega_{ct}^2} - \frac{2 \bar{\alpha}_s r}{\omega_{ct}^2} \right)
\]

\[
= -r + \frac{\bar{\alpha}_s r}{\omega_{ct}} + \omega_{ct} y + \Delta \frac{2 \bar{\alpha}_s(r)}{\omega_{ct}^2} \left( y - \frac{\bar{\alpha}_s r}{\omega_{ct}^2} \right) \frac{\omega_{ct}^2}{\omega_{ct}^2} - \left( \frac{\omega_{ct}^2}{\omega_{ct}^2} - t \right).
\]

(A.4)

Integration over \( \Delta \) gives \( \delta\left[ \frac{2\pi \omega_{ct}^2}{\omega_{ct}^2} \right] (t - t_0) \), which yields

\[
g_A(y, \xi = \eta_h, r, b_i) = \frac{\omega_{ct}^2}{2 \pi 2 \bar{\alpha}_s(r) t_0^2} \frac{\alpha_s \pi y A}{4 \pi R_h^2} e^{b_i/r} e^{u_{si}r - r + \pi_s y / \omega_{ct}} .
\]

(A.5)

References