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Mathematical Knowledge for Preservice Teachers

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> East Asian students, taught by competent teachers, acquire proficiency in mathematics. When they graduate and join the teacher force, they in turn become competent teachers. Once a good cycle starts, the positive effects accumulate and increasingly reproduce themselves. Unfortunately, this holds true for a vicious cycle as well. (Leung, 2006, p. 43).

> There is evidence of a vicious cycle in which too many [U.S.] prospective teachers enter college with an insufficient understanding of school mathematics, experience little college instruction focused on the mathematics they will teach, and then enter their classrooms inadequately prepared to teach mathematics to the following generations of students. (CBMS, 2001, p. 163)

U.S. mathematics educators are faced with the challenge of breaking this vicious cycle. The most viable place where the cycle can be broken is in the preparation of a new generation of teachers. This task is challenging because we have to work with preservice teachers who are mathematically unsophisticated (Seaman & Szydlik, 2007; Ma, 1999).

The Conference Board of the Mathematical Sciences (2001) made five recommendations with regard to mathematics curriculum and instruction for prospective teachers. Listed below are three of the five recommendations:

- Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach (p. 7).
- Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas. All courses designed for prospective teachers should develop careful reasoning and mathematical 'common sense' in analyzing conceptual relationships and in solving problems (p. 8).
- Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the *habits of mind of a mathematical thinker* (italics added) and demonstrate flexible, interactive styles of teaching. (p. 8).

These recommendations emphasize mathematical understanding as well as mathematical reasoning and thinking.

In this paper, the "habits of mind of a mathematical thinker" are explored in the context of mathematics knowledge for preservice teachers. Students' "deficient" habits of mind are discussed in the first section. Shulman's (1986) categorization of subject matter knowledge and Harel's distinction between ways of understanding and ways of thinking are presented in the second section. The relations among ways of understanding, ways of thinking, and pedagogical content knowledge are explored in the third section. Questions for research are suggested in the final section.

Students' Deficient Ways of Thinking

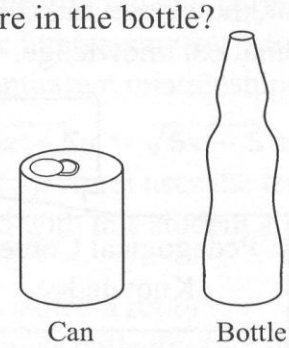
Many students apply procedures they have been taught without having to make sense of what they are doing. Teachers frequently witness inappropriate use of procedures, what Fischbein and Barash (1993) call improper application of algorithmic models. For example, consider the following problem that was posed by Cramer, Post and Currier (1993) to preservice teachers in a mathematics methods class: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* 32 out of 33 preservice teachers "solved" this problem by setting up a proportion such as $\frac{9}{3} = \frac{x}{15}$. These students had applied the proportion algorithm without making "sense," from an observer's perspective, of the problem situation. Because they have a tool, the proportion algorithm, they did not make sense of the situation.

The phenomenon of overusing proportional strategy is observed in my pilot study. After being taught fractions, ratios, and proportions, preservice elementary and middle-school teachers performed better in the post-test for all four direct-proportional items but worse in all three non-direct-proportional items. Figure 1 shows that (a) the number of students who chose the correct answer "b" for the direct-proportional item increased from 64% (out of 138 students) to 78% (out of 124 students), (b) the number of students who chose the correct answer "a" for the inverse-proportional item, on the other hand, dropped from 53% to 42% , and (c) the number of students who chose the incorrect proportional answer "d" increased from 24% to 40%.

Direct-Proportional Item

The ratio of the amount of soda in the can to the amount of soda in the bottle is 4:3. There are 12 fluid ounces of soda in the can, how many fluid ounces of soda are in the bottle?

	Pretest:	Posttest:
(a) 8 fluid ounces	3%	6%
(b) 9 fluid ounces	64%	78%
(c) 15 fluid ounces	6%	3%
(d) 16 fluid ounces	27%	11%
(e) None of the above	1%	2%



Inverse-Proportional Item

The ratio of the volume of a small glass to the volume of a large glass is 3:5. If it takes 15 small glasses to fill the container, how many large glasses does it take to fill the container?

	Pretest:	Posttest:
(a) 9 glasses	53%	42%
(b) 13 glasses	9%	13%
(c) 17 glasses	4%	2%
(d) 25 glasses	24%	40%
(e) None of the above	10%	2%

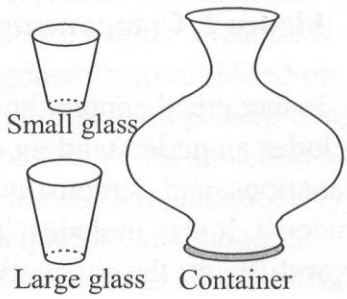


Figure 1. Pre- and post-test comparison for two items

In their study of 11 preservice elementary teachers' use of a web-based resource, Seaman and Szydlik (2007) found that these preservice teachers "displayed a set of values and avenues for doing mathematics so different from that of the mathematical community, and so impoverished, that they found it difficult to create fundamental mathematical understandings" (p. 167). These preservice teachers' deficient ways of thinking prevented them from utilizing mathematical resources to solve problems. For example, "they did not even attempt to make sense of the relevant definitions provided by a teacher resource" (p. 179) for solving this problem: *What is the least common multiple of 60 and 105?*

Habits of mind of a mathematical thinker are necessary not only for solving problems, but for learning mathematics as well as teaching mathematics effectively. Examples of efficacious ways of thinking include investigating the mathematical structure underlying an observed pattern, seeking analogous structure in different mathematical entities, making and testing conjectures, creating mental models, examples, and non-examples for mathematical objects, being precise with use of terms, symbols, and notations, and capitalizing on representational tools to organize information and structure relationships. Cuoco, Goldenberg, and Mark (1996) described in detail some general habits of minds and some habits of mind that are specific to mathematics. The latter includes thinking big and thinking small, using functions, using multiple points of view, and mixing deduction and experiment. Ideally, preservice teachers should gain appreciation for these ways of thinking and have a desire to inculcate them as they progress through the teacher preparation program and throughout their teaching career.

Categories of Mathematical Knowledge for Teaching Mathematics

According to Shulman (1986), teaching effectively requires more than mastery of subject matter content knowledge. He identifies two other categories: pedagogical content knowledge and curricular knowledge. Figure 2 shows the various components of mathematical knowledge.

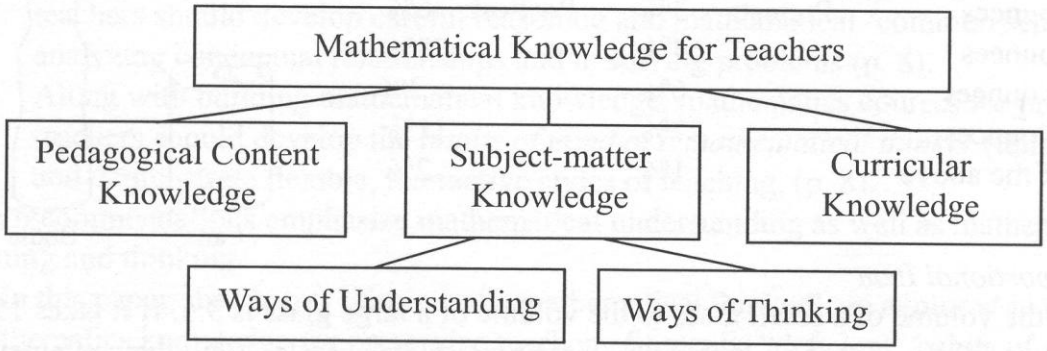


Figure 2. Components of mathematical knowledge

Pedagogical content knowledge refers to subject matter knowledge for teaching. It includes an understanding of effective forms of representations, examples, analogies, explanations, and demonstrations that make a particular piece of knowledge comprehensible to students. It also includes "an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and

backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 10).

Curricular knowledge refers to knowledge of "the full range of programs designed for the teaching of particular subjects and topics at a given level [and] the variety of instructional materials available in relation to those programs" (p. 10).

Subject matter content knowledge, for Shulman (1986), includes an understanding of the structure of the subject matter. He identifies two types of structures: "The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established" (p. 10). The two types of structures may be viewed as the "product" and "process" of doing mathematics, which coincide with NCTM's (2000) *Content Standards* and *Process Standards*. Harel's (in press) definition of mathematics accentuates the difference between these two aspects.

> Mathematics consists of two complementary subsets. The first subset is a collection of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the institutionalized *ways of understanding* (italics added) in mathematics throughout history. … The second subset consists of all the *ways of thinking* (italics added), which are characteristic of the mental acts whose products comprise the first set (p. 8).

A way of understanding refers to the cognitive product of a mental act carried out by a person whereas a way of thinking refers to the cognitive characteristic of the act. If the mental act is interpreting, then the way of understanding is the actual interpretation one gives to a term or a string of symbols. For example, ways of understanding for interpreting $y = \sqrt{6x - 5}y = \sqrt{6x - 5}$ include (a) an equation that constrains the values of x and y, (b) a function where x is the input and $\sqrt{6x - 5}\sqrt{6x - 5}$ is the output, and (c) a proposition-valued function that can be either true or false depending on the values of the ordered pair (x, y). "These ways of understanding manifest certain characteristics of the interpreting act – for example, that 'symbols in mathematics represent quantities and quantitative relationships'." (Harel, in press, p. 5). Alternatively a student may interpret $y = \sqrt{6x - 5}y = \sqrt{6x - 5}$ as "a thing where what you do on the left you do on the right" (ibid, p. 5). Harel uses the term "non-referential symbolic" to refer to the way of thinking associated with this student's interpretation.

With respect to the mental act of proving, a way of thinking is called a *proof scheme* (Harel and Sowder, 1998), which refers to the character of one's collective act of ascertaining for oneself and persuading others of the truth of something. As mathematics educators, we want students to advance from an authoritative proof scheme (reliance on a teacher or textbook for conviction) and empirical proof scheme (reliance on empirical cases) to deductive proof schemes (conviction via a logical argument). For the mental act of generalizing, we want students to advance from result-pattern generalization (based on numerical patterns) to process-pattern generalization (based on an established principle that can account for the observed patterns) (Harel, 2001). For the act of foreseeing an action to solve a problem, we want to advance students from impulsive anticipation (spontaneously proceeding with the first idea that comes to mind) to analytic anticipation (analyzing the problem situation) (Lim, 2006). For the act of predicting we want students to advance from

association-based prediction (based on associating two ideas without establishing the basis of the association) to coordination-based prediction (based on coordinating quantities and attending to relationships) (Lim, 2006). For the act of problem-solving we want students to progress from less desirable strategies such as relying on keywords to heuristics like drawing a figure and decomposing and recombining (Pólya, 1945).

Relating Ways of Understanding, Ways of Thinking and Pedagogical Content Knowledge

Harel (in press) asserts that "students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess" (p. 19). This principle underscores the importance of incorporating complementary ways of understanding and ways of thinking into cognitive objectives for instruction. Harel and Sowder (2005) observed that "teachers often focus on ways of understanding but overlook the goal of helping students abstract effective ways of thinking from these ways of understanding" (p. 29). For example, students may learn completing the square as a procedure for solving quadratic equations without recognizing that algebraic expressions are manipulated "with the purpose of arriving at a desired form and maintaining certain properties of the expression invariant" (Harel, in press).

Implementing the Duality Principle involves (a) attending to students' existing ways of understanding and ways of thinking; (b) identifying appropriate cognitive objectives, appropriate in the sense that they are aligned with students' current ways of understanding and ways of thinking, and that preserve the mathematical integrity of the content; and (c) designing activities, with an understanding of the interplay among various ways of understanding and ways of thinking to meet those objectives. These activities cannot be carried out effectively without pedagogical content knowledge.

Efficacious pedagogical content knowledge is in turn supported by ways of profound understanding and ways of thinking. Let us consider Ball and Bass's (2000) notion of decompression, which is the process of "deconstruct[ing] one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible" (p. 98). They argue that teachers must be able to "work backward from mature and compressed understanding of the content to unpack its constituent elements" and "decompose a mathematics task, considering its diverse possible trajectories of enactment and engagement" (p. 98). The ability to decompress one's knowledge requires teachers to have profound ways of understanding mathematics. According to Ma (1999), profound understanding of mathematics has three characteristics: breadth, depth, and thoroughness. "Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics" (p. 124).

Decompressing one's knowledge in a goal-oriented manner (to make it accessible for students while maintaining its mathematical integrity) is considered a desirable way of thinking with respect to the act of decomposing a concept into its components. An example of a less desirable way of thinking is to decompose division of fractions into loosely connected topics like division, fractions, and the invert-multiply rule. A more desirable way of thinking, on the other hand, involves establishing connections among various ways of

understanding and pedagocical content knowledge such as (a) quotitive view of division (as opposed to students' dominant view of sharing equally); (b) use of common denominator to facilitate repeated subtraction (an opportunity to reinforce students' understanding of common multiples); (c) the referent unit of the quotient is the divisor (students tend to operate on fractions without attending to its referent unit); (d) the correspondence between the divisor in quotitive division and the multiplier in multiplication (if the divisor is less than dividend, then the "multiplier" is actually a proper fraction); and (e) division by a unit fraction (an intermediate step to explain the invert-and-multiply algorithm). This example highlights the relationship between ways of understanding of dividing by a fraction, the pedagogical content knowledge for teaching the concept of dividing by a fraction, and the ways of thinking associated with the act of decomposing this concept.

Opportunities for Research

In conclusion, both solving a mathematical problem and planning a mathematical lesson involves ways of thinking. A teacher preparation program should aim to help preservice teachers advance ways of thinking, ways of understanding, and pedagogical content knowledge. However, specific relationships among these components of knowledge that are pertinent to teaching and learning have yet to be investigated. Listed below are some questions to initiate research in this area.

- How do preservice teachers' existing ways of thinking facilitate or interfere with their development of ways of understanding and pedagogical content knowledge?
- What specific ways of thinking should a teacher preparation program address? What mathematical tasks or instructional strategies are effective in helping preservice teachers advance their ways of thinking?
- What tasks are effective for accessing students' ways of understanding, ways of thinking, and pedagogical content knowledge associated with a particular mathematical topic?
- What are the differences in ways of thinking and understanding exhibited by U.S. and Chinese preservice teachers before and after completion of their teacher preparation program?

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